

Algebraic Geometry WS 26/27

RECAP Sheet 2: 21.10-27.10

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R — Recall

1. What is the ring of functions $\mathcal{O}(V)$ of an affine algebraic set V ?
2. What is a morphism of affine algebraic sets ?
3. Why are affine algebraic sets isomorphic if and only if their rings of functions are ?
4. What does it mean for an affine algebraic set to be irreducible ? In terms of its ring of functions ?
5. What is a noetherian topological space ?
6. What is an irreducible component ?



E — Explore

1. Given a finite set of points $X \subset \mathbb{A}^n$, show that it is an affine algebraic set. What is its ring of functions ? When is it irreducible ?
2. Consider the following affine algebraic sets in $\mathbb{A}^2$, which ones are irreducible, and what are their irreducible components ?

$$V(XY), V(X^2 - Y^2), V(X^2 + 2XY + Y^2), V(X^2 + Y^2 - 1), V(X^3 - Y^2X)$$

3. Is the topological space $\mathbb{R}$ (with its usual topology) noetherian ? Look at your favourite topological spaces and decide which are noetherian.
4. Find your own examples of maps between affine algebraic sets that is not a morphism of affine algebraic sets.
5. Does the topological space $\mathbb{A}^2$ agree with the product of topological spaces $\mathbb{A}^1 \times \mathbb{A}^1$? How about with the product of affine algebraic sets (in the category of affine algebraic sets) ?
6. What are some examples of morphisms of affine algebraic sets that are bijections but not isomorphisms ?



C — Connect

1. What is the relation between the notion of a noetherian space and that of an irreducible component ?
2. Explain why noetherian-ness of the ring of polynomials $k[X_1, \dots, X_n]$ is related to the noetherian-ness of the topological space k^n with the Zariski topology.
3. What is the logical relation between reducibility and zero divisors ?



C — Connect (continued)

- What is the logical relation between Hilbert's Nullstellensatz and the equivalence of categories between $\text{AffVar}_k^{\text{op}}$ and $\text{Alg}_k^{\text{f.g.,red}}$? Which one implies the other?



A — Apply

- Recall how $GL_n(k)$ is viewed as an affine algebraic set, and describe $GL_n(k) \times GL_n(k)$ similarly. Prove that the multiplication map $GL_n(k) \times GL_n(k) \rightarrow GL_n(k)$ is a morphism of affine algebraic sets, and similarly for the inversion map $M \mapsto M^{-1}, GL_n(k) \rightarrow GL_n(k)$.
 - Deduce that the version of $GL_n(k)$ as an affine algebraic set from the lectures is isomorphic, as an affine algebraic set, to $V((\sum_{k=1}^n X_{ik}Y_{kj} - \delta_{ij})_{1 \leq i,j \leq n}) \subset \mathbb{A}^{n^2+n^2}$.
- What are the irreducible components of $V(X^2 - XY - 2Y^2) \subset \mathbb{A}^2$?
- Let X, Y be topological spaces and $f : X \rightarrow Y$ a continuous map. Show that if X is irreducible, then so is $f(X)$.



P — Present

Possible starting questions for the tutorial:

- Why is the notion of irreducible component useful?
- Why is the definition of a morphism of affine algebraic sets what it is?
- How would you determine if $V(p)$ is irreducible, given a polynomial $p \in k[X_1, \dots, X_n]$?

The tutor may also ask about other central definitions, examples, arguments, or connections from this sheet.