A SHORT PROOF OF THE SEGAL CONJECTURE FOR C_p

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Let p be a prime and $C_p = \{1, \sigma, \dots, \sigma^{p-1}\}$ the cyclic group of order p. We give a new and direct proof of the following result. The version for finite spectra was proved by Jones and Wegmann [JW, Theorem A(ii) and the result on page 474 for odd p]. Lunøe-Nielsen and Rognes proved the result for bounded below spectra whose mod-p homology is of finite type [LNR, Theorem 5.13]. The statement in the generality given here is [NS, Theorem III.1.7].

Theorem 1. *For every bounded below spectrum X, the Tate diagonal*

$$X \longrightarrow (X^{\otimes p})^{tC_p},$$

is a p-completion. Here C_p acts by cyclic permutation on $X^{\otimes p} = X \otimes \dots \otimes X$.

For X the sphere spectrum this recovers the Segal conjecture for C_p, which was proved by Lin for $p = 2$ [Lin] and by Gunawardena for odd p [Gun]. The proof given in [NS] is a formal reduction of Theorem 1 to the special case $X = \mathbb{F}_p$, which then follows by Lunøe-Nielsen and Rognes' finite-type theorem since $\mathbb{F}_p$ is of finite type.
Here we give a direct proof of the case $X = \mathbb{F}_p$ and hence, by this reduction, of the full theorem. Our proof is inspired by Hahn and Wilson's descent argument for $p = 2$ [HW] and its extension to odd primes by Angelini-Knoll and Zou using spoke topological Hochschild homology [AKZ, Sections 5–6]. We however use a different descent here and do not need any computational input except the well-known calculation

$$\pi_* \operatorname{THH}(\mathbb{F}_p) = \mathbb{F}_p[u], \qquad \pi_* \operatorname{THH}(\mathbb{F}_p)^{tC_p} = \mathbb{F}_p[x^{\pm 1}]. \tag{1}$$

Here $|u| = |x| = 2$, and the cyclotomic Frobenius sends u to x. The first isomorphism in (1) is due to Bökstedt [Bök, Theorem 1.1(a), (d)]; the Tate computation and the identification of the cyclotomic Frobenius are due to Hesselholt–Madsen [HM, equation (33) and Proposition 5.3]. For a direct proof of Bökstedt periodicity from the dual Steenrod algebra and its Dyer–Lashof operations, see Krause–Nikolaus [KN, Theorems 1.1–1.2]. The statements (1) can also be deduced entirely without any computation using polynomial functoriality and TR as in [BGMN, Theorem 5.5 and Corollary 5.14].

In order to prove the theorem we use the C_p-equivariant map of $\mathbb{E}_\infty$-rings

$$A_{-1} = \mathbb{F}_p^{\otimes p} \longrightarrow \operatorname{THH}(\mathbb{F}_p) = A_0$$

induced by the inclusion of p points into the circle (equivalently by the inclusion of the term in simplicial degree $p-1$ into the realization of the cyclic bar construction or by the universal property of the source as the free $\mathbb{E}_\infty$-algebra under $\mathbb{F}_p$ with C_p-action). We consider the Amitsur complex of this map with i-th term $A_i := A_0 \otimes_{A_{-1}} \cdots \otimes_{A_{-1}} A_0$, with $i+1$ factors. This is an augmented cosimplicial $\mathbb{E}_\infty$-ring with C_p-action:

$$A_{-1} \xrightarrow{\eta} A_0 \underset{d^1}{\overset{d^0}{\rightrightarrows}} A_1 \underset{d^2}{\overset{d^0}{\overset{d^1}{\rightrightarrows}}} A_2 \;\substack{\rightarrow\\ \rightarrow\\ \rightarrow\\ \rightarrow}\; \cdots \tag{2}$$

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Lemma 2. *The diagram*

$$A_{-1}^{tC_p} \xrightarrow{\eta} A_0^{tC_p} \underset{d^1}{\overset{d^0}{\rightrightarrows}} A_1^{tC_p} \underset{d^2}{\overset{d^0}{\overset{d^1}{\rightrightarrows}}} A_2^{tC_p} \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \cdots$$

is a limit diagram.

Proof. The fibre I of $A_{-1} \to A_0$ is 1-connective. The iterated fibres in the associated Tot-tower are tensor powers of I over A_{-1}, so their connectivity tends to infinity. Thus (2) is a limit diagram; see also [DR]. Applying homotopy fixed points preserves this limit. Applying homotopy orbits also gives a limit diagram, since homotopy orbits preserve finite limits and connectivity. The claim then follows since Tate is the cofibre of the norm map from homotopy orbits to homotopy fixed points. □

The $i+1$ equivariant maps $A_0 \to A_i$ induce after taking the Tate construction classes $x_0, \dots, x_i$ in $\pi_2(A_i^{tC_p})$, each the image of $x \in \pi_2(A_0^{tC_p})$. Together these classes give rise to a map

$$\mathbb{F}_p[x_0^{\pm 1}, \dots, x_i^{\pm 1}] \longrightarrow \pi_*(A_i^{tC_p}), \qquad |x_j| = 2. \tag{3}$$

As i varies, these maps form a morphism of augmented cosimplicial graded commutative rings. The source is augmented by $\mathbb{F}_p$, and the map in degree -1 is induced by the Tate diagonal $\mathbb{F}_p \to A_{-1}^{tC_p}$. The cosimplicial maps in the source act on the indices of the generators. Consider the ideal

$$I_i = (x_s - x_t \mid 0 \le s, t \le i) \subseteq \mathbb{F}_p[x_0^{\pm 1}, \dots, x_i^{\pm 1}].$$

Lemma 3. *For $i \ge 0$ the map (3) exhibits $\pi_*(A_i^{tC_p})$ as the degreewise I_i-adic completion of $\mathbb{F}_p[x_0^{\pm 1}, \dots, x_i^{\pm 1}]$.*

The completion is taken in the graded sense. Set $z_j := (x_j - x_0)/x_0$ for $j = 1, \dots, i$. Then

$$C^i := \mathbb{F}_p[x_0^{\pm 1}, \dots, x_i^{\pm 1}]^\wedge_{I_i} = \mathbb{F}_p[\![z_1, \dots, z_i]\!][x_0^{\pm 1}], \qquad |z_j| = 0. \tag{4}$$

i.e. a 2-periodic version of the power series ring $\mathbb{F}_p[\![z_1, \dots, z_i]\!]$. These coordinates z_j of course depend on the choice of x_0 and the cosimplicial maps are a bit messy in this description.

Proof. The $\mathbb{E}_\infty$-ring $\mathrm{THH}(\mathbb{F}_p)$ is obtained by tensoring $\mathbb{F}_p$ with S^1 in the category of $\mathbb{E}_\infty$-rings, while $\mathbb{F}_p^{\otimes p}$ is obtained by tensoring with the free C_p-set of p points, embedded in S^1 as the pth roots of unity. Since tensoring preserves colimits, A_i is obtained by tensoring $\mathbb{F}_p$ with the C_p-space T_i formed by gluing $i+1$ circles along these marked points. The quotient T_i/C_p is a wedge of $i+1$ circles. Choosing the zeroth circle as the base circle gives the following picture:

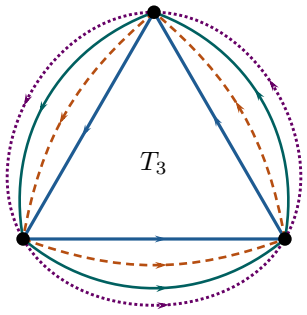

FIGURE 1. The space T_3 for $p = 3$: four circles glued at three marked points. The blue triangle is the chosen base circle. The other circles are orange, green and purple. The C_3 action is given by rotation.

The space T_i is equivariantly homotopy equivalent to the base circle with i free C_p-orbits of additional loops attached at the marked points:

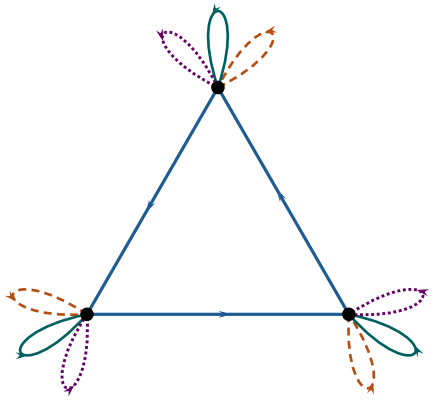

FIGURE 2. The homotopy equivalent space for $p = 3$, $i = 3$: the blue base circle with three loops at each marked point. Each colour forms a free C_3-orbit under rotation by 120°. The homotopy equivalence to the graph T_i of Figure 1 fixes the blue circle and sends each coloured loop to the corresponding coloured edge followed by the blue edge backwards.

Put $P_i = \mathrm{THH}(\mathbb{F}_p)^{\otimes_{\mathbb{F}_p} i}$ and $N(P_i) = P_i^{\otimes p}$, where C_p cyclically permutes the factors of $N(P_i)$. From the pushout of spaces we get

$$A_i \simeq A_0 \otimes_{A_{-1}} N(P_i). \tag{5}$$

Let $u_1, \dots, u_i$ be the Bökstedt classes of the attached loops at one of the vertices. Then $\pi_* P_i = \mathbb{F}_p[u_1, \dots, u_i]$ with $|u_j| = 2$ for all j. Since the tensor products defining $N(P_i)$ are taken over the sphere, we get

$$\pi_* N(P_i) = (\pi_* A_{-1})[u_1, \dots, u_i, \sigma u_1, \dots, \sigma u_i, \dots, \sigma^{p-1} u_1, \dots, \sigma^{p-1} u_i],$$

where σ is a generator of C_p. More precisely, $N(P_i)$ decomposes as a direct sum of suspensions of A_{-1} indexed by monomials. By (5), we obtain

$$\pi_*(A_i) = \pi_*(A_0)[u_1, \dots, u_i, \sigma u_1, \dots, \sigma u_i, \dots, \sigma^{p-1} u_1, \dots, \sigma^{p-1} u_i].$$

Thus A_i, as a spectrum, is a direct sum of shifted copies of A_0. On homotopy groups, the C_p-action cyclically permutes the variables as indicated. Since p is prime, every monomial is either fixed or lies in an orbit of length p. The fixed monomials are precisely

$$\prod_{j=1}^{i} \left(u_j \, \sigma u_j \cdots \sigma^{p-1} u_j\right)^{n_j}, \qquad n_1, \dots, n_i \geq 0.$$

The orbits of length p give summands that are induced, so that their Tate constructions vanish (the projection formula identifies the base change of each induced A_{-1}-module with an induced A_0-module).

For a fixed monomial, set $d = n_1 + \cdots + n_i$. Its summand is

$$\Sigma^{2d\rho_p} A_0, \qquad \rho_p = \mathbb{R}[C_p].$$

The representation suspension records the cyclic permutation of the p copies of $\mathbb{R}^{2d}$. The diagonal inclusion $\mathbb{R}^{2d} \to (\mathbb{R}^{2d})^p$ induces an equivalence $\Sigma^{2d} A_0^{tC_p} \xrightarrow{\simeq} \left(\Sigma^{2d\rho_p} A_0\right)^{tC_p}$. There are only finitely many monomials in each degree, and the connectivity of the summands tends to infinity with the

degree. Thus the direct sum of the summands agrees with their product and the Tate construction preserves products of uniformly bounded below spectra. Thus we obtain

$$A_i^{tC_p} \simeq \prod_{n_1,\dots,n_i \geq 0} \Sigma^{2(n_1+\cdots+n_i)} A_0^{tC_p}. \tag{6}$$

To write this more algebraically let y_j denote the resulting degree-two class corresponding to $u_j\, \sigma u_j \cdots \sigma^{p-1} u_j$. The multiplication is given by adding exponents. Thus together with (1) and the description of C^i above this gives an isomorphism of graded rings

$$\mathbb{F}_p[\![z_1,\dots,z_i]\!][x_0^{\pm 1}] \xrightarrow{\cong} \pi_*(A_i^{tC_p}), \qquad x_0 \longmapsto x_0, \quad z_j \longmapsto y_j/x_0.$$

It remains to check that this isomorphism extends (3) under the identification (4), i.e. that the generators really are what their name suggests. The class y_j is by construction the image of u_j under

$$P_i \longrightarrow N(P_i)^{tC_p} \longrightarrow A_i^{tC_p},$$

where the first map is the Tate diagonal. Indeed, apply naturality to a representative $S^2 \to P_i$: its p-fold tensor product defines the fixed monomial, and the diagonal $S^2 \to S^{2\rho_p}$ gives exactly the normalization used above.

The (indexed) Tate diagonal for the covering $T_i \to T_i/C_p$ [Law, Corollary 10.3 and Example 10.4], together with the Bökstedt classes, induces a linear map

$$H_1(T_i/C_p; \mathbb{F}_p) \cong \pi_2(\mathbb{F}_p^{\otimes (T_i/C_p)}) \longrightarrow \pi_2(A_i^{tC_p}).$$

If we write a_j for the j-th original quotient circle, then its class maps to x_j, since the construction restricts to the cyclotomic Frobenius. On the attached loop $a_j a_0^{-1}$ the covering is trivial, so the construction restricts to the ordinary Tate diagonal and its class maps to y_j. By linearity we get $y_j = x_j - x_0$, which proves the claim. □

Lemma 4. *The augmented cosimplicial graded abelian group $\mathbb{F}_p \to C^\bullet$ is a derived limit diagram. Equivalently,*

$$H^s(C^\bullet)_t = \begin{cases} \mathbb{F}_p, & s = t = 0, \\ 0, & \textit{otherwise.} \end{cases} \tag{7}$$

Without completion, the cosimplicial object $C^\bullet$ is the Amitsur complex of the faithfully flat map $\mathbb{F}_p \to \mathbb{F}_p[x^{\pm 1}]$ so it is a limit. Geometrically, this corresponds to the Čech nerve of $\mathbb{G}_m \to \mathrm{pt}$. Completion takes the formal neighbourhood of the diagonal in each term. We were hoping for some general formal geometry descent statement for this situation but have not been able to find one, so we give a direct proof by showing that degreewise completion preserves exactness by expressing it as an inverse limit of ordinary Amitsur complexes.

Proof. All limits below are taken levelwise in the internal grading. Put $R = \mathbb{F}_p[x^{\pm 1}]$ and $R_n = \mathbb{F}_p[x^{\pm p^n}] \subseteq R$ for $n \geq 0$, with $|x| = 2$. Following [Ogu, Remark 4.12] we consider the Amitsur complexes

$$C_n^i := R^{\otimes_{R_n}(i+1)} \cong \frac{\mathbb{F}_p[x_0^{\pm 1}, \dots, x_i^{\pm 1}]}{((x_1 - x_0)^{p^n}, \dots, (x_i - x_0)^{p^n})}.$$

Since $R_n \subseteq R$ is faithfully flat, these resolve R_n. The defining ideals are cofinal with the powers of I_i, and the transition maps are surjective in every degree. Thus $C^\bullet \simeq \operatorname{Rlim}_n C_n^\bullet$ as cosimplicial objects, and commuting derived limits gives

$$\operatorname{Rlim}_\Delta C^\bullet \simeq \operatorname{Rlim}_n \operatorname{Rlim}_\Delta C_n^\bullet \simeq \operatorname{Rlim}_n R_n \simeq \mathbb{F}_p.$$

For the last equivalence, the tower $(R_n)_{2k}$ is constant $\mathbb{F}_p$ for $k = 0$ and eventually zero for $k \neq 0$, since p^n eventually does not divide k. All odd degrees vanish. These equivalences identify the given augmentation with an isomorphism, proving the claim. □

Proof of Theorem 1. By the formal reduction in [NS, proof of Theorem III.1.7], it suffices to prove that the Tate diagonal is an equivalence for $X = \mathbb{F}_p$. Lemmas 3 and 4 show that

$$H^s\big(\pi_t(A_\bullet^{tC_p})\big) = \begin{cases} \mathbb{F}_p, & s = t = 0, \\ 0, & \text{otherwise.} \end{cases}$$

By Lemma 2, we get a conditionally convergent descent spectral sequence

$$E_2^{s,t} = H^s\big(\pi_t(A_\bullet^{tC_p})\big) \Rightarrow \pi_{t-s}(A_{-1}^{tC_p}).$$

This half-plane spectral sequence collapses at E_2, so it converges strongly by [Boa, Theorem 7.1]. Thus $\pi_*(A_{-1}^{tC_p}) \cong \mathbb{F}_p$, concentrated in degree zero. Since the Tate diagonal $\Delta_p : \mathbb{F}_p \to A_{-1}^{tC_p}$ is unital, it is an equivalence, completing the proof. □

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