

Algebraic Geometry WS 26/27

RECAP Sheet 1 14.10.-20.10.

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R — Recall

1. What is an affine algebraic set $V(M) \subseteq k^n$?
2. Why does $V(M)$ depend only on the ideal (M) generated by M ?
3. What does it mean for a ring to be noetherian? State Hilbert's basis theorem and explain its geometric consequence for affine algebraic sets.
4. What are the closed subsets in the Zariski topology on $\mathbb{A}^n$?
5. Define the radical $\sqrt{\mathfrak{a}}$ of an ideal and the vanishing ideal $I(S)$ of a subset $S \subseteq \mathbb{A}^n$.
6. State Hilbert's Nullstellensatz (Theorem 2.7) in the form given in the lecture.



E — Explore

1. Work over an algebraically closed field k . Compare the zero sets in $\mathbb{A}_k^2$

$$V(X), \quad V(X^2), \quad V(X^7), \quad V(XY), \quad V(X, Y).$$

Which different ideals give the same zero set? What are their radicals?

2. Compare the equation $X^2 + 1 = 0$ over $\mathbb{R}$ and over $\mathbb{C}$. What does this suggest about the assumption that k be algebraically closed?
3. Find your own examples of
 - (a) two different ideals with the same zero set;
 - (b) a non-radical ideal;
 - (c) an affine algebraic set that is visibly the union of two simpler algebraic sets.

Try to find examples different from the ones above.



C — Connect

1. For a set of polynomials $M \subseteq k[X_1, \dots, X_n]$, consider the constructions

$$M \longrightarrow (M) \longrightarrow \sqrt{(M)} \quad \text{and} \quad V(M) \longrightarrow I(V(M)).$$

Explain how these objects are related. Which constructions lose information, and which determine the same zero set? Which facts follow from the definitions, and where does the Nullstellensatz enter?



C — Connect (continued)

2. Relate the formulas

$$\bigcap_{i \in I} V(M_i) = V\left(\bigcup_{i \in I} M_i\right), \quad V(M) \cup V(N) = V(MN)$$

to the axioms for closed subsets of a topology. Reconstruct the proofs directly from the definition of $V(-)$.

3. Explain why radical ideals naturally appear in algebraic geometry. Connect the following facts:

$$V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}}), \quad I(S) \text{ is radical}, \quad I(V(\mathfrak{a})) = \mathfrak{a}$$

for radical $\mathfrak{a}$ over an algebraically closed field. Reconstruct the proof of $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$.

4. Explain the logical relations among the following statements: Hilbert's basis theorem; every affine algebraic set is defined by finitely many equations; the existence of the Zariski topology; Hilbert's Nullstellensatz. Which statements imply others, and which express different features of affine algebraic geometry?



A — Apply

1. Let k be algebraically closed. Give an explicit description of all open subsets of $\mathbb{A}^1(k)$ in the Zariski topology.
2. Determine the vanishing ideals in $\mathbb{C}[X, Y]$ of

$$\{(0, 0), (1, 1)\} \quad \text{and} \quad \{(n, n) \mid n \in \mathbb{Z}\}.$$

3. ★ **Optional challenge.** Show that the set of $m \times n$ matrices of rank at most r is Zariski closed. Find explicit polynomial equations defining it.



P — Present

Possible starting questions for the tutorial:

- Why is it useful to replace a collection of polynomial equations by the ideal it generates?
- Why do zero sets naturally lead to radical ideals?
- How do the algebraic identities for $V(-)$ produce the Zariski topology?
- What does Hilbert's Nullstellensatz say conceptually?
- A student claims

$$V(\mathfrak{a}) = V(\mathfrak{b}) \implies \mathfrak{a} = \mathfrak{b}.$$

Is this correct? If not, repair the statement.

The tutor may also ask about other central definitions, examples, arguments, or connections from this sheet.