Algebraic Geometry

An Introduction to Schemes

Lecture notes for a course at
the University of Münster
Winter term 2026/27

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These are lecture notes for a course on algebraic geometry taught in the winter term 2026/27 at the University of Münster. The plan is to begin with polynomial equations and develop the language and geometry of schemes, sheaves, and vector bundles.
These notes are a work in progress and will be revised as the course proceeds. Revisions may include substantial changes to the content and structure, as well as changes to the order and numbering of sections and results. Please always use the latest version, identified by the date on the title page.

General references we use are:

(1) R. Hartshorne: *Algebraic Geometry*. Graduate Texts in Mathematics 52, Springer [**Har**].
(2) D. Mumford: *The Red Book of Varieties and Schemes*. Lecture Notes in Mathematics 1358, Springer [**Mum**].
(3) U. Görtz and T. Wedhorn: *Algebraic Geometry. Part I: Schemes. With Examples and Exercises*. Vieweg+Teubner [**GW**].
(4) A. Grothendieck, with the collaboration of J. Dieudonné: *Éléments de géométrie algébrique, I–IV* [**EGA**].
(5) P. Scholze: *Algebraic Geometry I* and *Algebraic Geometry II*. Lecture notes written by Jack Davies [**SchI, SchII**].
(6) The Stacks Project Authors: *The Stacks Project* [**Stacks**]. [1]

For commutative algebra, our main reference is M. F. Atiyah and I. G. Macdonald: *Introduction to Commutative Algebra*, Addison-Wesley [**AM**]. We assume familiarity with the commutative algebra needed below, as covered in this book. A preceding course in Münster, taught by Christopher Deninger, covered this material. In particular, we use localization, noetherian rings, tensor products, and Nakayama's lemma as prerequisites. We recall their statements and indicate references where they are used; the geometric consequences are developed in detail. Primary decomposition is not assumed.
Basic topology is assumed. Adjunctions and the Yoneda lemma are recalled in Section 8. The theory of sheaves, schemes, Proj, and vector bundles is developed in the text.

[1] For references to individual results in the Stacks Project we use their permanent tags (which remain valid when chapter and lemma numbers change).

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1. Affine algebraic sets and an overview of the course

What is the idea of algebraic geometry? We first look at some analogies:

- Linear algebra: study systems of linear equations. The solutions of a homogeneous system form a vector subspace.
- Algebra: study a single polynomial equation, for example the zeros of a polynomial with rational coefficients in $\mathbb{C}$ and the field extensions generated by its roots, such as $\mathbb{Q}(\zeta_n)$.
- Algebraic geometry: study solutions to several polynomial equations, for example

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid f_1(x) = \cdots = f_k(x) = 0\}, \qquad f_i \in \mathbb{R}[X_1, \dots, X_n].$$

Think geometrically: consider $\{x^2+y^2 = 1\}$ as a circle, with its geometric properties, rather than just as a set of elements of $\mathbb{R}^2$. The aim is to transfer geometric intuition to solve algebraic questions, and vice versa.

REMARK 1.1. Recall that a smooth submanifold of $\mathbb{R}^{d+k}$ of dimension d can locally be described as the common zero set of k smooth functions:

$$M \cap U = \{x \in U \mid f_1(x) = \cdots = f_k(x) = 0\}, \qquad f_i : U \longrightarrow \mathbb{R},$$

where $U \subseteq \mathbb{R}^{d+k}$ is open and the Jacobian matrix $(\partial f_i/\partial x_j)$ has rank k at every point of $M \cap U$. The rank condition ensures that this zero set looks locally like $\mathbb{R}^d$.

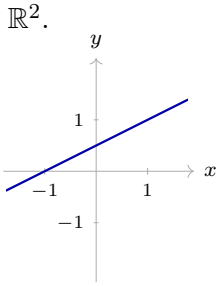
The main difference is that we restrict to polynomial equations instead of smooth functions (we are doing algebra after all), and drop the rank condition. This allows singularities such as crossings and cusps, where the local manifold picture breaks down. Polynomial equations also make sense over other fields, so we need not restrict ourselves to real geometry.
Now let k be a field; we will soon assume that k is algebraically closed.

DEFINITION 1.2. *An* affine algebraic set *is a subset $V \subseteq k^n$ which is the common zero set of a collection $M \subseteq k[X_1, \dots, X_n]$ of polynomials:*

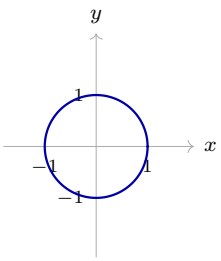
$$V = V(M) := \{(x_1, \dots, x_n) \in k^n \mid f(x_1, \dots, x_n) = 0 \text{ for every } f \in M\}.$$

EXAMPLE 1.3. Here are some equations in two variables, together with drawings of their real point sets. The same equations can also be considered over other fields. Connectedness in these pictures refers to the usual topology on $\mathbb{R}^2$.

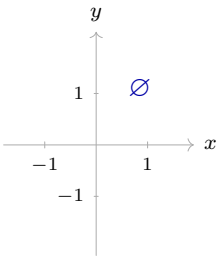
(a) $y = ax + b$ defines a line, for $a, b \in k$. The drawing uses $a = b = \frac{1}{2}$ over $\mathbb{R}$.



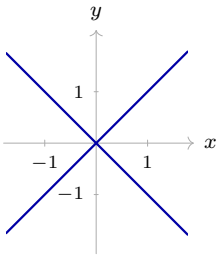
(b) $x^2 + y^2 = 1$ gives the unit circle over $\mathbb{R}$.



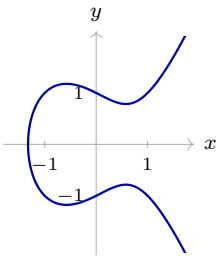
(c) $x^2+y^2=-1$ has no real points, but has complex points such as $(i,0)$. Thus a real picture need not reveal all the geometry of an equation.



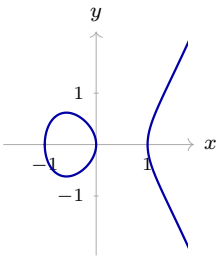
(d) $x^2=y^2$ factors as $(x-y)(x+y)=0$. Over a field of characteristic different from 2, it gives two distinct lines crossing at the origin.



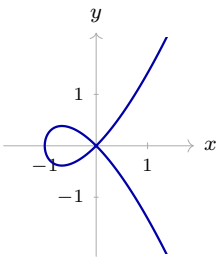
(e) $y^2=x^3-x+1$ is a smooth affine cubic over $\mathbb{R}$.



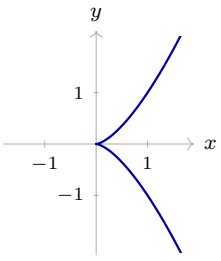
(f) $y^2=x^3-x$ is another smooth affine cubic over $\mathbb{R}$; its real point set has two connected components in the usual topology.



(g) $y^2=x^3+x^2=x^2(x+1)$ is a *nodal cubic*: two real branches cross at the origin, with tangent lines $y=x$ and $y=-x$.



(h) $y^2=x^3$ is a *cuspidal cubic*: the two real arcs meet in a cusp at the origin.



Recall that for every subset M of a commutative ring R there is a smallest ideal containing M. Its elements are the finite sums $\sum_i a_i m_i$ with $a_i \in R$ and $m_i \in M$.

For $M = \{r_1, \ldots, r_s\}$ we denote this ideal by $(r_1, \ldots, r_s)$. The first link between equations and algebra is that their common zero set depends only on this ideal.

PROPOSITION 1.4. *If $\mathfrak{a}$ is the ideal generated by $M \subseteq k[X_1, \ldots, X_n]$, then $V(M) = V(\mathfrak{a})$.*

PROOF. Since $M \subseteq \mathfrak{a}$, we have $V(\mathfrak{a}) \subseteq V(M)$. Conversely, if $x \in V(M)$ and $g = \sum_i a_i m_i \in \mathfrak{a}$, then

$$g(x) = \sum_i a_i(x) m_i(x) = 0.$$

Thus $x \in V(\mathfrak{a})$. □

THEOREM 1.5. *Every affine algebraic set $V \subseteq k^n$ is the common zero set of finitely many polynomials.*

This is the geometric consequence of Hilbert's basis theorem: the ideal generated by any collection of equations has a finite set of generators. This follows from basic properties of noetherian rings that we briefly recall. All rings in these notes are commutative with identity, and all ring homomorphisms preserve the identity. A ring R is *noetherian* if every ideal is finitely generated. We recall the equivalent formulations from [**AM**, Chapter 6].

PROPOSITION 1.6. *For a ring R the following are equivalent:*

(1) *Every ideal of R is finitely generated.*
(2) *Every ascending chain of ideals eventually becomes constant.*
(3) *Every nonempty collection of ideals has a maximal member under inclusion.*

These are prerequisites from commutative algebra. We will use them to turn ascending chains of ideals into descending chains of geometric zero loci.

THEOREM 1.7 (Hilbert's basis theorem). *If R is noetherian, then $R[X_1, \ldots, X_n]$ is noetherian. Quotients and localizations of noetherian rings are noetherian as well.*

For proofs see [**AM**, 7.1, 7.3, and 7.5–7.7]. In particular every finitely generated algebra over a field, and every quotient of $\mathbb{Z}[X_1, \ldots, X_n]$, is noetherian. This proves Theorem 1.5: an ideal defining an affine algebraic set has a finite set of generators, so finitely many equations suffice.

EXAMPLE 1.8. The ring $k[X_1, X_2, \ldots]$ is not noetherian: the chain $(X_1) \subsetneq (X_1, X_2) \subsetneq \cdots$ never stabilizes. Thus the finite generation hypothesis has geometric content; it is not automatic for rings of polynomial functions in infinitely many variables.

From equations to spaces: an overview of the course. The examples and the passage from equations to ideals give us starting points for the course. Before developing the theory, let us outline where these ideas will lead. We will move from affine algebraic sets to schemes and projective space, and conclude with line bundles and vector bundles.

Our first step will be to study the topology and polynomial functions of an affine algebraic set. The Zariski topology supplies open sets on which we can work locally (Section 2). Different polynomials may define the same function; passing to the ring of functions identifies these polynomials and turns geometric questions into algebraic ones. Maps pull functions back, while irreducible pieces correspond to prime ideals (Sections 3 and 4). Hilbert's Nullstellensatz (Theorem 2.7) makes the relationship

between zero sets and ideals precise over an algebraically closed field. One can think of this first part as providing a dictionary between the geometry of affine algebraic sets and the algebra of their coordinate rings.

A set of ordinary solutions does not retain all the information in the equations. The equations $x = 0$ and $x^2 = 0$ have the same solutions in a field, but their quotient rings are different: in $k[x]/(x^2)$ the class of x is nonzero and has square zero. Such elements will record infinitesimal information, for instance about tangencies. To retain this information, we enlarge our notion of a geometric space following Grothendieck. The spectrum of a ring supplies its points (Section 5), and the structure sheaf records functions on open sets (Section 6). This also brings arithmetic spaces such as $\mathrm{Spec}(\mathbb{Z})$ into the picture.

Besides allowing infinitesimal elements, we also 'globalize' these constructions. Here the analogy with manifolds guides the construction: a scheme is locally an affine spectrum, just as a manifold has local coordinate neighbourhoods. After developing the language of sheaves and locally ringed spaces, we define schemes in Section 9 and prove that affine pieces can be glued (Theorem 11.12). Points with values in a ring (Section 10) and fibre products (Section 11) let us study changing the ground field and intersections with their multiplicities. The double point returns in Examples 10.4 and 11.5 as a concrete reason to keep nilpotent elements.

Projective space is our main example beyond affine schemes (Section 12). Its affine charts explain how points at infinity fit into a space; Proj extends this construction to graded rings and homogeneous equations (Subsection 12.2). A conic gives an explicit example (Example 12.11). It will reappear as the image of the projective line under a map defined by sections of a line bundle (Example 15.10).

After developing quasi-coherent sheaves and their descent (Section 13), we turn to line bundles and vector bundles (Section 14). They describe families of vector spaces that can be trivialized locally. We prove their correspondence with finitely generated projective modules on affine schemes (Theorem 14.27), and describe maps to projective space by generating sections of a line bundle (Corollary 14.17). The final section connects graded modules with twisting sheaves on Proj (Section 15). There we also classify line bundles on projective space (Theorem 15.8) and compute their sections on the projective line (Proposition 15.9).

Three examples recur throughout the course: the crossing of two affine lines, the double point, and the projective line. They connect the definitions with calculations of points, functions, fibre products, and line bundles.

2. The Zariski topology

Okay, let's get back to affine algebraic sets. Recall the finite-equation consequence of Hilbert's basis theorem, Theorem 1.5. We know that affine algebraic sets $V \subseteq k^n$ are the zero sets of finitely many polynomials. Let us recall that a topology on a set X is a collection of subsets of X called *open subsets* such that

(1) $\emptyset$, X are open.
(2) Arbitrary unions of open sets are open.
(3) Finite intersections of open sets are open.

A subset $K \subset X$ is then called closed if the complement $X \setminus K$ is open. One immediately sees that

(1) $\emptyset$, X are closed.
(2) Arbitrary intersections of closed sets are closed.

(3) Finite unions of closed sets are closed.

One could clearly have defined a topological space equivalently by means of its closed subsets.

PROPOSITION 2.1. *There is a topology on k^n for which the closed subsets are the affine algebraic sets.*

PROOF. We have $\emptyset = V(1)$ and $k^n = V(\emptyset)$. Moreover we have that

$$\bigcap_{i\in I} V(M_i) = V(\bigcup_{i\in I} M_i)$$

so that the intersection of closed sets is closed. Finally we have to prove that the union $A \cup B$ of two closed sets

$$A = V(M) \qquad \text{and} \qquad B = V(N)$$

is again closed. We consider

$$M \cdot N = \{f \cdot g \mid f \in M, g \in N\}$$

and claim that $A \cup B = V(M \cdot N)$. The inclusion $\subseteq$ is clear. For the converse assume that $(x_1, ..., x_n)$ is a zero of all polynomials $f \cdot g$ and not in A, i.e. there is at least one $f_0 \in M$ with $f_0(x_1, ..., x_n) \neq 0$. For each $g \in N$ we get that $f_0(x_1, ..., x_n) \cdot g(x_1, ..., x_n) = 0$ so that $g(x_1, ..., x_n) = 0$ and thus $(x_1, ..., x_n) \in B$. □

DEFINITION 2.2. *The topology is called the* Zariski topology. *We denote the space k^n with the Zariski topology by $\mathbb{A}^n(k)$ or simply $\mathbb{A}^n$ if the ground-field is clear. It is called the n-dimensional affine space.*

EXAMPLE 2.3. Identify $\mathrm{Mat}(n \times n, k) = k^{n^2} = \mathbb{A}^{n^2}$. Then the subset

$$\mathrm{Gl}(n, k) \subseteq \mathrm{Mat}(n \times n, k)$$

of invertible matrices is Zariski open. To see this we note that the complement, i.e. the set of matrices with vanishing determinant, is closed. But this is true by definition since it is $V(\det)$ where $\det : \mathrm{Mat}(n \times n, k) \to k$ is considered as a polynomial in n^2 entries.
We can also consider $\mathrm{Gl}(n, k)$ as an affine algebraic set as follows: consider the identification

$$\mathrm{Gl}(n, k) \cong \{(A, y) \in \mathrm{Mat}(n \times n, k) \times k \mid \det(A) \cdot y = 1\} \subseteq k^{n^2} \times k = \mathbb{A}^{n^2+1}$$

in which case it is the vanishing locus of a single polynomial in $n^2 + 1$ entries, namely $\det \cdot Y - 1 \in k[X_1, ..., X_{n^2}, Y]$. We will consider the latter the 'standard way' of viewing it as an affine algebraic set.

Now we want to investigate how much the algebraic set $V = V(\mathfrak{a})$ remembers about the ideal $\mathfrak{a}$.

DEFINITION 2.4. *Let $\mathfrak{a} \subseteq R$ be an ideal. The* radical *of $\mathfrak{a}$ is the ideal*

$$\sqrt{\mathfrak{a}} = \{x \in R \mid x^n \in \mathfrak{a} \text{ for some } n \geq 1\}$$

An ideal is called radical (or radically closed) if $\sqrt{\mathfrak{a}} = \mathfrak{a}$.

LEMMA 2.5. *$\sqrt{\mathfrak{a}}$ is itself an ideal and we have $\sqrt{\sqrt{\mathfrak{a}}} = \sqrt{\mathfrak{a}}$.*

PROOF. Clearly for $x \in \sqrt{\mathfrak{a}}$ and $r \in R$ we have that $(rx)^n = r^n x^n \in \mathfrak{a}$. Thus $rx \in \sqrt{\mathfrak{a}}$.
For $x, y \in \sqrt{\mathfrak{a}}$ choose $n \geq 1$ with $x^n, y^n \in \mathfrak{a}$. We find that $(x+y)^{2n} \in \mathfrak{a}$ since each monomial $x^i y^j$ with $i + j = 2n$ has to contain at least n factors of x or n factors of y.
The other claim is clear. □

LEMMA 2.6. *For an ideal $\mathfrak{a} \subseteq k[X_1, ..., X_n]$ we have that*

$$V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$$

PROOF. The containment $V(\sqrt{\mathfrak{a}}) \subseteq V(\mathfrak{a})$ is obvious since $\mathfrak{a} \subseteq \sqrt{\mathfrak{a}}$. For the converse assume that $(x_1, ..., x_n) \in V(\mathfrak{a})$ and $p \in \sqrt{\mathfrak{a}}$. Then there exists an n such that $p^n \in \mathfrak{a}$, in particular $p^n(x_1, ..., x_n) = 0$. But then also $p(x_1, ..., x_n) = 0$. Thus $(x_1, ..., x_n) \in V(\sqrt{\mathfrak{a}})$. □

For any subset $M \subseteq \mathbb{A}^n$ we define the *vanishing ideal* $I(M) \subseteq k[X_1, .., X_n]$ as the set of elements

$$I(M) = \{f \in k[X_1, ..., X_n] \mid f(x_1, ..., x_n) = 0 \text{ for all } x \in M\} .$$

This is clearly a radical ideal.

THEOREM 2.7 (Hilbert's Nullstellensatz). *Assume that k is algebraically closed. Then the map*

$$V(-): \quad \{\textit{radical ideals } \mathfrak{a} \subseteq k[X_1, ..., X_n]\} \to \{\textit{Closed subsets of } \mathbb{A}^n\}$$

is a bijection with inverse $V \mapsto I(V)$.

From now on we shall always assume that k is algebraically closed. Why is this called Nullstellensatz?

COROLLARY 2.8. *Assume that $\mathfrak{a} \subsetneq k[X_1, ..., X_n]$ is a proper ideal. Then there is an element $(x_1, ..., x_n) \in k^n$ such that $f(x) = 0$ for all $f \in \mathfrak{a}$.*

PROOF. Since $\mathfrak{a}$ is proper we have $1 \notin \mathfrak{a}$. But then also $1 \notin \sqrt{\mathfrak{a}}$. Thus $\sqrt{\mathfrak{a}}$ is proper. Therefore we get from the injectivity of $V(-)$ that

$$V(\sqrt{\mathfrak{a}}) \neq V(k[X_1, ..., X_n]) = \emptyset .$$ □

COROLLARY 2.9. *Assume that we are given a finite set of polynomials $f_1, ..., f_k \in k[X_1, ..., X_n]$. Then exactly one of the following alternatives holds:*

1. *The polynomials have a common zero in k^n.*
2. *There are polynomials $g_1, ..., g_k \in k[X_1, ..., X_n]$ such that*

$$g_1 f_1 + ... + g_k f_k = 1 .$$

PROOF. First note that these two alternatives clearly exclude each other. By the last corollary we see that either $(f_1, ..., f_k)$ is the full ideal (which is the second alternative) or that the polynomials have a common zero. □

2.1. From commutative algebra to the Nullstellensatz. We use the algebraic Nullstellensatz from commutative algebra:

LEMMA 2.10 (Zariski's lemma). *If a field extension K/k is finitely generated as a k-algebra, it is a finite algebraic extension of k.*

See [**AM**, Corollary 5.24]. We retain the geometric deduction rather than repeat its algebraic proof.

COROLLARY 2.11. *Every nonzero finitely generated algebra R over an algebraically closed field k admits a k-algebra homomorphism $R \to k$.*

PROOF. Choose a maximal ideal $\mathfrak{m}$ of R. The field $R/\mathfrak{m}$ is a finitely generated k-algebra, so Zariski's lemma makes it a finite algebraic extension of k. Algebraic closedness identifies it with k. □

Recall also from [**AM**, 3.1–3.2 and the following examples] that localization satisfies its usual universal property and that

$$R[1/f] \cong R[T]/(fT-1), \qquad R[1/f] = 0 \iff f \text{ is nilpotent.}$$

The last equivalence is precisely the algebra needed in the following proof.

PROOF OF HILBERT'S NULLSTELLENSATZ (THEOREM 2.7). We will prove that for every radical ideal $\mathfrak{a}$ we have that

$$IV(\mathfrak{a}) = \mathfrak{a} \ . \tag{1}$$

This shows that $IV = \mathrm{id}$. But we have already seen that V is surjective, so that this finishes the proof.
In order to prove (1) we first note that for $f \in \mathfrak{a}$ we have by assumption that for each $x \in V(\mathfrak{a})$ that $f(x) = 0$. Thus $f \in IV(\mathfrak{a})$ and we get the inclusion $\supseteq$. Conversely assume that $f \in IV(\mathfrak{a})$. Consider the ring

$$R = (k[X_1, ..., X_n]/\mathfrak{a})\,[f^{-1}] \ .$$

which is a finitely generated k-algebra. If $R = 0$ then there would be an $n > 0$ such that $f^n \in \mathfrak{a}$. But then $f \in \mathfrak{a}$ since $\mathfrak{a}$ is radical, and we are done. If $R \neq 0$ then we find by the previous corollary a homomorphism $R \to k$ and thus elements $(x_1, ..., x_n) \in V(\mathfrak{a})$ such that $f(x_1, ..., x_n) \neq 0$. But this contradicts the assumption that $f \in IV(\mathfrak{a})$. □

COROLLARY 2.12. *Every maximal ideal of $k[X_1, ..., X_n]$ is of the form*

$$I(\{(\alpha_1, ..., \alpha_n)\}) = (X_1 - \alpha_1, ..., X_n - \alpha_n)$$

for an element $(\alpha_1, ..., \alpha_n) \in k^n$.

PROOF. For every maximal ideal $\mathfrak{m}$ there is a common zero $(\alpha_1, ..., \alpha_n)$ according to the Nullstellensatz. This means that the ideal is contained in the ideal of all polynomials that vanish at $(\alpha_1, ..., \alpha_n)$ and hence the two are equal (the latter is certainly not everything). But the latter is exactly the ideal $(X_1 - \alpha_1, ..., X_n - \alpha_n)$, which is clearly maximal as the quotient is k. □

3. The category of affine varieties

We now want to say things in more categorical language. As before we let k be an algebraically closed field.

DEFINITION 3.1. *Let $V \subseteq \mathbb{A}^n$ be a closed subset (i.e. an affine algebraic set). Then we define the* ring of functions *on V to be the ring*

$$\mathcal{O}(V) := k[X_1, ..., X_n]/I(V) \ .$$

EXAMPLE 3.2. We have that $\mathcal{O}(\mathbb{A}^n) = k[X_1, ..., X_n]$. We think of $\mathcal{O}(\mathbb{A}^n)$ as functions on $\mathbb{A}^n$. In general, every element $[p] \in \mathcal{O}(V)$ should be interpreted as a function $V \to k$ represented by the polynomial p. The point is that two polynomials $p, q \in k[X_1, ..., X_n]$ agree on V precisely if they differ by an element in $I(V)$, i.e. represent the same class in $\mathcal{O}(V)$.

DEFINITION 3.3. *A ring R is reduced if it does not admit any nonzero nilpotent elements, i.e. if $x^n = 0$ implies that $x = 0$.*

PROPOSITION 3.4. *The ring $\mathcal{O}(V)$ is a finitely generated and reduced k-algebra. Conversely any finitely generated, reduced k-algebra is of the form $\mathcal{O}(V)$ for some affine algebraic set $V \subseteq k^n$.*

PROOF. The ring $\mathcal{O}(V)$ is finitely generated, and it is reduced since $I(V)$ is radical. Conversely, any finitely generated algebra R is of the form $k[X_1, ..., X_n]/\mathfrak{a}$ for some ideal $\mathfrak{a}$ (where n is the number of generators). It is reduced precisely if $\mathfrak{a}$ is radical. In that case the Nullstellensatz gives $I(V(\mathfrak{a})) = \mathfrak{a}$, so $R \cong \mathcal{O}(V(\mathfrak{a}))$. □

Now we define morphisms of affine algebraic sets. These are informally given by maps that can be given by polynomials (think of our example of smooth manifolds, where morphisms would be smooth maps).

DEFINITION 3.6. *Let $V \subseteq \mathbb{A}^n$ and $W \subseteq \mathbb{A}^m$ be affine algebraic sets. Then a morphism $V \to W$ is given by a map of sets $f : V \to W$ for which there are polynomials $p_1, ..., p_m \in k[X_1, ..., X_n]$ such that*

$$f(x) = (p_1(x), ..., p_m(x)) \ .$$

for all $x \in V$. This notion of morphisms is clearly closed under composition and contains all identities. Thus affine algebraic sets and their morphisms form a category which we denote by AffVar_k.

WARNING 3.7. *The polynomials $p_1, ..., p_m$ are not uniquely determined by the map f. They are not part of the datum of a morphism, only their existence is required.*

EXAMPLE 3.8. For $V \subseteq \mathbb{A}^n$ the set $\mathrm{Hom}_{\mathrm{AffVar}_k}(V, \mathbb{A}^1)$ consists of functions $V \to k$ that can be written as polynomials $p \in k[X_1, ..., X_n]$. Thus by the comparison with Example 3.2 we get a canonical bijection

$$\mathrm{Hom}_{\mathrm{AffVar}_k}(V, \mathbb{A}^1) = \mathcal{O}(V) \ .$$

so that $\mathcal{O}(V)$ indeed consists of functions.

EXAMPLE 3.9. More generally we see that a morphism from $V \subseteq \mathbb{A}^n$ to $\mathbb{A}^m$ can be represented by polynomials $p_1, ..., p_m \in k[X_1, ..., X_n]$ modulo $I(V)$, i.e.

$$\mathrm{Hom}_{\mathrm{AffVar}_k}(V, \mathbb{A}^m) = \mathcal{O}(V)^m \ .$$

From Example 3.8 we in particular see that for a given morphism $f : V \to W$ we get an induced map

$$\mathcal{O}(W) \to \mathcal{O}(V)$$

by pullback (or precomposition) which is easily verified to be a k-algebra homomorphism as follows: we find $(p_1, ..., p_m)$ representing f. Then an element $g \in \mathcal{O}(W)$ is sent to the class $[g(p_1, ..., p_m)] \in \mathcal{O}(V)$. This assignment obviously preserves products and sums (of g's). Together this defines a contravariant functor

$$\mathcal{O} : \mathrm{AffVar}_k^{\mathrm{op}} \to \mathrm{Alg}_k^{\mathrm{f.g.,red.}}$$

to the category of finitely generated, reduced k-algebras.

THEOREM 3.10. *This functor is an equivalence of categories.*

PROOF. We have already seen that it is essentially surjective by Proposition 3.4. Thus we only have to show that it is fully faithful, i.e. that the induced map

$$\mathrm{Hom}_{\mathrm{AffVar}_k}(V,W) \to \mathrm{Hom}_{\mathrm{Alg}_k}(\mathcal{O}(W),\mathcal{O}(V))$$

is a bijection. Explicitly this maps the class $[p_1,...,p_m]$ to the morphism obtained by precomposition as described above. In the special case $W=\mathbb{A}^m$, this map is a bijection as explained in Example 3.9: An element on the left hand side is exactly given by an m-tuple of elements $([p_1],\dots,[p_m])\in\mathcal{O}(V)^m$, and the map takes it to the homomorphism $k[X_1,\dots,X_m]\to\mathcal{O}(V)$ which takes $X_i\mapsto[p_i]$.
For arbitrary $W\subseteq\mathbb{A}^m$ we have the diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{AffVar}_k}(V,W) & \hookrightarrow & \mathrm{Hom}_{\mathrm{AffVar}_k}(V,\mathbb{A}^m) \\
\downarrow & & \downarrow\cong \\
\mathrm{Hom}_{\mathrm{Alg}_k}(\mathcal{O}(W),\mathcal{O}(V)) & \hookrightarrow & \mathrm{Hom}_{\mathrm{Alg}_k}(k[x_1,\dots,x_m],\mathcal{O}(V))
\end{array}$$

where the upper horizontal arrow exhibits $\mathrm{Hom}_{\mathrm{AffVar}_k}(V,W)$ as subset of those maps represented by polynomials $(p_1,\dots,p_m)$ which map V to W, and the lower horizontal map exhibits $\mathrm{Hom}_{\mathrm{Alg}_k}(\mathcal{O}(W),\mathcal{O}(V))$ as subset of those maps which vanish on $I(W)$. Since the upper right composite is injective, the left vertical map is also injective. To see that it is an isomorphism, it therefore suffices to check that it is surjective. Thus we have to see that $[p_1,\dots,p_m]\in\mathrm{Hom}_{\mathrm{AffVar}_k}(V,\mathbb{A}^m)$ takes V to W if the corresponding map $k[X_1,\dots,X_m]\to\mathcal{O}(V)$, $X_i\mapsto[p_i]$, vanishes on $I(W)$. The latter condition spelled out means that for each $f\in I(W)$ we have that $f(p_1,...,p_m)$ has to lie in $I(V)$. In particular if $x\in V$ then $f(p_1(x),...,p_m(x))=0$, i.e. $p(x)\in W$. □

COROLLARY 3.11. *Two affine algebraic sets V and W are isomorphic precisely if their rings of functions $\mathcal{O}(V)$ and $\mathcal{O}(W)$ are isomorphic as k-algebras.*

EXAMPLE 3.12. Let k be algebraically closed and consider the cusp

$$C=\{(a,b)\in k^2 \mid b^2=a^3\}.$$

The polynomial map

$$\nu:\mathbb{A}^1_k\longrightarrow C,\qquad t\longmapsto(t^2,t^3)$$

is bijective on points. The origin comes from $t=0$; if $a\neq 0$, then the unique parameter of (a,b) is $t=b/a$, since $t^2=a$ and $t^3=b$.
Nevertheless, ν is not an isomorphism. Its map on coordinate rings is

$$\nu^*:k[x,y]/(y^2-x^3)\longrightarrow k[t],\qquad x\longmapsto t^2,\quad y\longmapsto t^3.$$

This map is injective: write a class as $p(x)+yq(x)$; its image is $p(t^2)+t^3q(t^2)$, whose even and odd powers cannot cancel. In particular the ideal (y^2-x^3) is prime, so the displayed quotient is indeed $\mathcal{O}(C)$. The image of ν^* is $k[t^2,t^3]$, which does not contain t. Thus ν^* is not surjective. The equivalence of categories shows that a bijective polynomial map need not have a polynomial inverse.

4. Irreducible varieties

COROLLARY 4.1 (of the Nullstellensatz). *Let V be an affine algebraic set over an algebraically closed field. Then the assignment*

$$v \in V \mapsto \{[f] \in \mathcal{O}(V) \mid f(v) = 0\}$$

is a bijection between points of V and maximal ideals of $\mathcal{O}(V)$.

PROOF. Ideals in $\mathcal{O}(V)$ are given by ideals in $k[X_1, ..., X_n]$ that contain $I(V)$. Thus the maximal ones are precisely the maximal ideals of $k[X_1, ..., X_n]$ which contain $I(V)$, i.e. points $v \in \mathbb{A}^n$ such that $v \in V$. □

Now since $\mathbb{A}^n$ is a topological space, we get a subspace topology on $V \subseteq \mathbb{A}^n$ in which a subset

$$V' \subseteq V$$

is closed precisely if $V' \subseteq \mathbb{A}^n$ is closed, i.e. V' is itself an affine algebraic set contained in V. Thus we get:

PROPOSITION 4.2. *The closed subsets of V are in one-to-one correspondence with radical ideals in $\mathcal{O}(V)$.*

PROOF. Ideals in $\mathcal{O}(V)$ correspond to ideals in $k[X_1, ..., X_n]$ which contain $I(V)$. If $\mathfrak{a} \subseteq \mathcal{O}(V)$ has preimage $\mathfrak{a}' \subseteq k[X_1, \ldots, X_n]$, then $\mathfrak{a}$ is radical precisely if $\mathfrak{a}'$ is radical, since

$$\left(k[X_1, ..., X_n]/I(V)\right)/\mathfrak{a} = k[X_1, ..., X_n]/\mathfrak{a}'$$ □

DEFINITION 4.3. *A topological space X is called* irreducible *if it is nonempty and cannot be written as the union of two closed subspaces (neither of which agrees with X).*
An affine algebraic set $V \subseteq \mathbb{A}^n$ is called irreducible (or an affine variety, but we try to avoid this terminology here to prevent confusion) if it is irreducible when considered as a topological space.

Equivalently, a nonempty V is irreducible if we cannot write it as $V = V_0 \cup V_1$ for affine algebraic subsets $V_0, V_1 \subsetneq V$. Also equivalent: if $V \subseteq V_0 \cup V_1$ for closed subsets $V_0, V_1 \subseteq \mathbb{A}^n$ then either $V \subseteq V_0$ or $V \subseteq V_1$.

EXAMPLE 4.4. The affine algebraic set

$$\{(x, y) \in \mathbb{C}^2 \mid x^2 - y^2 = 0\}$$

is not irreducible, since we can write it as the union of

$$V_0 = \{(x, y) \in \mathbb{C}^2 \mid x - y = 0\} \qquad \text{and} \qquad V_1 = \{(x, y) \in \mathbb{C}^2 \mid x + y = 0\}\ .$$

EXAMPLE 4.5. The affine algebraic set $k = \mathbb{A}^1$ is irreducible, since each proper closed subset is finite (solutions to a polynomial equation since every ideal is principal) and k is infinite, being algebraically closed. Thus it cannot be written as a union of two proper closed subsets.

PROPOSITION 4.6. *For an affine algebraic set $V \subseteq \mathbb{A}^n$ the following are equivalent:*

1. *V is irreducible*
2. *$I(V) \subseteq k[X_1, ..., X_n]$ is a prime ideal*
3. *$\mathcal{O}(V)$ is a domain, i.e. a nonzero ring without zero divisors.*

PROOF. The equivalence of (2) and (3) is the usual ideal–quotient correspondence. If $V = \emptyset$, then $I(V) = k[X_1, \ldots, X_n]$ and $\mathcal{O}(V) = 0$, so none of the three conditions holds. We may therefore assume that $V \neq \emptyset$. If $V = V_0 \cup V_1$ with V_i proper closed subsets, choose $f_i \in I(V_i) \setminus I(V)$. Their product vanishes on V, whereas neither factor does. Thus $I(V)$ is not prime. Conversely, if $fg \in I(V)$ with $f, g \notin I(V)$, then

$$V = (V \cap V(f)) \cup (V \cap V(g))$$

is a union of two proper closed subsets. Notice that the vanishing ideal of a union is an *intersection*: $I(V_0 \cup V_1) = I(V_0) \cap I(V_1)$. □

COROLLARY 4.7. *$\mathbb{A}^n(k)$ is irreducible.*

PROOF. The polynomial ring is a domain. □

EXAMPLE 4.8. For every irreducible polynomial $p \in k[X_1, ..., X_n]$ the hypersurface $\{p = 0\}$ is irreducible.

DEFINITION 4.9. *A topological space X is called noetherian if every descending chain*

$$V_1 \supseteq V_2 \supseteq V_3 \supseteq ...$$

of closed subsets eventually becomes constant.

LEMMA 4.10. *Every affine algebraic set V is noetherian as a topological space, i.e. every descending chain of affine algebraic subsets of V*

$$V_1 \supseteq V_2 \supseteq V_3 \supseteq ...$$

becomes constant.

PROOF. The corresponding vanishing ideals form an ascending chain in $\mathcal{O}(V)$, which is noetherian as a quotient of $k[X_1, ..., X_n]$. The ideals therefore stabilize, and so do their zero sets. □

DEFINITION 4.11. *Let X be a topological space. A subset $V \subseteq X$ is called irreducible if it is irreducible in the subspace topology, i.e. V is nonempty and for every inclusion $V \subseteq V_0 \cup V_1$ with V_i closed in X we have $V \subseteq V_0$ or $V \subseteq V_1$. An irreducible component is a maximal irreducible subset of X.*

LEMMA 4.12. *Irreducible components are closed and every irreducible subset is contained in an irreducible component.*

PROOF. For the first part we note that the closure of an irreducible set is irreducible. This is clear by definition. Thus if V is maximal it has to agree with its closure. For the second part assume that we are given an irreducible subset $M \subseteq X$. Then the set of all irreducible subsets between M and X is inductively ordered (i.e. every chain has an upper bound): given a nonempty chain $(M_i)_{i \in I}$ of such subsets, consider

$$M_\infty = \bigcup_{i \in I} M_i.$$

If $M_\infty \subseteq A_0 \cup A_1$ with A_0, A_1 closed in X, then every M_i is contained in one of them. If all M_i lie in A_0, so does their union. Otherwise choose $M_{i_0} \not\subseteq A_0$. Every member of the chain is either contained in $M_{i_0} \subseteq A_1$, or contains M_{i_0} and hence must itself lie in A_1. Thus the union lies in A_1 and is irreducible. The empty chain has M as an upper bound. Thus by Zorn's lemma a maximal element exists, i.e. M is contained in an irreducible component.

□

PROPOSITION 4.13. *Let X be a noetherian topological space. Then it admits finitely many, uniquely determined irreducible components $V_1, ..., V_r$. We have*

$$X = V_1 \cup .... \cup V_r \ .$$

No V_i can be omitted from this union. For $X = \emptyset$ we take $r = 0$.
Conversely, if we can write X as a finite union of closed irreducible subsets with no redundant member (equivalently: no one is contained in another), then these are already the irreducible components.

PROOF. First every closed subset of X is a finite union of irreducible closed subsets: otherwise, by the descending chain condition, choose a minimal closed subset Y for which this fails. It is nonempty and reducible, so $Y = Y_0 \cup Y_1$ with Y_0, Y_1 proper closed subsets. By minimality each is a finite union of irreducible closed subsets, a contradiction. In particular we can write

$$X = V_1 \cup ... \cup V_r.$$

By discarding redundant members, we can assume that none is contained in another. Now any irreducible subset $M \subseteq X$ has to be contained in one of the V_i by definition of irreducible. If M is an irreducible component, then it has to agree with the respective V_i. Thus every irreducible component has to agree with one of the V_i. Since every V_i is contained in an irreducible component this also shows that all of the V_i are irreducible components. □

Note that irreducible components of V correspond to minimal prime ideals in $\mathcal{O}(V)$. Thus the last result can be translated into the existence of minimal prime ideals in this ring.

EXAMPLE 4.14. The irreducible components of

$$\{(x, y) \in k^2 \mid xy = 0\}$$

are the axes $\{x = 0\}$ and $\{y = 0\}$, since V is their union and neither can be omitted. More algebraically, the vanishing ideal is

$$I(V) = (xy) = (x) \cap (y), \qquad B := \mathcal{O}(V) = k[x, y]/(xy).$$

Indeed, a polynomial is divisible by both x and y precisely if it is divisible by xy. The two minimal prime ideals of B are (x) and (y), corresponding to the two axes. The maximal ideal (x, y) corresponds to their common origin. We will use this example repeatedly to compare geometric constructions with operations on rings.

EXAMPLE 4.15. More generally, for a polynomial $p \in k[X_1, ..., X_n]$ we consider the zero set

$$V = \{x \in \mathbb{A}^n \mid p(x) = 0\} = \{p = 0\} \ .$$

If p is nonzero and nonconstant, write $p = cp_1^{n_1} \cdots p_r^{n_r}$ with $c \in k^\times$, pairwise nonassociate irreducible polynomials p_i, and $n_i \geq 1$. Then V is a hypersurface, and with $p' = \prod_i p_i$ we have $V = \{p' = 0\}$. In fact, $(p') = \sqrt{(p)}$. We also have

$$V = \{p_1 = 0\} \cup \{p_2 = 0\} \cup \ldots \cup \{p_r = 0\}.$$

These are the irreducible components: they are not contained in one another, since the irreducible polynomials cannot divide one another. If $p = 0$, then $V = \mathbb{A}^n$; if p is a nonzero constant, then $V = \emptyset$.

We finish this section with the following table:

Geometry	Algebra
V: affine algebraic set	$\mathcal{O}(V)$: reduced, finitely generated k-algebra
$x \in V$ point	$I(x)$ maximal ideal in $\mathcal{O}(V)$
$V' \subseteq V$ closed subsets	$I(V')$ radical ideals in $\mathcal{O}(V)$
irreducible subvarieties	prime ideals
irreducible components	minimal prime ideals

5. The spectrum of a ring

5.1. The prime spectrum and its topology. Recall that the points of an affine algebraic set V are the maximal ideals in $\mathcal{O}(V)$. We now want to do this for general rings A replacing $\mathcal{O}(V)$.

Definition 5.1. *Let A be a commutative ring. Then we define the* spectrum *of A, denoted by* $\mathrm{Spec}(A)$*, as the set of prime ideals of A (remember that prime ideals are proper ideals). The maximal spectrum is the subset*

$$\mathrm{mSpec}(A) \subseteq \mathrm{Spec}(A)$$

consisting of the maximal ideals. For a given point $x \in \mathrm{Spec}(A)$ we define the residue field at x *to be*

$$\kappa(x) := \mathrm{Frac}(A/x)$$

the fraction field of the domain A/x. [2] *An element $f \in A$ gives rise to an element*

$$f(x) \in \kappa(x)$$

represented by the class $[f]$.

We informally want to think of an element $f \in A$ as a function on $\mathrm{Spec}(A)$ which takes x to an element in the residue field $\kappa(x)$, thus the field of values varies. For $f, g \in A$ we clearly have $(f \cdot g)(x) = f(x)g(x)$ and $(f + g)(x) = f(x) + g(x)$.

Example 5.2. If $A = \mathcal{O}(V)$ for an affine algebraic set V over an algebraically closed field k, then $V \cong \mathrm{mSpec}(A)$ and for every $x \in V$ we have that $\kappa(x) \cong k$, the isomorphism is given by

$$\mathcal{O}(V)/I(x) \to k \qquad [f] \mapsto f(x).$$

Thus for a given $f \in \mathcal{O}(V)$ we get that under this identification $f(x)$ is simply the evaluation of f at the point x!
The spectrum $\mathrm{Spec}(A)$ now consists of the prime ideals, which correspond to the irreducible closed subsets of V. For a given point $x \in \mathrm{Spec}(A)$ corresponding to an irreducible variety $V' \subseteq V$ the fraction field $\kappa(x)$ is given by the fraction field of the (domain) $\mathcal{O}(V')$. The evaluation $f(x)$ for an element $f \in \mathcal{O}(V)$ at x is then given by considering the restriction of f to V' as an element in the fraction field of $\mathcal{O}(V')$.

Example 5.3. Let's consider the case $V = \mathbb{A}^1$, i.e. $A = \mathcal{O}(V)$. Then $\mathrm{Spec}(A)$ consists of all the points of V corresponding to the maximal ideals and additionally the nonmaximal prime ideals. But the ring $k[x]$ has only one prime ideal that is not maximal, namely (0). Geometrically this corresponds to the irreducible variety V itself. Thus $\mathrm{Spec}(A)$ has the points $\mathbb{A}^1$ and one additional point (0), which is morally 'arbitrarily close' to any other point. We will make the latter precise soon.

[2] That means we invert every nonzero element in A/x. Note that for a maximal ideal x we have that A/x is already a field, thus $\kappa(x) = A/x$.

EXAMPLE 5.4. Let's consider the case $V = \mathbb{A}^2$, i.e. $A = \mathcal{O}(V)$. Then $\mathrm{Spec}(A)$ consists of all the points of V corresponding to the maximal ideals and additionally the points corresponding to irreducible closed subsets $V' \subseteq V$ that are not points. There is again a point for the full subset V (corresponding to the 0-ideal) and for every irreducible polynomial (up to multiplication by a nonzero scalar) $p \in k[X_1, X_2]$ there is a further point corresponding to the zero locus $\{p = 0\}$. One can show that these are all points (but we will not do this here).

EXAMPLE 5.5. Consider the ring $A = \mathbb{Z}$. Then the prime ideals are given by $p\mathbb{Z}$ for p a prime, together with (0). The maximal ideals among them are the $p\mathbb{Z}$, i.e. everything except (0) is maximal. Thus $\mathrm{Spec}(\mathbb{Z})$ has a point for every prime and an additional point (0), which one should think of as a point which is 'arbitrarily close to all other points'. The residue fields are given by

$$\kappa(p) = \mathbb{F}_p$$

and

$$\kappa(0) = \mathbb{Q}$$

An element $n \in \mathbb{Z}$ acts as a function on $\mathrm{Spec}(\mathbb{Z})$ whose value at each point x is the image of n in $\kappa(x)$.

EXAMPLE 5.6. For any field k we have that $\mathrm{Spec}(k)$ consists of a single point (corresponding to the zero ideal), whose residue field is k itself.

PROPOSITION 5.7. *For any ring homomorphism $f : A \to B$ the induced map*

$$\mathrm{Spec}(f) : \mathrm{Spec}(B) \to \mathrm{Spec}(A) \qquad \mathfrak{p} \mapsto f^{-1}\mathfrak{p}$$

is well-defined.

PROOF. We only have to show that the ideal $f^{-1}\mathfrak{p}$ is prime. It is proper since $f(1) = 1 \notin \mathfrak{p}$. Thus assume $ab \in f^{-1}\mathfrak{p}$. Then $f(a)f(b) = f(ab) \in \mathfrak{p}$, therefore either $f(a)$ or $f(b)$ lies in $\mathfrak{p}$, which implies that either a or b lies in $f^{-1}\mathfrak{p}$. □

WARNING 5.8. *While* $\mathrm{mSpec}(A)$ *seems in some sense more natural than* $\mathrm{Spec}(A)$*, it is not functorial. For example the inclusion $\mathbb{Z} \to \mathbb{Q}$ induces the map*

$$\mathrm{Spec}(\mathbb{Q}) \to \mathrm{Spec}(\mathbb{Z})$$

which sends (0) *to* (0)*. But* $\mathrm{mSpec}(\mathbb{Q}) = \mathrm{Spec}(\mathbb{Q})$ *while* $\mathrm{mSpec}(\mathbb{Z})$ *does not contain the zero point. Said differently: the pullback of a maximal ideal is in general only a prime ideal but not maximal.*

DEFINITION 5.9. *Let M be a subset of a ring A. Then we define the vanishing locus of M as*

$$V(M) = \{x \in \mathrm{Spec}(A) \mid f(x) = 0 \ \forall f \in M\} = \{x \in \mathrm{Spec}(A) \mid M \subseteq x\} \ .$$

We leave the equality as an exercise to the reader to get used to the definitions.

PROPOSITION 5.10. (1) *We have that $V(M) = V(\sqrt{(M)})$.*
(2) *There is a topology on* $\mathrm{Spec}(A)$ *called the* Zariski topology *such that the closed sets are exactly the vanishing loci $V(M)$.*
(3) *For any ring map $f : A \to B$ the map* $\mathrm{Spec}(f) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is continuous.*[3]

[3]That is preimages of open sets are open, or equivalently preimages of closed sets are closed.

(4) *For any ideal* $\mathfrak{a}$ *the map*

$$\mathrm{Spec}(A/\mathfrak{a}) \to \mathrm{Spec}(A)$$

induced by the quotient map is a homeomorphism onto the closed subset $V(\mathfrak{a})$.

PROOF. (1) is clear. The proof of (2) proceeds exactly as in Proposition 2.1:

$$\bigcap_{i\in I} V(M_i) = V(\cup_{i\in I} M_i) \qquad V(M)\cup V(N) = V(M\cdot N).$$

For (3) note that

$$\mathrm{Spec}(f)^{-1}(V(M)) = \{x \in \mathrm{Spec}(B) \mid M \subseteq f^{-1}(x)\} = V(f(M))\ .$$

For (4), the prime-ideal correspondence gives a bijection $\mathrm{Spec}(A/\mathfrak{a}) \to V(\mathfrak{a})$. For any ideal $\mathfrak{b} \supseteq \mathfrak{a}$, put $\overline{\mathfrak{b}} = \mathfrak{b}/\mathfrak{a}$. Under this bijection $V(\overline{\mathfrak{b}})$ corresponds to $V(\mathfrak{b}) \subseteq V(\mathfrak{a})$. Thus the bijection and its inverse preserve closed sets. □

THEOREM 5.11. *Closed subsets of* $\mathrm{Spec}(A)$ *are in one-to-one correspondence with radical ideals of* A. *The correspondence sends an ideal* $\mathfrak{a}$ *to its vanishing locus and conversely sends a closed subset* $V \subseteq \mathrm{Spec}(A)$ *to the ideal*

$$\{f \in A \mid f(v) = 0\ \forall v \in V\} = \bigcap_{\mathfrak{p}\in V} \mathfrak{p}$$

PROOF. By definition every closed subset is $V(\mathfrak{a})$, and $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$. The algebraic identity [**AM**, Proposition 1.14]

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{a}\subseteq\mathfrak{p}} \mathfrak{p}$$

says precisely that taking the ideal of functions vanishing on $V(\mathfrak{a})$ recovers $\sqrt{\mathfrak{a}}$. Thus the two constructions are inverse on radical ideals and closed subsets. □

REMARK 5.12. This implies similarly to the Nullstellensatz (Corollary 2.8) that for every ideal $\mathfrak{a} \subsetneq A$ we have a point $x \in \mathrm{Spec}(A)$ with $f(x) = 0$ for all $f \in \mathfrak{a}$. But one can do better: for every $\mathfrak{a} \neq A$ we find a maximal ideal $\mathfrak{m}$ containing $\mathfrak{a}$. Thus $\mathfrak{m} \in V(\mathfrak{a})$, i.e. all $f \in \mathfrak{a}$ vanish on $\mathfrak{m}$, i.e. we even have a zero in the maximal spectrum.

Recall that a space X is called compact or quasi-compact (if one wants to emphasize the fact that it is not Hausdorff) if for every open cover $X = \bigcup_{i\in I} U_i$ there is a finite set of open sets $U_{i_0}, ..., U_{i_n}$ such that $X = U_{i_0} \cup ... \cup U_{i_n}$. In words: every cover has a finite subcover. Equivalently: if there are closed sets V_i with $\cap V_i = \emptyset$ then there are already finitely many whose intersection is empty.

PROPOSITION 5.13. *The topological space* $\mathrm{Spec}(A)$ *is quasi-compact.*

PROOF. Assume that

$$\emptyset = \bigcap_{i\in I} V(M_i) = V(\cup_{i\in I} M_i)$$

If $\mathfrak{a}$ is the ideal generated by all the M_i then $V(\mathfrak{a}) = \emptyset$ so that $\mathfrak{a} = A$. But this means that $1 \in \mathfrak{a}$. But then 1 is already contained in the ideal generated by finitely many M_i and thus the intersection of those $V(M_i)$ is already empty. □

For a subset $M \subseteq A$ we denote the open set

$$D(M) = V(M)^c \subseteq \mathrm{Spec}(A) \ .$$

This is the set of all $x \in \mathrm{Spec}(A)$ with $f(x) \neq 0$ for at least one $f \in M$.

PROPOSITION 5.14. *Given $f \in A$ the map*

$$\mathrm{Spec}(A[1/f]) \to \mathrm{Spec}(A)$$

induced by the canonical map $i : A \to A[1/f]$ is an open embedding with image $D(f)$.

PROOF. Recall the prime-ideal correspondence [**AM**, Proposition 3.11(iv)]. Prime ideals in $A[1/f]$ are in one-to-one correspondence with isomorphism classes of quotients $A[1/f] \to C$ which are domains. These are in one-to-one correspondence with domain quotients $A \to B$ which send f to a nonzero element (by localizing or passing to the image of A). So we see that prime ideals in $A[1/f]$ are in one-to-one correspondence with prime ideals of A which do not contain f (and the bijection is given by contracting an ideal along i, i.e. by $\mathrm{Spec}(i)$). This tells us that $\mathrm{Spec}(i)$ is a bijection from $\mathrm{Spec}(A[1/f])$ onto the complement of $V(f)$ in $\mathrm{Spec}(A)$. Thus it remains to prove that $\mathrm{Spec}(i)$ is an embedding, i.e. that $\mathrm{Spec}(A[1/f])$ carries the subspace topology. Since $\mathrm{Spec}(i)$ is continuous, the intersection of a closed set in $\mathrm{Spec}(A)$ with $\mathrm{Spec}(A[1/f])$ is closed in $\mathrm{Spec}(A[1/f])$, so it suffices to show that conversely, every closed set in $\mathrm{Spec}(A[1/f])$ arises as intersection with a closed set in $\mathrm{Spec}(A)$.
To this end consider a closed set

$$V(M) \subseteq \mathrm{Spec}(A[1/f])$$

defined by some set $M = \{\frac{p_i}{f^{n_i}} \mid i \in I\} \subseteq A[1/f]$, where $n_i \geq 0$. Then we consider the set $M' := \{p_i \mid i \in I\} \subseteq A$. We have that

$$\begin{aligned} V(M') \cap \mathrm{Spec}(A[1/f]) &= \{x \in \mathrm{Spec}(A[1/f]) \mid i(M') \subseteq x\} \\ &= \{x \in \mathrm{Spec}(A[1/f]) \mid M \subseteq x\} = V(M) \ . \end{aligned}$$

This finishes the proof. □

DEFINITION 5.15. *We call an open set $U \subseteq \mathrm{Spec}(A)$ a principal open if it is of the form $D(f) \cong \mathrm{Spec}(A[1/f])$ for some $f \in A$. In the principal case we set*

$$\mathcal{O}(U) = A[1/f]$$

REMARK 5.16. Note that $\mathcal{O}(U)$ is well-defined up to a canonical isomorphism: $V(f) = V(f')$ precisely if $\sqrt{(f)} = \sqrt{(f')}$, and then there are $n, m \geq 1$ and $g, g' \in A$ such that $f^n = gf'$ and $(f')^m = g'f$. Thus inverting f inverts f' and vice versa. The universal property gives a unique A-algebra isomorphism $A[1/f] \cong A[1/f']$ compatible with the canonical maps from A.

PROPOSITION 5.17. *Every open set $U \subseteq \mathrm{Spec}(A)$ is a union of principal opens (that is, the principal opens form a basis of the topology) and every finite intersection of principal opens is a principal open.*

PROOF. We have

$$U = V(M)^c = (\bigcap_{f \in M} V(f))^c = \bigcup_{f \in M} D_f.$$

and

$$D_f \cap D_g = (V(f) \cup V(g))^c = V(fg)^c = D_{fg} \ . \qquad \square$$

5.2. Generic points.

DEFINITION 5.18. *Let X be a topological space and $V \subseteq X$ be a closed subset. A point $x \in V$ is called a generic point of V if $\overline{\{x\}} = V$. If $V = X$ then it is simply called a generic point (of X).*
A space X is called a T_0-space if for each pair of points $x \neq y$ there is an open set which contains exactly one of the two points.

If V admits a generic point, then it is irreducible, since points are certainly irreducible and closures preserve irreducibility. If X is T_0 then generic points are, if they exist, unique since if we can separate x and y by an open set then $\overline{\{x\}} \neq \overline{\{y\}}$.

EXAMPLE 5.19. For every domain A the point $(0) \in \operatorname{Spec}(A)$ is generic since every closed set $V(M) = \{x \in \operatorname{Spec}(A) \mid M \subseteq x\}$ containing (0) has to be everything. Morally this means that (0) lies infinitely close to any other point in $\operatorname{Spec}(A)$ (as it is contained in each neighborhood). This is for example true for $\operatorname{Spec}(\mathbb{Z})$ which has a bunch of closed points (all the primes) and the point (0). This is also true for $\operatorname{Spec}(k[X])$ which is in fact homeomorphic to $\operatorname{Spec}(\mathbb{Z})$ as long as k is countable.

DEFINITION 5.20. *A space is called* sober *if it is T_0 and every irreducible closed subset has a generic point.*

EXAMPLE 5.21. Every Hausdorff space is sober (exercise). The space $\mathbb{A}^n = k^n$ with the Zariski topology is not sober for $n \geq 1$, since it is irreducible but does not possess a generic point (all points are in fact closed).
Every finite T_0-space is sober. To see this it suffices to show that a finite irreducible T_0-space has a generic point. Such a space is the finite union of the closures of its points, so irreducibility forces one of these closures to be the whole space.

PROPOSITION 5.22. *For any ring A the space $\operatorname{Spec}(A)$ is sober.*

PROOF. Assume that $x \neq y$ are two distinct points in $\operatorname{Spec}(A)$. Then there has to be an element $f \in A$ which is contained in one of the two ideals, but not the other. But then the open set $D_f \subseteq \operatorname{Spec}(A)$ which is the subset of all prime ideals which do not contain the element f contains exactly one of the two prime ideals.
Assume that $V(M)$ is an irreducible closed subset. Replace M by the radical ideal $\mathfrak{a} = \sqrt{(M)}$. Since $V(\mathfrak{a})$ is nonempty, $\mathfrak{a}$ is proper. If $fg \in \mathfrak{a}$, then $V(\mathfrak{a}) \subseteq V(fg) = V(f) \cup V(g)$. Irreducibility gives $V(\mathfrak{a}) \subseteq V(f)$ or $V(\mathfrak{a}) \subseteq V(g)$. By Theorem 5.11, this means $f \in \mathfrak{a}$ or $g \in \mathfrak{a}$, so $\mathfrak{a}$ is prime. Thus every irreducible closed subset is of the form $V(\mathfrak{p})$ for $\mathfrak{p}$ prime. Moreover $\mathfrak{p} \in V(\mathfrak{p})$ and $V(\mathfrak{p})$ is the closure of $\{\mathfrak{p}\}$, since every closed subset containing $\mathfrak{p}$ also contains $V(\mathfrak{p})$. Thus $\mathfrak{p}$ is a generic point. □

EXAMPLE 5.23. Let k be algebraically closed and return to the coordinate axes of Example 4.14, now considered as

$$X = \operatorname{Spec}(B), \qquad B = k[x, y]/(xy).$$

Every prime ideal contains x or y, since $xy = 0$ in B. The primes containing y correspond to the primes of $B/(y) = k[x]$, and similarly for x. Thus the points of X are

$$\eta_x = (y), \qquad \eta_y = (x), \qquad \mathfrak{m}_{a,b} = (x - a, y - b) \quad \text{with } ab = 0.$$

The last points are closed, and the origin is $o = \mathfrak{m}_{0,0}$. The other two points are generic points:

$$\overline{\{\eta_x\}} = V(y), \qquad \overline{\{\eta_y\}} = V(x).$$

These are exactly the two irreducible components. In particular, o lies in both closures. The whole space has no generic point, since it is reducible. Passing from the classical algebraic set to its spectrum has added one generic point for each axis.

REMARK 5.24. We have now seen that $\mathrm{Spec}(R)$ is quasi-compact and sober, and admits a basis of quasi-compact open subsets closed under finite intersections: the principal opens are homeomorphic to $\mathrm{Spec}(R[1/f])$. It is a theorem of Hochster that the spaces homeomorphic to spectra of rings are precisely the quasi-compact sober spaces which possess a basis of quasi-compact opens closed under finite intersections. Such spaces are called *spectral spaces*.

Geometry	Algebra
$\mathrm{Spec}(A)$	A any ring
$x \in \mathrm{Spec}(A)$	Prime ideals
closed points	maximal ideals
$V \subseteq \mathrm{Spec}(A)$ closed	radically closed ideals
irreducible closed subsets	prime ideals

CONSTRUCTION 5.25. Let X be any topological space. We can form a new topological space $S(X)$, called its sobrification, which is sober and comes with a continuous map $X \to S(X)$ universal among continuous maps from X to sober spaces. Concretely,

$$S(X) = \{V \subseteq X \mid V \text{ irreducible and closed}\} .$$

We define a topology on $S(X)$ by letting the closed sets be the subsets of $S(X)$ of the form

$$\{V \in S(X) \mid V \subseteq A\}$$

for closed sets $A \subseteq X$. The map $X \to S(X)$ is given by $x \mapsto \overline{\{x\}}$. We leave it as an exercise to check that this has the desired properties, concretely that:

(1) $S(X)$ is a topological space
(2) The map $X \to S(X)$ is continuous
(3) $S(X)$ is sober
(4) Every continuous map $X \to Y$ with Y sober factors uniquely through $S(X)$
(5) If X is T_0 then the map $X \to S(X)$ is an embedding.

We now have the following immediate consequence:

THEOREM 5.26. *Let V be an affine algebraic set over an algebraically closed field k. Then the map*

$$V = \mathrm{mSpec}(\mathcal{O}(V)) \to \mathrm{Spec}(\mathcal{O}(V))$$

is a continuous embedding and exhibits $\mathrm{Spec}(\mathcal{O}(V))$ *as the sobrification of V.*

This in particular shows that in passing from V to the prime spectrum we exactly add 'generic points' of irreducible subvarieties.

REMARK 5.27. In general it is not true that $\mathrm{mSpec}(A) \to \mathrm{Spec}(A)$ is a sobrification. The inclusion exhibits $\mathrm{Spec}(A)$ as the sobrification of $\mathrm{mSpec}(A)$ precisely when every prime ideal of A is the intersection of the maximal ideals containing it. Such rings are called Jacobson rings. Hilbert's theorem can then be interpreted as saying that quotients of $k[X_1, ..., X_n]$ are Jacobson rings. We will however from now on work with the prime spectrum.

6. Sheaves and the structure sheaf

6.1. Presheaves and sheaves. For a topological space X we can consider the partially ordered set of open subsets of X. We can consider this as a category which we denote $\mathrm{Open}(X)$. Concretely the objects of $\mathrm{Open}(X)$ are the open sets $U \subseteq X$ and the sets of morphisms are given by

$$\mathrm{Hom}_{\mathrm{Open}(X)}(U,V) = \begin{cases} \mathrm{pt} & \text{if } U \subseteq V \\ \emptyset & \text{otherwise} \end{cases}$$

Every continuous map $f : X \to Y$ gives rise to a functor

$$\mathrm{Open}(Y) \to \mathrm{Open}(X) \qquad U \mapsto f^{-1}U \ .$$

Remark 6.1. One can try to reconstruct the topological space X from the category $\mathrm{Open}(X)$. It turns out that this is possible precisely if the space X is sober. The idea is the following: any point $\mathrm{pt} \xrightarrow{x} X$ gives rise to a functor

$$x^{-1} : \mathrm{Open}(X) \to \mathrm{Open}(\mathrm{pt})$$

which preserves arbitrary unions and finite intersections (which can be characterised as colimits and limits in the categories). Conversely one can consider the set of all functors

$$\mathrm{Open}(X) \to \mathrm{Open}(\mathrm{pt})$$

preserving arbitrary unions and finite intersections. It turns out that these are precisely in bijection with closed, irreducible subsets $V \subseteq X$ for any topological space X. Thus the assignment which sends each point x to the associated functor x^{-1} is a bijection precisely if X is sober.

Let $\mathcal{C}$ be any category. Examples to have in mind are $\mathcal{C} = \mathrm{Set}, \mathrm{Ab}, \mathrm{Ring}, \ldots$

Definition 6.2. *A $\mathcal{C}$-valued* presheaf *on a topological space X is a functor*

$$F : \mathrm{Open}(X)^{\mathrm{op}} \to \mathcal{C} \ .$$

We will refer to $F(U)$ as the value on U or as the sections of F over U. *For an inclusion of open sets $U \subseteq V$ we call the induced map $F(V) \to F(U)$ the restriction map (or restriction along $U \subseteq V$). This map is sometimes written as $(-)|_U$. The category of presheaves is the functor category*

$$\mathrm{PSh}(X;\mathcal{C}) := \mathrm{Fun}(\mathrm{Open}(X)^{\mathrm{op}}, \mathcal{C}) \ .$$

Example 6.3. We consider the functor

$$F_1(U) := C^0(U, \mathbb{R})$$

with the restriction of functions. This is a presheaf of rings. We also have the presheaf of all set maps

$$F_2(U) := \mathrm{Hom}_{\mathrm{Set}}(U, \mathbb{R})$$

This is also a presheaf of rings, and the inclusion of continuous maps in all maps induces a morphism of presheaves $F_1 \to F_2$. We also have the constant presheaf $F_3(U) = \mathbb{R}$ for all U and all restrictions the identity. Thinking of this as constant functions with value in $\mathbb{R}$ we get a further morphism

$$F_3 \to F_1 \ .$$

DEFINITION 6.4. *Assume that the category $\mathcal{C}$ has all small products. A presheaf $F \in \mathrm{PSh}(X;\mathcal{C})$ is called a* sheaf *if for each collection $(U_i)_{i\in I}$ of open subsets in X with $U = \cup_{i\in I} U_i$ the induced diagram*

$$F(U) \xrightarrow{(-|_{U_i})_{i\in I}} \prod_{i\in I} F(U_i) \underset{(-|_{U_i\cap U_j}\circ \mathrm{pr}_j)_{i,j\in I}}{\overset{(-|_{U_i\cap U_j}\circ \mathrm{pr}_i)_{i,j\in I}}{\rightrightarrows}} \prod_{i,j\in I} F(U_i\cap U_j)$$

is a limit cone in $\mathcal{C}$. Here the two maps project to the i-th or j-th factor, respectively, and then restrict to the respective subset.
The category of sheaves $\mathrm{Shv}(X;\mathcal{C})$ *is the full subcategory of* $\mathrm{PSh}(X;\mathcal{C})$ *spanned by the sheaves.*

REMARK 6.5. Assume that $\mathcal{C} = \mathrm{Set}$ (or any category like Ab or Ring consisting of sets with some algebraic extra structure for which limits are computed on underlying sets). Then the sheaf condition is explicitly saying that for any family of elements

$$s_i \in F(U_i)$$

such that $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$ there is a unique element $s \in F(U)$ with $s|_{U_i} = s_i$. Morally this means that one can 'glue' elements from local ones or elements are determined locally. In this case, if the map from $F(U)$ to the equalizer is just injective, we say that F is separated.

EXAMPLE 6.6. Consider the presheaf C^0. This is a sheaf, since if we have locally defined continuous functions $U_i \to \mathbb{R}$ which agree on double intersections $U_i\cap U_j \to \mathbb{R}$ then we get a unique continuous function $U \to \mathbb{R}$. The important fact is that being continuous can be checked locally (check this as an exercise if you are not sure).
The constant presheaf is in general not a sheaf, since the sheaf condition applied to the *empty* cover of $U =$ requires that

$$F() \longrightarrow \prod_{i\in\emptyset} F(U_i) \rightrightarrows \prod_{i,j\in\emptyset} F(U_i\cap U_j)$$

is a limit cone. Since empty products are simply pt, this precisely means that $F(\emptyset) = \mathrm{pt}$ if F is a sheaf, which is not satisfied by the constant presheaf.
If we fix this by defining instead a presheaf with $F(\emptyset) = \mathrm{pt}$, $F(U) = \mathbb{R}$ for $U \neq \emptyset$, this is still not a sheaf in general: Assume that we have disjoint nonempty open subsets $U, V \subseteq X$. Then we can consider constant functions $\lambda_1 : U \to \mathbb{R}$ and $\lambda_2 : V \to \mathbb{R}$. They agree on the intersection $U \cap V = \emptyset$ but there cannot be a constant function extending them unless their constant values agree.

PROPOSITION 6.7. *Assume that $F : \mathrm{Open}(X)^{\mathrm{op}} \to \mathcal{C}$ is a sheaf.*

(1) *Then $F(\emptyset)$ is terminal.*
(2) *For disjoint open sets $U, V \subseteq X$ we have that the map*

$$F(U\cup V) \xrightarrow{(-|_U,-|_V)} F(U)\times F(V)$$

is an isomorphism.
(3) *If we have a sequence of open subsets $U_0 \subseteq U_1 \subseteq U_2 \subseteq \ldots$ with $\bigcup_{i=0}^{\infty} U_i = U$, then the map*

$$F(U) \to \varprojlim F(U_i)\ .$$

is an isomorphism.

(4) *More generally assume that we have a map of posets*

$$I \to \mathrm{Open}(X) \qquad i \mapsto U_i$$

where I has infima for pairs of objects and the map sends those to intersections (i.e. preserves infima). Then with $U := \cup U_i$ we have that

$$F(U) = \lim_{I^{\mathrm{op}}} F(U_i)$$

PROOF. The reader who is not yet completely versed in the language of category theory might want to give these proofs first for the case $\mathcal{C} = \mathrm{Set}$.
1) For the empty set we have the covering consisting of no open subsets. Then we get that $F(\emptyset)$ is the equalizer of the two empty products. These are terminal. Thus we get that $F(\emptyset)$ is itself terminal (think about why the equalizer is itself terminal).
2) For $U \cup V$ we take the cover consisting of the two open sets U and V. Applying the sheaf condition we get a limit diagram

$$F(U \cup V) \longrightarrow F(U) \times F(V) \rightrightarrows F(U) \times F(V) \times F(U \cap V) \times F(V \cap U) \ .$$

Here since U and V are disjoint we find that $F(U \cap V) = \mathrm{pt}$ and thus we get that

$$F(U \cup V) \longrightarrow F(U) \times F(V) \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightrightarrows}} F(U) \times F(V) \ .$$

But the equaliser of the identity map with itself is simply the object $F(U) \times F(V)$.

3) We consider the covering of U given by the U_i. Then we get that

$$\begin{aligned} F(U) &= \mathrm{Eq}\left(\prod_i F(U_i) \Rightarrow \prod_{i,j} F(U_{\min(i,j)}) \right) \\ &= \mathrm{Eq}\left(\prod_i F(U_i) \Rightarrow \prod_{i \leq j} F(U_i) \right) \end{aligned}$$

where the equality holds by universal properties (the upper product is strictly larger than the lower one, but the remaining terms are just copies of terms that are already there and the map factors accordingly). (4) works exactly as (3). □

EXAMPLE 6.8. If $\mathcal{B}$ is a basis of the topology of X closed under intersections, we see from (4) that for a sheaf F on X we have

$$F(U) = \lim_{V \subseteq U, V \in \mathcal{B}} F(V) \ .$$

6.2. Sheaves on a basis. In this subsection we assume that $\mathcal{C}$ has all small limits, as do Set, Ab, and Ring.
Using this we shall construct a specific sheaf on $\mathrm{Spec}(A)$ called the structure sheaf and denoted $\mathcal{O}_{\mathrm{Spec}(A)}$ or simply $\mathcal{O}$. It will have the property that

$$\mathcal{O}(D_f) := A[1/f].$$

as in Definition 5.15. In order to turn this into a proper definition we will show that one can define sheaves on a basis of the topology.

DEFINITION 6.9. *Let X be a topological space and $\mathcal{B}$ be a basis of the topology stable under finite intersections. A presheaf on $\mathcal{B}$ is a functor*

$$\mathcal{B}^{\mathrm{op}} \to \mathcal{C}$$

where we consider $\mathcal{B}$ as a full subcategory of $\mathrm{Open}(X)$. It is called a sheaf whenever it satisfies the sheaf condition for all $U \in \mathcal{B}$ and all open covers by elements in $\mathcal{B}$.

LEMMA 6.10. *A sheaf on $\mathcal{B}$ extends essentially uniquely to a sheaf on X. More precisely the forgetful functor from sheaves on X to sheaves on $\mathcal{B}$*

$$U : \mathrm{Shv}(X; \mathcal{C}) \to \mathrm{Shv}_{\mathcal{B}}(X; \mathcal{C})$$

which simply forgets the value on all opens not belonging to $\mathcal{B}$ is an equivalence of categories.

PROOF. We construct an inverse functor

$$\mathrm{Shv}_{\mathcal{B}}(X; \mathcal{C}) \to \mathrm{Shv}(X; \mathcal{C}) \qquad F \mapsto \overline{F}.$$

We simply define $\overline{F}(U) := \lim_{V \subseteq U, V \in \mathcal{B}} F(V)$ where the limit is taken over the opposite of the category associated with the poset of all open subsets $V \subseteq U$ belonging to $\mathcal{B}$. This construction defines a functor $\mathrm{Open}(X)^{\mathrm{op}} \to \mathcal{C}$ in the following way: for a given inclusion $U \subseteq U'$ we can consider elements in the poset $\{V \subseteq U \mid V \in \mathcal{B}\}$ also as elements of $\{V \subseteq U' \mid V \in \mathcal{B}\}$ and therefore get a 'projection'

$$\lim_{V \subseteq U', V \in \mathcal{B}} F(V) \to \lim_{V \subseteq U, V \in \mathcal{B}} F(V) \ .$$

This clearly defines a functor $\overline{F} : \mathrm{Open}(X)^{\mathrm{op}} \to \mathcal{C}$. To verify that it is a sheaf, first consider the case $\mathcal{C} = \mathrm{Set}$. Suppose that sections of $\overline{F}$ on an open cover (U_i) of U agree on the intersections. For each basis element $V \subseteq U$, the basis elements $W \subseteq V \cap U_i$, for varying i, cover V. The given sections provide compatible elements of $F(W)$, which glue uniquely to an element of $F(V)$. These elements are compatible as V varies, so they give a unique element of $\overline{F}(U)$ restricting to the given sections. For general $\mathcal{C}$, apply this argument to the set-valued sheaf $\mathrm{Hom}_{\mathcal{C}}(T, F(-))$ for each $T \in \mathcal{C}$. Since these functors preserve limits and detect limit cones, this proves the sheaf condition.
Now for an open U in $\mathcal{B}$ we have

$$\overline{F}(U) = F(U)$$

because in this case the opposite of the poset $\{V \subseteq U \mid V \in \mathcal{B}\}$ has an initial object (namely U), so the limit is simply evaluation at this object. We see that $U\overline{F}$ is naturally equivalent to F (the fact that all these constructions are natural is easily verified).
Conversely if we start with a sheaf F on X we get a natural map

$$F \to \overline{UF}$$

given on an open set $U \subseteq X$ by the morphism

$$F(U) \to \lim_{V \subseteq U, V \in \mathcal{B}} F(V) \tag{2}$$

induced by the restrictions $F(U) \to F(V)$. This is an isomorphism as seen before. □

6.3. The structure sheaf on $\mathrm{Spec}(A)$.

THEOREM 6.11. *There is a sheaf $\mathcal{O}$ of commutative rings on $\mathrm{Spec}(A)$, unique up to unique isomorphism compatible with the following identifications, which takes a principal open $U = D_f$ to*

$$\mathcal{O}(U) = A[1/f]$$

and an inclusion $U = D_f \subseteq D_g = V$ to the localization

$$\mathcal{O}(V) = A[1/g] \to A[1/f] = \mathcal{O}(U) \ .$$

PROOF. First we note that this makes sense, since for $D_f \subseteq D_g$ we have that $\sqrt{(f)} \subseteq \sqrt{(g)}$, thus $f^n = hg$ for some n and h. Thus inverting f also inverts g and we get a unique map $A[1/g] \to A[1/f]$. This uniqueness also gives functoriality, i.e. we get a functor

$$\mathcal{O} : \mathcal{B}^{\mathrm{op}} \to \mathrm{Ring} \ .$$

Thus in order to apply the previous lemma we need to verify that $\mathcal{O}$ is a sheaf on $\mathcal{B}$. We need to verify the sheaf condition for $U = D_f$ and $U = \bigcup_i D_{f_i}$. Since U itself is given by $\mathrm{Spec}(A')$ with $A' = A[1/f]$ we can without loss of generality assume $f = 1$, i.e. that $\mathrm{Spec}(A) = \bigcup_i D_{f_i}$. We will work under the additional assumption that I is finite. The next lemma will show that this is indeed enough.
We have to show that

$$A \longrightarrow \prod_{i\in I} A[1/f_i] \rightrightarrows \prod_{i,j\in I} A[1/(f_i f_j)]$$

is an equalizer. The open set D_{f_i} does not change if we replace f_i by a power $f_i^{n_i}$. Then the fact that $\bigcup D_{f_i^{n_i}} = \mathrm{Spec}(A)$ shows that $A = \sqrt{(f_i^{n_i})}$, i.e. some power of 1 lies in $(f_i^{n_i})$, i.e. 1 does. In particular, we have that for every sequence of natural numbers $(n_i)_{i\in I}$ that $1 \in A$ can be written as a sum

$$1 = \sum_I g_i f_i^{n_i} \tag{3}$$

for elements $g_i \in A$.

Assume now that $s \in A$ lies in the kernel of the map

$$A \to \prod_{i\in I} A[1/f_i] \ .$$

Then we know that there exist n_i such that $f_i^{n_i} s = 0$. But then choosing the g_i as in (3) we get that

$$s = 1 \cdot s = \sum_I g_i f_i^{n_i} s = 0 \ .$$

This shows that the map from A into the equalizer is injective. For surjectivity assume that we have elements

$$s_i = a_i/f_i^{n_i} \in A[1/f_i]$$

such that the images of $a_i/f_i^{n_i}$ and $a_j/f_j^{n_j}$ agree in $A[1/(f_i f_j)]$, that is

$$a_i f_j^{n_j} f_i^m f_j^m = a_j f_i^{n_i} f_i^m f_j^m$$

in A, for some $m = m(i,j)$. In particular in $A[1/f_j]$ we have

$$(a_i f_i^m) f_j^{n_j} = a_j f_i^{n_i+m}$$

Since I is finite, choose a common $M \geq m(i,j)$ for all pairs i,j. Replacing every a_i by $a_i f_i^M$ and every n_i by $n_i + M$ leaves s_i unchanged. The preceding equations then show that we can assume, simultaneously for all i,j, that in $A[1/f_j]$ we have

$$a_i f_j^{n_j} = a_j f_i^{n_i} \tag{4}$$

Now we again choose g_i as above and set

$$s := \sum g_i a_i \ .$$

Now in $A[1/f_j]$ we have that

$$f_j^{n_j} s = \sum_i g_i f_j^{n_j} a_i = \sum g_i f_i^{n_i} a_j = a_j,$$

thus $s = s_j$ in $A[1/f_j]$ which finishes the proof modulo the next Lemma. □

LEMMA 6.12. *A presheaf $F \in \mathrm{PSh}_{\mathcal{B}}(X; \mathrm{Ring})$ of rings on principal opens in $\mathrm{Spec}(A)$ that satisfies the sheaf condition for all finite open covers $\{U_i\}_{i\in I}$ of a principal open U is already a sheaf.*

PROOF. We want to show that F is separated first, that is for an arbitrary cover $\{U_i\}_{i\in I}$ of U (not necessarily finite) the map

$$F(U) \to \prod_I F(U_i) \tag{5}$$

is injective. To see this we observe that U is quasi-compact, thus we have a finite subcover $\{U_i\}_{i\in I_0}$ that covers U. But then the composition

$$F(U) \to \prod_I F(U_i) \to \prod_{I_0} F(U_i)$$

is injective, thus also the map (5).
Now we want to show that for a given element $(s_i)_{i\in I}$ in the equalizer

$$\textstyle\prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \cap U_j)$$

we find an s. Again we pick a finite subcover and observe that our element $(s_i)_{i\in I}$ gives rise to an element $(s_i)_{i\in I_0}$. By the sheaf condition for finite covers we conclude that there is some $s \in F(U)$ such that $s|_{U_i} = s_i$ for all $i \in I_0$. We want to show that we have this for all $i \in I$. Thus for a fixed i we observe that the restriction of s_i to $U_i \cap U_{i_0} \subseteq U_i$ agrees with the restriction of $s|_{U_i}$, since we have

$$(s|_{U_i})|_{U_i\cap U_{i_0}} = (s|_{U_{i_0}})|_{U_i\cap U_{i_0}} = (s_{i_0})|_{U_i\cap U_{i_0}} = (s_i)|_{U_i\cap U_{i_0}} \ .$$

This holds for all i_0. But the sets $\{U_i \cap U_{i_0}\}_{i_0\in I_0}$ for varying i_0 cover U_i, so that it follows from the separatedness that the elements s_i and $s|_{U_i}$ actually agree. (Note that here we are using that the U_i themselves are quasi-compact, since they are principal opens and thus themselves of the form $\mathrm{Spec}(A')$.) □

EXAMPLE 6.13. Consider $\mathrm{Spec}(\mathbb{Z})$. For $n \neq 0$, the open $D(n)$ contains the generic point (0) and the closed points (p) for primes p not dividing n. Every nonempty open is of this form, while $D(0) = \emptyset$. Its ring of functions is

$$\mathcal{O}_{\mathrm{Spec}\,\mathbb{Z}}(D(n)) = \mathbb{Z}[1/n].$$

For nonzero m, n, the inclusion $D(n) \subseteq D(m)$ holds exactly when every prime divisor of m divides n; powers do not matter. The restriction is the natural map $\mathbb{Z}[1/m] \to \mathbb{Z}[1/n]$.

EXAMPLE 6.14. Let k be algebraically closed and let $X = \mathrm{Spec}(B)$ with $B = k[x,y]/(xy)$ as in Example 5.23. On the principal open $D(x)$, the element x is invertible. The equation $xy = 0$ therefore forces $y = 0$, so

$$B[1/x] \cong k[x, x^{-1}], \qquad \mathcal{O}_X(D(x)) \cong k[x, x^{-1}].$$

Thus $D(x)$ is the x-axis with its origin removed. Similarly,

$$\mathcal{O}_X(D(y)) \cong k[y, y^{-1}].$$

Their intersection is $D(xy) = D(0) = \emptyset$. These two opens cover $X \setminus \{o\}$, not X. The sheaf condition gives

$$\mathcal{O}_X(X \setminus \{o\}) \cong k[x, x^{-1}] \times k[y, y^{-1}].$$

For comparison, a global section on X is represented uniquely by

$$c + xp(x) + yq(y), \qquad c \in k, \quad p \in k[x], \quad q \in k[y].$$

It gives a polynomial on each axis, and the two polynomials must have the same value c at the origin. The map $B \to B[1/x]$ is restriction to $D(x)$; it sends y to zero. This is a concrete example of a restriction map of the structure sheaf which is not injective.

7. Locally ringed spaces

7.1. Stalks and local rings. From now on we shall work with the categories $\mathcal{C} = \mathrm{Set}, \mathrm{Ring}$, or Ab.

DEFINITION 7.1. *Let $F \in \mathrm{PSh}(X;\mathcal{C})$ be a presheaf and $x \in X$ a point. We define the* stalk *of F at x to be the colimit*

$$F_x := \varinjlim_{U \in \mathrm{Open}(X), x \in U} F(U) \ .$$

Here the neighbourhoods are ordered by reverse inclusion, so the maps in the diagram are restriction maps. For a given element $s \in F(U)$ with $x \in U$ we say that the image of s in F_x is the germ of s.

Note that this is a filtered colimit since the diagram is filtered: for any pair of open sets U, V with $x \in U, V$ we can take the intersection $U \cap V$ which still contains x and is contained in both of them.
Concretely an element of F_x is given by an equivalence class of pairs (U, f) where U is open containing x and $f \in F(U)$. Two such pairs (U, f) and (V, g) are declared equivalent if there exists an open $W \subseteq U \cap V$ containing x such that $f|W = g|W$. So informally we think of elements as germs of 'functions around x'.

EXAMPLE 7.2. For $F = C^0(-, \mathbb{R})$ we have that we can really think of the stalk F_x as continuous functions defined on a very small neighborhood of x (and identified if they agree even closer to x).

REMARK 7.3. For the case of a basis of the topology $\mathcal{B}$ we have that we can also write the stalk as

$$F_x := \varinjlim_{U \in \mathcal{B}, x \in U} F(U) \ .$$

This follows simply since each open neighborhood of x contains a smaller one in $\mathcal{B}$.

EXAMPLE 7.4. Let us consider the case of the structure sheaf $\mathcal{O}$ on $\mathrm{Spec}(A)$. Then the stalk at $x \in \mathrm{Spec}(A)$ is given by

$$\mathcal{O}_x = \mathrm{colim}_{x \in D_f} A[1/f] \ .$$

Here we have that $x \in D_f$ translates into $f \notin x$. The structure maps of the colimit are the maps $A[1/f] \to A[1/g]$ for $D_g \subseteq D_f$ (i.e. $\sqrt{(g)} \subseteq \sqrt{(f)}$). We claim that this ring is the localization

$$A[1/f \mid f \notin x] =: A_x,$$

i.e. the universal ring obtained from A by inverting all elements in $A \setminus x$. Note that we can form the ring A_x for every prime ideal $x \subseteq A$ and it is called the *localization of A at x*.
This abstract ring has a universal property, namely that the elements in $A \setminus x$ are mapped to units in this ring and for any other ring B we have that the induced map

$$\mathrm{Hom}_{\mathrm{Ring}}(A_x, B) \to \mathrm{Hom}_{\mathrm{Ring}}(A, B)$$

obtained by composition with $A \to A_x$ is injective with image those morphisms $A \to B$ that send $A \setminus x$ to units. Thus in order to prove the isomorphism

$$\mathcal{O}_{\mathrm{Spec}(A),x} \cong A_x$$

we have to verify this universal property for $\mathrm{colim}_{x \in D_f} A[1/f]$: first of all, every element $f \in A$ that does not lie in x gets mapped to a unit in this colimit since it gets mapped to a unit in a finite stage. By definition we have that a morphism from this colimit to B is given by compatible maps $A[1/f] \to B$ for all D_f. Thus a morphism from this colimit to B consists exactly of a single morphism $A \to B$ such that all $f \notin x$ get mapped to units.

DEFINITION 7.5. *A ring A is called a* local ring *if it has a unique maximal ideal $\mathfrak{m} \subseteq A$.*

EXAMPLE 7.6. Fields are local rings with the zero ideal being maximal. The integers $\mathbb{Z}$ are not a local ring, since there are many different maximal ideals. Similarly, if k is a field and $n \geq 1$, the rings $k[x_1, ..., x_n]$ are not local.

LEMMA 7.7. *For a ring A the following are equivalent:*

(1) *A is local*
(2) *The non-units $A \setminus A^\times$ form an ideal*
(3) *There exists a proper ideal $\mathfrak{a}$ such that $A \setminus \mathfrak{a}$ consists of units*
(4) $\mathrm{Spec}(A)$ *has a unique closed point.*

PROOF. (1) $\Rightarrow$ (2): assume A is local. We claim that $\mathfrak{m} = A \setminus A^\times$. The inclusion $\subseteq$ is clear, since no maximal ideal can contain a unit. Conversely assume that $a \in A$ is not a unit. Then $(a) \neq A$ and (a) is contained in a maximal ideal, hence in $\mathfrak{m}$.
(2) $\Rightarrow$ (3): Take $\mathfrak{a} = A \setminus A^\times$.
(3) $\Rightarrow$ (1): We have to show that any ideal $I \subsetneq A$ is contained in $\mathfrak{a}$. Given an $i \in I$, the element i is not a unit. Thus i is not contained in $A \setminus \mathfrak{a}$, hence lies in $\mathfrak{a}$. Consequently $\mathfrak{a}$ is the unique maximal ideal.
The equivalence of (1) and (4) is immediate, since the closed points of $\mathrm{Spec}(A)$ correspond to the maximal ideals. □

EXAMPLE 7.8. Consider for a topological space X with the sheaf $C^0_X \in \mathrm{Shv}(X; \mathrm{Ring})$ of continuous $\mathbb{R}$-valued functions on X the stalk $C^0_{X,x}$. This is a local ring. An

element $[f : U \to \mathbb{R}]$ is a unit precisely if $f(x) \neq 0$ since we can then find a neighborhood on which f is not zero and thus invert it. The complement consists of all f that vanish at x and is clearly an ideal.

Recall that for a prime ideal $x \subseteq A$ in a ring we define the localization of A at x denoted A_x by inverting all elements in $A \setminus x$.

PROPOSITION 7.9. *For every prime ideal $x \in \operatorname{Spec}(A)$ the stalk $\mathcal{O}_{\operatorname{Spec}(A),x} = A_x$ is a local ring. The map*

$$\operatorname{Spec}(A_x) \to \operatorname{Spec}(A)$$

exhibits $\operatorname{Spec}(A_x)$ *as a subspace given by the intersection of all opens $U \subseteq \operatorname{Spec}(A)$ that contain x.*

PROOF. Generally, the prime ideals in a ring $A[S^{-1}]$ for some set $S \subseteq A$ are given by the prime ideals of A disjoint from S, which follows as in Proposition 5.14. We conclude that in our case the prime ideals of A_x are in 1-1 correspondence to prime ideals of A that are contained in x. It therefore immediately follows that there is a unique maximal ideal (namely xA_x) and that A_x is local. It also follows that

$$\operatorname{Spec}(A_x) = \{y \in \operatorname{Spec}(A) \mid y \subseteq x\}$$

is a subset and even a subspace of $\operatorname{Spec}(A)$ similarly to the proof of Proposition 5.14. Again this is a general statement about inverting a subset $S \subseteq A$.

Since principal opens form a basis, we have

$$\bigcap_{\substack{U \text{ open} \\ x \in U}} U = \bigcap_{f \notin x} D(f) = \{y \in \operatorname{Spec}(A) \mid y \subseteq x\},$$

which identifies the desired subset. □

Note that in particular the rings $\mathbb{Z}_{(p)}$ are local. These can canonically be considered as subsets of $\mathbb{Q}$. The maximal ideal is then given by (p).

EXAMPLE 7.10. Let k be algebraically closed and consider again $X = \operatorname{Spec}(B)$, where $B = k[x, y]/(xy)$. At the generic point $\eta_x = (y)$ of the x-axis, the stalk is

$$\mathcal{O}_{X,\eta_x} = B_{(y)} \cong k(x).$$

Indeed, localization inverts x, so $y = 0$, and then it inverts every nonzero polynomial in x. Similarly,

$$\mathcal{O}_{X,\eta_y} \cong k(y).$$

At the origin the stalk is instead

$$\mathcal{O}_{X,o} = B_{(x,y)}.$$

Both x and y remain nonzero in this ring: the quotient maps to $k[x]_{(x)}$ and $k[y]_{(y)}$ respectively show this. Their product is still zero. Thus the local ring at the origin has zero divisors, whereas the local ring at either generic point is a field. The stalk at the origin remembers the two branches meeting there.

7.2. Locally ringed spaces and residue fields.

DEFINITION 7.11. *A* ringed space *is a pair $(X, \mathcal{O}_X)$ of a topological space and $\mathcal{O}_X$ a sheaf of rings. It is called a locally ringed space if all the stalks $\mathcal{O}_{X,x}$ are local rings.*

EXAMPLE 7.12. For any ring A the pair $(\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}(A)})$ is a locally ringed space. We will also simply denote this pair as $\mathrm{Spec}(A)$ and leave the sheaf of rings implicit (so be careful if we mean $\mathrm{Spec}(A)$ as a plain topological space or a locally ringed space).
For any topological space X we have the locally ringed space (X, C^0_X).

EXAMPLE 7.13. The pair (X, Abb), where Abb denotes the sheaf $\mathrm{Hom}_{\mathrm{Set}}(-, \mathbb{R})$, is not a locally ringed space unless X carries the discrete topology (in which case it agrees with (X, C^0)). To see this assume that X does not carry the discrete topology and pick a point $x \in X$ for which every open U containing x has at least one other point. Then we consider at x the germs of the two functions

$$f_1(y) = \begin{cases} 0 & \text{for } y = x \\ 1 & \text{else} \end{cases} \qquad f_2(y) = \begin{cases} 1 & \text{for } y = x \\ 0 & \text{else} \end{cases}$$

These two germs $f_1, f_2 \in \mathrm{Abb}_x$ are not units (being a unit means that a function has to be non-zero in a neighborhood of x). But their sum is a unit, since it is constant 1. Therefore the ring cannot be local, by criterion (2) of Lemma 7.7.
Given any ring R which is not local, we can also consider the pair (pt, R) where we consider R as a sheaf on the one point space pt. Concretely we have that this sheaf sends $\emptyset$ to the zero-ring (the terminal object in rings) and the full subset pt to R. Then the stalk of this sheaf is simply R which is not local.

DEFINITION 7.14. *For a locally ringed space $(X, \mathcal{O}_X)$ and $x \in X$ we define the residue field $\kappa(x)$ at x as the quotient*

$$\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$$

where $\mathfrak{m}_x$ is the maximal ideal. For any $f \in \mathcal{O}(U)$ and $x \in U$ we define $f(x)$ to be the class of $[U, f]$ in $\kappa(x)$.

EXAMPLE 7.15. For the locally ringed space (X, C^0_X) all the residue fields are $\mathbb{R}$ and for $f : U \to \mathbb{R}$ and $x \in U$ this simply evaluates at x.
For $\mathrm{Spec}(A)$ we have that

$$\kappa(x) = A_x/\mathfrak{m}_x = A[(A \setminus x)^{-1}]/\mathfrak{m}_x = (A/x)[(A \setminus x)^{-1}] = \mathrm{Quot}(A/x).$$

This was the definition of $\kappa(x)$ as given before (Definition 5.1). For $f \in A = \mathcal{O}_{\mathrm{Spec}(A)}(\mathrm{Spec}(A))$ we also find that under this identification the two definitions of $f(x)$ agree.

7.3. Morphisms of locally ringed spaces. Before we can talk about morphisms now we need the following construction:

CONSTRUCTION 7.16. Let $f : X \to Y$ be a continuous map of topological spaces and $F \in \mathrm{PSh}(X; \mathcal{C})$. We construct a presheaf $f_*(F) \in \mathrm{PSh}(Y; \mathcal{C})$ by setting

$$f_*(F)(U) = F(f^{-1}(U)) \ .$$

This is a presheaf, i.e. functorial in U. This is clear, but a more formal way of saying this is to observe that f induces a functor $f^{-1} : \mathrm{Open}(Y) \to \mathrm{Open}(X)$ and

we have simply composed with this functor:

$$f_*(F) : \mathrm{Open}(Y)^{\mathrm{op}} \xrightarrow{f^{-1}} \mathrm{Open}(X)^{\mathrm{op}} \xrightarrow{F} \mathcal{C} \ .$$

This assignment defines a functor

$$f_* : \mathrm{PSh}(X;\mathcal{C}) \to \mathrm{PSh}(Y;\mathcal{C}) \ .$$

LEMMA 7.17. *Assume that $F \in \mathrm{PSh}(X;\mathcal{C})$ is a sheaf. Then so is f_*F.*

PROOF. Assume that we have an open covering U_i of U in Y. Then the pullback $f^{-1}U_i$ is an open covering of $f^{-1}U$ and we have that $f^{-1}U_i \cap f^{-1}U_j = f^{-1}(U_i \cap U_j)$. Therefore the sheaf condition for f_*F is simply given by the diagram

$$F(f^{-1}U) \longrightarrow \prod_{i\in I} F(f^{-1}U_i) \rightrightarrows \prod_{i,j\in I} F(f^{-1}U_i \cap f^{-1}U_j)$$

which is satisfied since F is a sheaf. □

DEFINITION 7.18. *A morphism of ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is given by a pair $(f, f^\sharp)$ where $f : X \to Y$ is a continuous map and $f^\sharp : \mathcal{O}_Y \to f_*(\mathcal{O}_X)$ is a morphism of sheaves of rings over Y.*

EXAMPLE 7.19. Let $f : X \to Y$ be a continuous map. Then we get an induced morphism

$$f^\sharp : C^0_Y \to f_*(C^0_X)$$

which on an open set $U \subseteq Y$ is given by the map

$$C^0(U;\mathbb{R}) \ \to C^0(f^{-1}U;\mathbb{R})$$

which composes a map $\varphi : U \to \mathbb{R}$ with the map $f : f^{-1}U \to U$ to get a map $f^\sharp(\varphi) : f^{-1}U \to \mathbb{R}$. This is a morphism of sheaves as one directly sees. Thus the morphisms $f^\sharp$ should be seen as the pullback of functions.

EXAMPLE 7.20. Let $f : X \to Y$ be a smooth map between smooth manifolds. Then we get an induced morphism

$$f^\sharp : C^\infty_Y \to f_*(C^\infty_X)$$

which on an open set $U \subseteq Y$ is given by the composition map

$$C^\infty(U;\mathbb{R}) \ \to C^\infty(f^{-1}(U);\mathbb{R})$$

as before. In fact, one can show that for a continuous map $f : X \to Y$ the induced morphism

$$f^\sharp : C^0_Y \to f_*(C^0_X)$$

restricts to a morphism $C^\infty_Y \to f_*(C^\infty_X)$ precisely if f is smooth. In other words: a continuous map $f : X \to Y$ between smooth manifolds X and Y is smooth precisely if for each smooth function $\varphi : U \to \mathbb{R}$ with $U \subseteq Y$ the composition $\varphi \circ f : f^{-1}(U) \to U \to \mathbb{R}$ is smooth.

EXAMPLE 7.21. Let $f : A \to B$ be a morphism of rings. Then we claim that we get an induced morphism of ringed spaces

$$\mathrm{Spec}(f) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

which on underlying topological spaces is given by the morphism $\mathrm{Spec}(f)$ defined before (see Proposition 5.7) and which simply takes preimages of prime ideals. Now we want to extend this to a morphism of ringed spaces, i.e. we have to define

$$\mathrm{Spec}(f)^\sharp : \mathcal{O}_{\mathrm{Spec}(A)} \to \mathrm{Spec}(f)_*\mathcal{O}_{\mathrm{Spec}(B)} \ .$$

By the equivalence of Lemma 6.10 it suffices to define such a morphism on principal opens, i.e. for each principal open $D_g \subseteq \mathrm{Spec}(A)$ with $g \in A$ a morphism

$$\mathcal{O}_{\mathrm{Spec}(A)}(D_g) \to \mathcal{O}_{\mathrm{Spec}(B)}(\mathrm{Spec}(f)^{-1}(D_g))$$

natural in the open set. Note that $\mathrm{Spec}(f)^{-1}(D_g) = D_{f(g)}$, as we proved when proving continuity of $\mathrm{Spec}(f)$. So we need to give a map

$$A[1/g] \to B[1/f(g)]$$

which is simply the map induced by f. Naturality is then straightforward.

Now we would like to define morphisms of locally ringed spaces. To this end we first have to talk about morphisms of local rings. For every ring morphism $f : A \to B$ between local rings we clearly have that $f^{-1}(\mathfrak{m}_B)$ is an ideal in A. Since $f^{-1}(\mathfrak{m}_B) \neq A$ we have that $f^{-1}(\mathfrak{m}_B) \subseteq \mathfrak{m}_A$. In general there is no equality as the example of the map

$$\mathbb{Z}_{(p)} \to \mathbb{Q}$$

shows. The preimage of 0 is given by 0 which is not the maximal ideal of $\mathbb{Z}_{(p)}$.

Definition 7.22. *A morphism of local rings is a morphism $f : A \to B$ such that $f^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.*

Remark 7.23. Note that the definition is chosen in such a way that morphisms of local rings $f : A \to B$ induce maps between residue fields $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$. (It is equivalent to $f(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$.) In fact, this gives us a functor from the category of local rings to the category of fields. Residue fields are not functorial for arbitrary ring homomorphisms between local rings.

Lemma 7.24. *A ring homomorphism between local rings is a morphism of local rings iff the induced map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of topological spaces maps the closed point to the closed point.*

Proof. Clear. □

Example 7.25. Every isomorphism $A \to B$ between local rings is a morphism of local rings.

Construction 7.26. Now let $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. For every point $x \in X$ there is a canonical morphism of rings

$$\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

induced by $(f, f^\sharp)$. This morphism sends (U, φ) to $(f^{-1}U, f^\sharp\varphi)$. By this requirement the morphism is completely determined and it is in fact easy to verify the claim here directly (one simply needs to check well-definedness, but by assumption $f^\sharp$ is compatible with restriction, so this is clear). But we will say things a bit more systematically in the next section.

Definition 7.27. *A morphism of locally ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of the underlying ringed spaces such that the induced morphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is a morphism of local rings for every $x \in X$.*

Example 7.28. For a continuous map $f : X \to Y$ of topological spaces, the morphism $(f, f^\sharp) : (X, C^0_X) \to (Y, C^0_Y)$ from Example 7.19 induces on stalks the map

$$C^0_{Y,f(x)} \to C^0_{X,x}$$

which sends a germ $\varphi : U \to \mathbb{R}$ to the composition $\varphi \circ f : f^{-1}U \to U \to \mathbb{R}$. If φ is in the maximal ideal (i.e. vanishes at $f(x)$) then this composition also lies in the maximal ideal. Therefore this is a map of locally ringed spaces.
Similarly we claim that $\mathrm{Spec}(f) : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ for $f : A \to B$ is a morphism of locally ringed spaces. Under the appropriate definitions the induced morphism on stalks is simply given by the map

$$f : A_{f^{-1}(x)} \to B_x$$

for $x \in \mathrm{Spec}(B)$. Clearly, this takes the maximal ideal $f^{-1}(x)A_{f^{-1}(x)}$ into the maximal ideal xB_x.

REMARK 7.29. If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a map of locally ringed spaces, the map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ induces a map $\kappa(f(x)) \to \kappa(x)$ of fields. If we think of $f^\sharp$ as precomposing a function $g \in \mathcal{O}_Y(U)$ with f, then $\kappa(f(x)) \to \kappa(x)$ tells us how the value $g(f(x)) \in \kappa(f(x))$ relates to the value of the restriction $f^\sharp(g)(x) \in \kappa(x)$.
For a map $f : X \to Y$ of topological spaces considered as a map of locally ringed spaces $(X, C^0_X) \to (Y, C^0_Y)$ the maps $\kappa(f(x)) \to \kappa(x)$ are all the identity $\mathbb{R} \to \mathbb{R}$, so in that case this simply expresses that $(g \circ f)(x) = g(f(x))$.

THEOREM 7.30. *The functor* $\mathrm{Spec}(-) : \mathrm{Ring}^{\mathrm{op}} \to \{$*locally ringed spaces*$\}$ *is fully faithful.*

We will prove this later, in fact deduce it from a more general statement.

8. Sheafification and adjunctions

8.1. Sheafification.

LEMMA 8.1. *A morphism of sheaves $F \to G$ with values in rings/sets/abelian groups on a space X is an isomorphism precisely if it is an isomorphism on stalks.*

PROOF. The only-if statement is clear. Thus assume that $F \to G$ induces an isomorphism on stalks.
Observe that the map $F(U) \to G(U)$ is injective for each U: If $f, g \in F(U)$ have the same image in $G(U)$, they in particular have the same image in the stalk G_x for each $x \in U$. Since the map on stalks $F_x \to G_x$ is injective, this shows that f, g have the same image in F_x for each $x \in U$. By the explicit description of stalks, this shows that for each $x \in U$, there is a neighbourhood $U_x \subseteq U$ such that f, g agree already in $F(U_x)$. The U_x cover U, so f, g agree in $F(U)$.
For surjectivity, pick some $f \in G(U)$. For each $x \in U$, we find some element of F_x which maps to the same element as f in G_x. Representing this by some $f_x \in F(U_x)$ for a neighbourhood, we see that the image of f_x in $G(U_x)$ has the same germ at x as f. Thus, we can make U_x smaller to assume that the image of f_x in $G(U_x)$ agrees with $f|_{U_x}$.
What we have achieved is that we have found an open cover U_x of U and preimages $f_x \in F(U_x)$ of $f|_{U_x} \in G(U_x)$. In particular, the images of f_x and f_y in $G(U_x \cap U_y)$ agree. By injectivity this shows that $f_x|_{U_x \cap U_y} = f_y|_{U_x \cap U_y}$. It follows that the f_x glue to an element of $F(U)$, whose image in $G(U)$ agrees with f since it does on the cover U_x. □

REMARK 8.2. Note that the injectivity portion of the proof only required injectivity on stalks, i.e. if $F_x \to G_x$ is injective for all $x \in U$, $F(U) \to G(U)$ is, too.

The corresponding statement for surjectivity does not hold! As an example, consider $X = \mathbb{C} \setminus \{0\}$, with the sheaf $C^0(-; \mathbb{C}^\times)$ of nowhere-zero continuous complex-valued functions, regarded as a sheaf of abelian groups under multiplication. We have the squaring map

$$C^0(U; \mathbb{C}^\times) \to C^0(U; \mathbb{C}^\times), \quad f \mapsto f^2,$$

which is surjective on stalks: near any point, a nowhere-zero continuous function takes values in a small disc avoiding 0, on which a continuous square root exists. But the map is not surjective on X. Otherwise, the identity function $z \mapsto z$ would have a continuous square root on $\mathbb{C} \setminus \{0\}$, which is impossible by restricting to the unit circle.

So isomorphisms of sheaves are detected on stalks. As we will now see, there is also a universal way of turning a presheaf into a sheaf without changing the stalks.

THEOREM 8.3. *For every presheaf F of rings/sets/abelian groups there exists a universal sheaf F' with a morphism $F \to F'$, i.e. such that every other morphism $F \to G$ with G a sheaf factors uniquely through F'. The morphism $F \to F'$ induces an isomorphism on stalks, and F' is characterized by this property.*

PROOF. We set

$$F'(U) \subseteq \{(f_x \in F_x)_{x \in U}\}$$

where an element lies in $F'(U)$ if each x admits an open neighbourhood $U_x \subseteq U$ and $g \in F(U_x)$ such that for all $y \in U_x$ the germ of g agrees with f_y. So $F'(U)$ consists of "functions on U with values in the stalks" with the property that they locally come from actual sections of F.

This is functorial in the obvious way. For rings or abelian groups, the pointwise operations preserve local representability, so $F'(U)$ has the same algebraic structure. We get a presheaf map $F \to F'$ which takes an actual element $f \in F(U)$ to the collection of its germs. On stalks, this map is surjective, since by definition every element of $F'(U)$ with $x \in U$ lifts to $F(V)$ on some smaller neighbourhood V of x. It is also injective, since if two $f, g \in F_x$ have the same image in F'_x, their representatives coincide in $F'(V)$ for some neighbourhood V of x, so in particular in F_x.

We now directly check that F' is indeed a sheaf: A section on U consists of compatible sections on all U_i for some cover, since both amount to giving an element $f_x \in F_x$ for each $x \in U$, with the property that this assignment locally comes from actual sections in F. But since this last condition is local anyway, it can be checked on the cover.

We now check the universal property. Given another map $F \to G$ with G a sheaf, we get a diagram

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & \nearrow & \downarrow \cong \\ F' & \longrightarrow & G' \end{array}$$

The lower horizontal map sends each tuple of germs to its image in the stalks of G; local representability is preserved. The right vertical map is an isomorphism of sheaves by the preceding lemma. Thus there exists a dashed morphism of sheaves making the upper left triangle commute. It is unique: since $F \to F'$ is an isomorphism on stalks, any such morphism has the same maps on stalks, and morphisms of sheaves agreeing on stalks are equal.

The final claim is that F' is also characterized simply by being a sheaf with a map $F \to F'$ which is an isomorphism on stalks. Given another morphism $F \to G$ which is an isomorphism on stalks, with G a sheaf, by the universal property we get a morphism $F' \to G$ of sheaves which also is an isomorphism on stalks. So it is an isomorphism, again by the preceding lemma.

□

REMARK 8.4. The theorem extends, with suitable hypotheses and a different proof, to many other categories $\mathcal{C}$. Here we will only need sets, rings, and abelian groups.

REMARK 8.5. The assignment $F \mapsto F'$ defines a functor $\mathrm{PSh}(X;\mathcal{C}) \to \mathrm{Shv}(X;\mathcal{C})$ by the universal property. This is the *sheafification functor* a, we will therefore write F' as aF usually.

8.2. Adjunctions and the Yoneda lemma.

DEFINITION 8.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. An* adjunction *between a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ consists of a bijection*

$$\mathrm{Hom}_{\mathcal{D}}(Fx, y) \cong \mathrm{Hom}_{\mathcal{C}}(x, Gy)$$

natural in x and y (i.e. we think of both sides as functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathrm{Set}$.)
In this situation we call F the left adjoint and G the right adjoint, and say things like "G is right adjoint to F". We write $F \dashv G$ to express the relationship.

EXAMPLE 8.7. Sheafification is left adjoint to the forgetful functor $i : \mathrm{Shv}(X;\mathcal{C}) \to \mathrm{PSh}(X;\mathcal{C})$, i.e. $a \dashv i$, so for each pair $F \in \mathrm{PSh}(X;\mathcal{C})$, $G \in \mathrm{Shv}(X;\mathcal{C})$ we have

$$\mathrm{Hom}_{\mathrm{PSh}(X;\mathcal{C})}(F, iG) \cong \mathrm{Hom}_{\mathrm{Shv}(X;\mathcal{C})}(aF, G).$$

This is precisely the universal property of sheafification.

EXAMPLE 8.8. Sobrification is left adjoint to the forgetful functor from sober spaces to spaces.

EXAMPLE 8.9. The functor taking a local ring to its residue field is left adjoint to the forgetful functor from fields to the category of local rings. This is because we have

$$\mathrm{Hom}_{\mathrm{locRing}}(R, k) \cong \mathrm{Hom}_{\mathrm{field}}(R/\mathfrak{m}, k)$$

EXAMPLE 8.10. The forgetful functor $\mathrm{Ring} \to \mathrm{Set}$ has a left adjoint. It is given by the functor which takes $X \in \mathrm{Set}$ to the polynomial ring $\mathbb{Z}[X] = \mathbb{Z}[x_i \mid i \in X]$. This is the universal property of polynomial rings. (Similarly, the functor taking a set to the free group, free abelian group, free K-vector space,... can be described as a left adjoint to a forgetful functor.)

We now recall the Yoneda Lemma. This is about the category of functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ (sometimes also referred to as presheaves on $\mathcal{C}$ since it is the obvious generalization). Every object $c \in \mathcal{C}$ gives rise to such a functor denoted $\underline{c}$ as follows:

$$\underline{c} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set} \qquad x \mapsto \mathrm{Hom}_{\mathcal{C}}(x, c)\ .$$

PROPOSITION 8.11 (Yoneda Lemma). *For $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ and $c \in \mathcal{C}$ there is a (natural) isomorphism*

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathrm{Set})}(\underline{c}, F) \cong F(c)\ .$$

PROOF. We explicitly give a map $\phi : \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathrm{Set})}(\underline{c}, F) \to F(c)$: for a natural transformation $\eta : \underline{c} \to F$ we let $\phi(\eta)$ be the value

$$\mathrm{id}_c \in \underline{c}(c) \xrightarrow{\eta} F(c) \ .$$

Conversely, for a given $\xi \in F(c)$ we consider the natural transformation $\eta_\xi : \underline{c} \to F$ which sends $f : d \to c$ to $F(f)(\xi) \in F(d)$, using the map

$$F(c) \xrightarrow{F(f)} F(d) \ .$$

Functoriality of F gives naturality of η_ξ, and $F(\mathrm{id}_c)(\xi) = \xi$. Conversely, naturality of η applied to $f : d \to c$ gives $\eta_d(f) = F(f)(\eta_c(\mathrm{id}_c))$. Thus these two maps are inverse to each other. □

COROLLARY 8.12. *The Yoneda embedding*

$$\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set}) \qquad c \mapsto \underline{c}$$

is fully faithful. In particular if for two objects $c, d \in \mathcal{C}$ the functors $\underline{c}, \underline{d}$ are naturally equivalent, then c and d are isomorphic.

We say that a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ is representable if it is in the essential image of the Yoneda embedding, that is if it is isomorphic to $\underline{c}$ for some c. This object is then determined up to isomorphism; with the representing isomorphism fixed, it is determined up to unique isomorphism.
Of course we also have the dual version, obtained by applying this to $\mathcal{C}^{\mathrm{op}}$: the functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}, \mathrm{Set})$ sending $c \in \mathcal{C}$ to the functor $d \mapsto \mathrm{Hom}_{\mathcal{C}}(c, d)$ is fully faithful. Thus mapping out of an object also characterises it up to isomorphism.

LEMMA 8.13. *If a functor has a right adjoint, it is unique up to canonical natural isomorphism, and dually for left adjoints.*

PROOF. Assume G and G' are two right adjoints to $F : \mathcal{C} \to \mathcal{D}$. We get a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(x, Gy) \cong \mathrm{Hom}_{\mathcal{D}}(Fx, y) \cong \mathrm{Hom}_{\mathcal{C}}(x, G'y),$$

which, by the Yoneda lemma, gives rise to a natural isomorphism $Gy \to G'y$ (we get the map by looking at what happens to $\mathrm{id}_{Gy} \in \mathrm{Hom}_{\mathcal{C}}(Gy, Gy)$ on the left, checking that it gives an isomorphism is a little diagram chase). □

LEMMA 8.14. *Right adjoint functors preserve limits. Left adjoint functors preserve colimits.*

PROOF. We again show one direction. Assume G is right adjoint to $F : \mathcal{C} \to \mathcal{D}$. Let $y \to (y_i)_{i \in I}$ be some limit cone over a diagram $I \to \mathcal{D}$. We want to show that $Gy \to (Gy_i)_{i \in I}$ is a limit cone. This means we have to show that given any cone $x \to (Gy_i)_{i \in I}$, it factors uniquely through Gy. By adjunction, the maps in the cone give a cone $Fx \to (y_i)_{i \in I}$ in $\mathcal{D}$. By assumption, this factors uniquely through a map $Fx \to y$. By adjunction, the corresponding map $x \to Gy$ is the unique factorization of the given cone. □

For example, this explains why limits of rings are formed "underlying", i.e. the limit in the category of rings is a ring whose underlying set is just the limit of the sets. This is because the forgetful functor Ring $\to$ Set has a left adjoint. The analogous statement for colimits is of course not true (and the forgetful functor does not have a right adjoint).

Limits of sheaves are computed sectionwise: the resulting presheaf satisfies the sheaf condition because limits commute with limits. Thus these limits exist, and the right adjoint $i : \mathrm{Shv} \to \mathrm{PSh}$ preserves them. The analogous statement for colimits is typically not true; one has to sheafify the sectionwise colimit.

LEMMA 8.15. *Given functors $F_1 : \mathcal{C}_1 \to \mathcal{C}_2$ and $F_2 : \mathcal{C}_2 \to \mathcal{C}_3$ which both admit right adjoints G_1 and G_2, $(G_1 \circ G_2)$ is right adjoint to $(F_2 \circ F_1)$. (Analogously for left adjoints).*

PROOF. We have natural isomorphisms

$$\mathrm{Hom}_{\mathcal{C}_3}(F_2F_1(x), y) \cong \mathrm{Hom}_{\mathcal{C}_2}(F_1(x), G_2(y)) \cong \mathrm{Hom}_{\mathcal{C}_1}(x, G_1G_2(y))$$

which proves the claim. □

8.3. Inverse images of sheaves. We will now turn back to the study of sheaves on spaces. Recall that we described for every continuous map $f : X \to Y$ the direct image functor $f_* : \mathrm{PSh}(X; \mathcal{C}) \to \mathrm{PSh}(Y; \mathcal{C})$ which took sheaves to sheaves, i.e. we also have $f_* : \mathrm{Shv}(X; \mathcal{C}) \to \mathrm{Shv}(Y; \mathcal{C})$.

LEMMA 8.16. *The functor $f_* : \mathrm{PSh}(X; \mathcal{C}) \to \mathrm{PSh}(Y; \mathcal{C})$ has a left adjoint f^+, which takes F to $F^+ = f^+(F)$ with*

$$f^+(F)(U) = \mathrm{colim}_{V \supseteq f(U)} F(V).$$

Here V ranges over open subsets of Y, ordered by reverse inclusion.

PROOF. For each F, the construction $F^+ = f^+(F)$ defines a presheaf, with restriction map coming from the observation that if $U' \subseteq U$, then every $V \supseteq f(U)$ also satisfies $V \supseteq f(U')$, so we can map the colimits to each other. It is also clearly functorial in F. To see that it is left adjoint, we need to check that a map of presheaves $F^+ \to G$ is the same as a map $F \to f_*G$. Unwrapping the definitions, a map $F^+ \to G$ consists of maps

$$\mathrm{colim}_{V \supseteq f(U)} F(V) \to G(U),$$

i.e. of compatible maps $F(V) \to G(U)$ for each pair of $U \subseteq X$, $V \subseteq Y$ with $f(U) \subseteq V$, or $U \subseteq f^{-1}(V)$. By the same logic, such a collection of compatible maps also is the same as a map

$$F(V) \to \lim_{U \subseteq f^{-1}(V)} G(U) = G(f^{-1}V) = f_*G(V),$$

as desired. □

REMARK 8.17. Note that the colimit in the definition of $F^+(U)$ is filtered so that even in rings it is formed on underlying sets.

This construction $F \mapsto f^+(F)$ does *not* preserve sheaves. Instead, we can get a similar adjoint also on the level of sheaves, by postcomposing with sheafification. This is the functor we are really after:

THEOREM 8.18. *The direct image functor $f_* : \mathrm{Shv}(X; \mathcal{C}) \to \mathrm{Shv}(Y; \mathcal{C})$ admits a left adjoint f^{-1}. It can be explicitly described as taking F to the sheafification of*

$$U \mapsto \mathrm{colim}_{V \supseteq f(U)} F(V).$$

PROOF. We first observe that $f_* \circ i : \mathrm{Shv}(X;\mathcal{C}) \to \mathrm{PSh}(Y;\mathcal{C})$ has a left adjoint $f^{-1} : \mathrm{PSh}(Y;\mathcal{C}) \to \mathrm{Shv}(X;\mathcal{C})$, given by the functor from Lemma 8.16 followed by sheafification, by Lemma 8.15.
Explicitly, the adjunction gives a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Shv}(X;\mathcal{C})}(f^{-1}(F), G) \cong \mathrm{Hom}_{\mathrm{PSh}(Y;\mathcal{C})}(F, f_*G),$$

where we suppress i. We know that f_*G is also a sheaf, so if we consider the above natural isomorphism only for $F \in \mathrm{Shv}(Y;\mathcal{C})$, we also get a natural isomorphism

$$\mathrm{Hom}_{\mathrm{Shv}(X;\mathcal{C})}(f^{-1}(F), G) \cong \mathrm{Hom}_{\mathrm{Shv}(Y;\mathcal{C})}(F, f_*G),$$

as desired. □

The inverse image functor is a common generalisation of restriction and of stalks.

EXAMPLE 8.19. For an open inclusion $i : U \to X$, the inverse image functor $\mathrm{Shv}(X;\mathcal{C}) \to \mathrm{Shv}(U;\mathcal{C})$ simply restricts a sheaf: For each open $V \subseteq U$, we have

$$i^{-1}F(V) = \mathrm{colim}_{V' \supseteq i(V)} F(V') = F(i(V)),$$

since for open $i(V)$ the diagram has terminal object $i(V)$. A priori we need to sheafify this, but if i is an inclusion, this is already a sheaf. More generally this description works for any open map (not necessarily injective), but then we *do* need to sheafify! In this case we shall also sometimes simply write the functor $i^{-1}F$ as $F|_U$ and think of it simply as restriction of sheaves.

EXAMPLE 8.20. For the inclusion of a point $i_x : \mathrm{pt} \to Y$, $\mathrm{Shv}(\mathrm{pt};\mathcal{C})$ is just $\mathcal{C}$, and the explicit formula for the inverse image functor tells us that $i_x^{-1}F$ is F_x under that identification.

EXAMPLE 8.21. We first note that the constant presheaf F on a space X with value R (a ring) is in general not a sheaf, as we have seen in Example 6.6. But we can form the sheafification aF and we claim that on an open $U \subseteq X$ this is given by locally constant functions $U \to R$, that is functions for which every point of U has a neighborhood on which the function is constant. The latter is clearly a sheaf and the canonical map from the constant presheaf to this sheaf of locally constant functions is a morphism of presheaves which induces an isomorphism on stalks. Thus this is aF. We refer to this sheaf also as the *constant sheaf with value R*.

For the constant map $c : X \to \mathrm{pt}$, $c^{-1}R$ is canonically isomorphic to the sheafification of the constant presheaf

$$U \mapsto R,$$

since the inverse-image presheaf differs from this only at the empty open set, where it has the terminal value. Thus, $c^{-1}R$ is the "sheaf of locally constant functions with values in R". Note that the direct image also has a very important description: $c_*(F)$, regarded as an object in $\mathcal{C}$, is given by the *global sections* $F(X)$. The adjunction thus gives us the important identity

$$\mathrm{Hom}_{\mathrm{Shv}(X;\mathcal{C})}(c^{-1}R, F) \cong \mathrm{Hom}_{\mathcal{C}}(R, F(X)),$$

i.e. maps out of the sheaf of locally constant functions into any sheaf are uniquely determined by what they do on the global constant sections.

LEMMA 8.22. *For a continuous map $f : X \to Y$ and a sheaf F on Y, we have for every $x \in X$ a natural isomorphism $(f^{-1}F)_x \cong F_{f(x)}$.*

PROOF. For the composition pt $\xrightarrow{i_x} X \xrightarrow{f} Y$ we have $f_* \circ (i_x)_* = (f \circ i_x)_* = (i_{f(x)})_*$. By Lemma 8.15 and uniqueness of left adjoints, this gives $(i_x)^{-1} \circ f^{-1} \cong (i_{f(x)})^{-1}$, which by Example 8.20 proves the claim. □

COROLLARY 8.23. *For a locally ringed space $(Y, \mathcal{O}_Y)$ and a continuous map $f : X \to Y$ the pair $(X, f^{-1}\mathcal{O}_Y)$ is also a locally ringed space. In particular for an open subset $U \subseteq Y$ the pair $(U, \mathcal{O}_Y|_U)$ is again a locally ringed space.*

EXAMPLE 8.24. Consider the open subset $D(f) \subseteq \mathrm{Spec}(A)$. Then this becomes a locally ringed space. We claim that as such it is isomorphic to $\mathrm{Spec}(A[1/f])$. We have shown this as topological spaces already in Proposition 5.14. Now we need to also compare the structure sheaves. But note that a principal open in $\mathrm{Spec}(A[1/f])$ is simply a principal open in $\mathrm{Spec}(A)$ of the form $D(g)$ such that $\sqrt{(g)} \subseteq \sqrt{(f)}$. But then

$$\mathcal{O}_{\mathrm{Spec}(A)}(D(g)) = A[1/g] = A[1/f][1/g] = \mathcal{O}_{\mathrm{Spec}(A[1/f])}(D(g)) \ .$$

These identifications are compatible with restriction, so they give the required isomorphism of structure sheaves.

DEFINITION 8.25. *A morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of locally ringed spaces is called an* open immersion *if it induces an isomorphism between X and $(U, \mathcal{O}_Y|_U)$ for some open $U \subseteq Y$.*

Note that a morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ was defined as a pair $f : X \to Y$ and a morphism $f^\sharp : \mathcal{O}_Y \to f_*(\mathcal{O}_X)$. By adjunction we can now also think of $f^\sharp$ as a morphism $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$. We will do that very often and also denote this map as $f^\sharp$ abusively. Then this morphism on stalks induces the map

$$\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$$

that we have been using in the Definition of morphisms of locally ringed spaces (Definition 7.27). Thus we could rephrase this definition as saying such a morphism is a pair $(f, f^\sharp)$ with $f^\sharp : f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ such that $f^\sharp$ is stalkwise a morphism of local rings.

9. Schemes and affinization

Recall that we stated as Theorem 7.30 that the functor $\mathrm{Spec}(-) : \mathrm{Ring}^{\mathrm{op}} \to \{\text{locally ringed spaces}\}$ is fully faithful. In this section we will deduce this from the more general result that we describe now.

DEFINITION 9.1. *The global sections functor Γ is the functor*

$$\Gamma : \{\textit{locally ringed spaces}\} \to \mathrm{Ring}^{\mathrm{op}}$$

which takes a locally ringed space $(X, \mathcal{O}_X)$ to the global sections $\mathcal{O}_X(X)$. It sends a morphism $(f, f^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ to the map $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ induced by $f^\sharp$ on Y.

THEOREM 9.2. *The functor $\Gamma : \{\textit{locally ringed spaces}\} \to \mathrm{Ring}^{\mathrm{op}}$ is left adjoint to the functor $\mathrm{Spec} : \mathrm{Ring}^{\mathrm{op}} \to \{\textit{locally ringed spaces}\}$, that is we have a natural isomorphism*

$$\mathrm{Hom}\Big((X, \mathcal{O}_X), \mathrm{Spec}(A)\Big) \cong \mathrm{Hom}_{\mathrm{Ring}^{\mathrm{op}}}\Big(\mathcal{O}_X(X), A\Big) \cong \mathrm{Hom}_{\mathrm{Ring}}\Big(A, \mathcal{O}_X(X)\Big).$$

Let us first explain how this proves Theorem 7.30. We immediately get an isomorphism

$$\mathrm{Hom}\Big(\mathrm{Spec}(B), \mathrm{Spec}(A)\Big) \cong \mathrm{Hom}_{\mathrm{Ring}}\Big(A, \mathcal{O}_{\mathrm{Spec}(B)}(\mathrm{Spec}(B))\Big) = \mathrm{Hom}_{\mathrm{Ring}}(A, B)$$

but this does not quite show the claim, since we need to show that this bijection is inverse to the map induced by $\mathrm{Spec}(-)$. In order to see this one has to go into the analysis of adjunctions a bit more. Let

$$\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$$

be an adjunction. Then we get natural morphisms $\epsilon : LR(d) \to d$ as the image of the identity under the bijection

$$\mathrm{Hom}_{\mathcal{C}}(R(d), R(d)) \cong \mathrm{Hom}_{\mathcal{D}}(LR(d), d)$$

which is part of the adjunction. The naturality in $d \in \mathcal{D}$ is immediately verified. This transformation is called the *counit* of the adjunction. Dually there is also a *unit* $\eta : c \to RL(c)$ (but we focus on the counit for now). Now assume that we are given a transformation $\epsilon : LR \to \mathrm{id}$ as above, then we can define a natural map

$$\mathrm{Hom}_{\mathcal{C}}(c, Rd) \xrightarrow{L} \mathrm{Hom}_{\mathcal{D}}(Lc, LRd) \xrightarrow{\epsilon} \mathrm{Hom}_{\mathcal{D}}(Lc, d)\ .$$

and if ϵ came from an adjunction we claim that this is indeed the natural isomorphism which is part of the adjunction. This follows from the Yoneda Lemma: for fixed d any morphism $\mathrm{Hom}_{\mathcal{C}}(-, Rd) \to \mathrm{Hom}_{\mathcal{D}}(L-, d)$ has to be induced by an element in $\mathrm{Hom}_{\mathcal{D}}(LRd, d)$, which is the counit and the way the morphism is induced is then the upper composition.

Remark 9.3. One can even revert the logic and give the following equivalent definition of an adjunction: it is given by a natural transformation $\epsilon : LR \to \mathrm{id}$ such that the induced natural transformation

$$\mathrm{Hom}_{\mathcal{C}}(c, Rd) \to \mathrm{Hom}_{\mathcal{D}}(Lc, d)$$

is an isomorphism. Using this definition is sometimes beneficial.

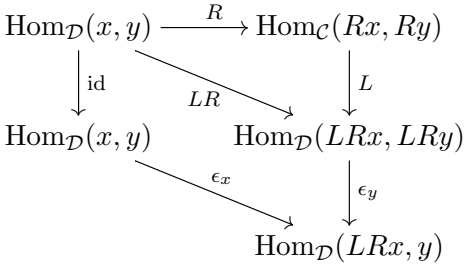
Proposition 9.4. *For an adjunction $L \dashv R$ the functor R is fully faithful precisely if the counit $LR \to \mathrm{id}$ is an isomorphism.*

Proof. We claim that the following triangle commutes

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(x, y) & \xrightarrow{R} & \mathrm{Hom}_{\mathcal{C}}(Rx, Ry) \\ & \searrow^{\epsilon^*} & \downarrow \cong \\ & & \mathrm{Hom}_{\mathcal{D}}(LR(x), y) \end{array}$$

This then immediately implies the claim since then the first map is a bijection precisely if the latter one is, which by Yoneda is equivalent to the assertion that $LR(x) \to x$ is an isomorphism. To see the commutativity we consider the larger

diagram



where the commutativity of the lower square is the naturality of $\epsilon : LR \to \mathrm{id}$. □

Now in order to deduce that $\mathrm{Spec}(-)$ is fully faithful we need to show that the counit of the adjunction is an isomorphism. In fact, we will show in the proof of Theorem 9.2 that the counit of the adjunction is given by the natural isomorphism $\Gamma(\mathrm{Spec}(A)) \cong A$. Before we give the proof we will give the main definition of the course, the definition of schemes. It is similar to that of a manifold, which is a topological space locally isomorphic to $\mathbb{R}^n$:

Definition 9.5. *An* affine scheme *is a locally ringed space $(X, \mathcal{O}_X)$ isomorphic to $\mathrm{Spec}(A)$ for some ring A. The category of affine schemes* AffSch *is the full subcategory of locally ringed spaces on affine schemes.*
A scheme *is a locally ringed space $(X, \mathcal{O}_X)$ which is locally affine, that is for which there exists an open cover U_i of X such that for each i the locally ringed space $(U_i, \mathcal{O}_X|_{U_i})$ is an affine scheme (i.e. isomorphic to $\mathrm{Spec}(A_i)$ for some ring A_i). The category of schemes* Sch *is the full subcategory of locally ringed spaces on schemes.*

We will often write a scheme simply as X instead of $(X, \mathcal{O}_X)$ and leave the sheaf implicit.

Corollary 9.6. (1) *The functors* Spec *and* Γ *induce inverse equivalences*

$$\mathrm{AffSch} \simeq \mathrm{Ring}^{\mathrm{op}} \ .$$

(2) *The inclusion* $\mathrm{AffSch} \to \mathrm{Sch}$ *admits a left adjoint given by sending $X \in \mathrm{Sch}$ to $\mathrm{Spec}(\Gamma(X))$. Concretely a morphism from a scheme X to an affine scheme Y will factor uniquely through the morphism $X \to \mathrm{Spec}(\Gamma(X))$.*
(3) *A scheme X is affine precisely if the morphism $X \to \mathrm{Spec}(\Gamma(X))$ is an isomorphism.*
(4) *The scheme $\mathrm{Spec}(\mathbb{Z})$ is terminal in the category of schemes, that is for every scheme X there is a unique morphism $X \to \mathrm{Spec}(\mathbb{Z})$.*

Proof. Immediate from the previous statements and Theorem 9.2. □

Lemma 9.7. *Let $(X, \mathcal{O}_X)$ be a scheme and $U \subseteq X$ be an open subset. Then $(U, \mathcal{O}_X|_U)$ is also a scheme.*

Proof. To see this note that for a given point $x \in U$ we find an open neighborhood $V \subseteq X$ of x such that $(V, \mathcal{O}_X|_V)$ is isomorphic to $\mathrm{Spec}(A)$. But we need an affine neighborhood of x that lies in U, i.e. in $V \cap U$. The latter is open in $\mathrm{Spec}(A)$, so it contains a principal open which also contains x. This principal open is affine. □

Definition 9.8. *We will denote the affine scheme $\mathrm{Spec}(k[x_1, \ldots, x_n])$ also by $\mathbb{A}^n_k$ for any ground ring k.*

Note that this definition is slightly dangerous since, for an algebraically closed field k, we earlier denoted the affine algebraic set k^n by $\mathbb{A}^n_k$. In that case $\mathrm{Spec}(k[x_1,\dots,x_n])$ is its sobrification and only adds generic points. The categories of sheaves on these two spaces are equivalent by Exercise 3 of Sheet 5, so that the structure sheaf can be viewed on either space. This comparison is specific to algebraically closed fields, whereas the scheme $\mathbb{A}^n_k$ is defined for every ground ring k.

EXAMPLE 9.9. We have that for any scheme X

$$\mathrm{Hom}_{\mathrm{Sch}}(X,\mathbb{A}^1_{\mathbb{Z}}) = \mathrm{Hom}_{\mathrm{Ring}}(\mathbb{Z}[x],\mathcal{O}(X)) = \mathcal{O}(X)\ .$$

That is global sections are functions $X \to \mathbb{A}^1_{\mathbb{Z}}$.

EXAMPLE 9.10. Let k be a field. Consider the affine scheme $\mathbb{A}^2_k = \mathrm{Spec}(k[X,Y])$ and let U be the open set

$$D(X,Y) = \mathbb{A}^2_k \setminus 0$$

given by the complement of the closed point 0. We consider U as a scheme. By the first exercise of sheet 5 we see that the global sections of the scheme U are given by $k[X,Y]$. Thus if U was affine the morphism

$$U \to \mathrm{Spec}(\Gamma(U)) = \mathbb{A}^2_k$$

would have to be an isomorphism, which it is not. This morphism is in fact simply the inclusion which is not even surjective.

9.1. Proof of the affine adjunction. Now we would like to prove Theorem 9.2. We have seen that we need to verify that the morphism

$$\mathrm{Hom}\Big((X,\mathcal{O}_X),\mathrm{Spec}(A)\Big) \xrightarrow{\Gamma} \mathrm{Hom}\Big(A,\mathcal{O}_X(X)\Big) \tag{6}$$

is a bijection.
For injectivity assume two maps $f,g : (X,\mathcal{O}_X) \to \mathrm{Spec}(A)$ with corresponding morphisms

$$f^\sharp : f^{-1}\mathcal{O}_{\mathrm{Spec}(A)} \to \mathcal{O}_X \qquad g^\sharp : g^{-1}\mathcal{O}_{\mathrm{Spec}(A)} \to \mathcal{O}_X$$

such that f and g induce the same map $\varphi : A \to \mathcal{O}_X(X)$ on global sections. Then for any $x \in X$ we have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f^\sharp=\varphi} & \mathcal{O}(X) \\ \downarrow & & \downarrow \\ A_{f(x)} & \xrightarrow{f^\sharp_x} & \mathcal{O}_{X,x} \end{array}$$

The preimage of the maximal ideal $\mathfrak{m}_{\mathcal{O}_{X,x}} \subseteq \mathcal{O}_{X,x}$ under the counterclockwise composition is $f(x)$. For this we use that $f^\sharp_x$ is a local morphism! From the clockwise composition we see that this preimage does not depend on f but only on φ. Thus we get for each x that $f(x) = g(x)$, i.e. the underlying maps of topological spaces agree. The diagram also shows that the maps $f^\sharp : f^{-1}\mathcal{O}_{\mathrm{Spec}(A)} \to \mathcal{O}_X$ and $g^\sharp : g^{-1}\mathcal{O}_{\mathrm{Spec}(A)} \to \mathcal{O}_X$ agree on stalks (there is at most one factorization of a map $A \to B$ over $A_{f(x)}$). Thus the maps agree, since any two maps of sheaves that agree on stalks are equal.

Now to prove surjectivity of (6) we have to construct for a given morphism $\varphi : A \to \mathcal{O}_X(X)$ a morphism $(f, f^\sharp)$ of locally ringed spaces. We consider the composition $\varphi_x : A \to \mathcal{O}_X(X) \to \mathcal{O}_{X,x}$ and have to set

$$f(x) := \varphi_x^{-1}(\mathfrak{m}_{X,x})$$

This is a prime ideal and this way we define a map $f : X \to \mathrm{Spec}(A)$ of topological spaces. We claim that it is continuous. To see this we have to verify that $f^{-1}(D(a))$ for $a \in A$ is open. But we find that

$$\begin{aligned} f^{-1}(D(a)) &= \{x \in X \mid a \notin \varphi_x^{-1}(\mathfrak{m}_{X,x})\} \\ &= \{x \in X \mid \varphi_x(a) \notin \mathfrak{m}_{X,x}\} \\ &= \{x \in X \mid \varphi(a)(x) \neq 0 \text{ in } \kappa(x)\} \end{aligned}$$

This subset is open (as in Exercise 3 on sheet 7): if a germ is a unit, choose a representative of its inverse; their product is 1 on a sufficiently small neighborhood. Thus $\varphi(a)$ is invertible throughout that neighborhood, and f is continuous. Now we want to define $f^\sharp : \mathcal{O}_{\mathrm{Spec}(A)} \to f_*\mathcal{O}_X$ on principal opens $D(a)$. To this end we have to construct an extension

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & \mathcal{O}_X(X) \\ \downarrow & & \downarrow \mathrm{res} \\ A[1/a] & \dashrightarrow & \mathcal{O}_X(f^{-1}(D(a))) \end{array}$$

which exists precisely if the restriction of $\varphi(a)$ is invertible in $\mathcal{O}_X(f^{-1}(D(a)))$. But invertibility of sections of a sheaf can be checked on stalks: a section $s \in \mathcal{O}_X(W)$ is invertible if and only if multiplication by s is an isomorphism of sheaves of abelian groups $\mathcal{O}_X|_W \to \mathcal{O}_X|_W$, which can be checked on stalks. By the definition of $f^{-1}(D(a))$, this applies to the restriction of $\varphi(a)$.
We therefore get our transformation $f^\sharp$ which clearly is compatible with restriction. The last thing we have to verify is that it induces local maps on stalks, so that we have a map of locally ringed spaces.
For this we note that we have by construction the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f^\sharp = \varphi} & \mathcal{O}(X) \\ \downarrow & & \downarrow \\ A_{f(x)} & \xrightarrow{f_x^\sharp} & \mathcal{O}_{X,x} \end{array}$$

The preimage of the maximal ideal along the clockwise composition is $\varphi_x^{-1}(\mathfrak{m}_{X,x})$, which by definition is $f(x)$. Since every element of $A \setminus f(x)$ is inverted in $A_{f(x)}$, this shows that the preimage under $f_x^\sharp$ is exactly $f(x)A_{f(x)}$. Thus the map on stalks is local, which finishes the proof.

10. k-valued points

Let k be a field (or even a commutative ring).

Definition 10.1. *For a scheme X we define the k-valued points $X(k)$ to be the set of homomorphisms* $\mathrm{Spec}(k) \to X$.

EXAMPLE 10.2. Assume that $X = \mathrm{Spec}(R)$ for some ring R. Then we have that $X(k) = \mathrm{Hom}_{\mathrm{Ring}}(R, k)$. For example if $R = \mathbb{Z}[X_1, \ldots, X_n]/(f_1, \ldots, f_r)$ then we have that

$$X(k) = \{x \in k^n \mid f_1(x) = \cdots = f_r(x) = 0\} .$$

Thus $X(k)$ is the set of solutions to the equations given by the polynomials f_i. For affine varieties over a fixed field, we recover our earlier notion of points when we require the ring maps to preserve that field. We will make this relative version precise below. We could for example evaluate at $k = \mathbb{C}$ or $\mathbb{R}$ and get some intuition what our scheme 'looks like' by evaluating at those points.

EXAMPLE 10.3. Assume that $X = \mathrm{Spec}(\mathbb{Z}[X]/(X^2))$. Then we get that

$$X(k) = \{x \in k \mid x^2 = 0\} = \{0\}$$

for all fields k. Thus by k-valued points we cannot even distinguish the schemes $\mathrm{Spec}(\mathbb{Z}[X]/(X^2))$ and $\mathrm{Spec}(\mathbb{Z})$.

EXAMPLE 10.4. The preceding example becomes especially transparent over a field k. Set

$$D = \mathrm{Spec}(k[t]/(t^2)), \qquad P = \mathrm{Spec}(k).$$

In this example all ring maps are required to preserve k. Thus for a k-algebra A we write

$$D(A) = \mathrm{Hom}_{\mathrm{Alg}_k}(k[t]/(t^2), A) = \{a \in A \mid a^2 = 0\}, \qquad P(A) = \{*\}.$$

For every field extension K/k, the set $D(K)$ consists only of 0, so D and P have the same field-valued points. Their underlying topological spaces also agree: D has just the prime ideal (t).
Now take the k-algebra $A = k[\varepsilon]/(\varepsilon^2)$. An element $a + b\varepsilon$ has square zero only if $a = 0$, and every $b\varepsilon$ has square zero. Consequently,

$$D(A) = \{b\varepsilon \mid b \in k\} \cong k, \qquad P(A) = \{*\}.$$

The corresponding map $k[t]/(t^2) \to A$ sends t to $b\varepsilon$. These are distinct maps for distinct b. Ring-valued points therefore distinguish the two schemes. Their structure sheaves already distinguish them as well: their rings of global sections are $k[t]/(t^2)$ and k.

DEFINITION 10.5. *A scheme X is called* reduced *if for each open $U \subseteq X$ the ring $\mathcal{O}_X(U)$ is a reduced ring (i.e. has no nonzero nilpotent elements).*

PROPOSITION 10.6. *An affine scheme* $\mathrm{Spec}(A)$ *is reduced iff A is reduced. A general scheme X is reduced precisely if it admits an open cover $(U_i)_{i \in I}$ by affine reduced schemes U_i. A scheme is reduced precisely if the local rings $\mathcal{O}_{X,x}$ are reduced.*

PROOF. If $\mathrm{Spec}(A)$ is reduced then clearly also $A = \mathcal{O}_{\mathrm{Spec}(A)}(\mathrm{Spec}(A))$ is reduced. Conversely assume that A is reduced. Then for any $a \in A$ we have that $A[a^{-1}]$ is also reduced: if $(b/a^k)^n = 0$ in $A[a^{-1}]$ then $a^m b^n = 0$ in A for some m. Then $(a^m b)^n = a^{mn} b^n = 0$ (enlarging m if necessary), so $a^m b = 0$ since A is reduced. Hence $b/a^k = 0$ in $A[a^{-1}]$.
For a general open set $U \subseteq \mathrm{Spec}(A)$ we find a cover $U_i = \mathrm{Spec}(A[1/f_i])$ by principal opens and this way we write $\mathcal{O}_{\mathrm{Spec}(A)}(U)$ as a subring of the reduced ring $\prod A[1/f_i]$ which shows that it is also reduced.
For the assertion about open covers, assume $(X, \mathcal{O}_X)$ is reduced. By definition we can cover it by opens $U_i \cong \mathrm{Spec}(A_i)$ and we get that $A_i = \mathcal{O}_X(U_i)$ is reduced.

Conversely assume that X is covered by open, reduced subschemes X_i (e.g. the X_i affine). Then we claim that X is reduced. To see this take $U \subseteq X$ open. Then U is covered by $U \cap X_i$ and all those sets are open in X_i so that we get $\mathcal{O}_X(U \cap X_i)$ is reduced. Using the sheaf condition we get that $\mathcal{O}(U)$ is a subring of $\prod_{i \in I} \mathcal{O}(U \cap X_i)$. For the statement about stalks, suppose first that every ring of sections is reduced. If a germ is nilpotent, represent it by a section s; after shrinking the neighborhood, some power of s is zero. Thus s and its germ are zero. Conversely, if all stalks are reduced, every nilpotent section has zero germ at every point, and is therefore zero by the sheaf condition. □

EXAMPLE 10.7. For an affine algebraic set V the associated scheme $\mathrm{Spec}(\mathcal{O}(V))$ is reduced.

Recall that for any ring A we can form a reduced ring $A_{red} := A/\sqrt{0}$ which is reduced (as one directly verifies). For a topological space X with a sheaf $\mathcal{O}$ of rings, we define a new sheaf $\mathcal{O}_{red}$ as the sheafification of the presheaf $U \mapsto \mathcal{O}(U)_{red}$.

PROPOSITION 10.8. *For a scheme $X = (X, \mathcal{O}_X)$, set $\mathcal{O}_{X_{red}} := (\mathcal{O}_X)_{red}$. Then the ringed space $X_{red} := (X, \mathcal{O}_{X_{red}})$ is a reduced scheme and we have a canonical morphism $i : X_{red} \to X$ which exhibits X_{red} as the universal reduced scheme with a morphism to X. If $X = \mathrm{Spec}(A)$ then $X_{red} \cong \mathrm{Spec}(A_{red})$.*

PROOF. For $X = (\mathrm{Spec}(A), \mathcal{O}_{\mathrm{Spec}\, A})$ we have a canonical homeomorphism $\mathrm{Spec}(A_{red}) \cong \mathrm{Spec}(A)$: ideals in A_{red} correspond to ideals of A containing $\sqrt{0}$, so prime ideals (describing points) and radical ideals (describing closed subsets) correspond in A and A_{red}. Under this homeomorphism the open sets $D(f)$ correspond to each other for $f \in A$ (for $f \in \sqrt{0}$ the open set $D(f) = \emptyset$ so that we see that there is no difference). The sheaf $(\mathcal{O}_{\mathrm{Spec}\, A})_{red}$ is given by the sheafification of

$$D(f) \mapsto (A[1/f])_{red} = (A_{red})[1/f]$$

thus it agrees with the structure sheaf of $\mathrm{Spec}(A_{red})$.
Now for a general X we observe that for any open $U \subseteq X$ we have that the reduction of $(U, \mathcal{O}_X|_U)$ is given by first reducing X and then restricting to U (this follows since sheafification commutes with restriction to opens). Therefore we conclude that if U is affine in X then U is also affine in X_{red} which shows that X_{red} is a scheme. Moreover the map of ringed spaces $i : X_{red} \to X$ given by the identity on spaces and $i^\sharp : \mathcal{O}_X \to \mathcal{O}_{X_{red}}$ is a map of locally ringed spaces since it is locally induced by $A \to A_{red}$.
Finally we note that a morphism $Y \to X$ with Y reduced is given by a pair $(f, f^\sharp)$ with

$$f^\sharp : \mathcal{O}_X \to f_* \mathcal{O}_Y.$$

Since Y is reduced, every ring of sections of $f_* \mathcal{O}_Y$ is reduced. Thus the map factors uniquely through the presheaf $U \mapsto \mathcal{O}_X(U)_{red}$ and then, by the universal property of sheafification, through $\mathcal{O}_{X_{red}}$. This gives a unique map of ringed spaces $Y \to X_{red}$. It is local on stalks: the stalk of $\mathcal{O}_{X_{red}}$ at x is $(\mathcal{O}_{X,x})_{red}$, whose maximal ideal is the image of $\mathfrak{m}_x$, and the original map on stalks was local. □

In particular this means that we cannot distinguish the schemes X and X_{red} as long as we only consider k-valued points for fields k (or more generally reduced rings) since the induced map

$$X_{red}(k) \to X(k)$$

is a bijection.
For example we have that $\mathrm{Spec}(\mathbb{Z}[x]/(x^2))_{red} = \mathrm{Spec}(\mathbb{Z})$, and thus k-valued points of $\mathrm{Spec}(\mathbb{Z}[x]/(x^2))$ and $\mathrm{Spec}(\mathbb{Z})$ agree.
For any commutative ring A we consider the A-valued points $X(A)$ and these assemble together into a functor

$$X(-) : \mathrm{Ring} \to \mathrm{Set}$$

called the *functor of points* of X. We will now see that the functor of points completely determines X:

PROPOSITION 10.9. *The assignment $X \mapsto X(-)$ defines a fully faithful functor*

$$\mathrm{Sch} \to \mathrm{Fun}(\mathrm{Ring}, \mathrm{Set}) \ .$$

PROOF. We first note that if we replace Ring by Sch then we have for every scheme X the functor $S \in \mathrm{Sch} \mapsto \mathrm{Hom}_{\mathrm{Sch}}(S, X) = \underline{X}$. The Yoneda Lemma implies that this functor

$$\mathrm{Sch} \to \mathrm{Fun}(\mathrm{Sch}^{\mathrm{op}}, \mathrm{Set})$$

is fully faithful and our functor is the composition

$$\mathrm{Sch} \to \mathrm{Fun}(\mathrm{Sch}^{\mathrm{op}}, \mathrm{Set}) \xrightarrow{\mathrm{res}} \mathrm{Fun}(\mathrm{AffSch}^{\mathrm{op}}, \mathrm{Set})$$

(where we have implicitly identified $\mathrm{AffSch}^{\mathrm{op}} \simeq \mathrm{Ring}$). So our proof roughly proceeds like the proof of the Yoneda lemma: we want to show that every morphism $\eta : \underline{X}|_{\mathrm{AffSch}} \to \underline{Y}|_{\mathrm{AffSch}}$ is induced by a unique morphism $X \to Y$ of schemes. If X is affine this follows from Yoneda's Lemma (applied to AffSch). In general given a natural transformation η

$$\eta_S : \{S \to X\} \to \{S \to Y\}$$

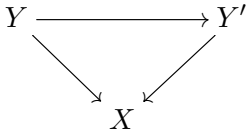
for S affine we note that we can cover X by affines X_i and evaluate η_{X_i} on the inclusion $X_i \to X$, obtaining a morphism $f_i : X_i \to Y$. For every affine open $W \subseteq X_i \cap X_j$, naturality shows that $f_i|_W$ and $f_j|_W$ both equal $\eta_W(W \to X)$. Since such W cover the intersection, the maps agree on $X_i \cap X_j$. Thus, using Exercise 2 of sheet 8, we can glue them to a unique map $f : X \to Y$. Finally, for any $g : S \to X$ with S affine, cover S by affine opens S_α each mapping into some X_i (refine the preimages of the X_i by affine opens). Naturality gives

$$\eta_S(g)|_{S_\alpha} = \eta_{S_\alpha}(g|_{S_\alpha}) = f_i \circ g|_{S_\alpha}.$$

Gluing shows that $\eta_S(g) = f \circ g$. Thus η is induced by f, and the values on the affine opens X_i also show uniqueness. □

Now finally we also want to talk about schemes over some fixed base. To this end we need a general definition from category theory:

DEFINITION 10.10. *Let $\mathcal{C}$ be a category and $X \in \mathcal{C}$ an object. Then we define the* slice over X *or the* overcategory *to be the category $\mathcal{C}_{/X}$ whose objects are morphisms $Y \to X$ and whose morphisms from $Y \to X$ to $Y' \to X$ are commutative diagrams*



Composition is defined in the evident way. Similarly we define the undercategory *or* slice under X *to be the category* $\mathcal{C}_{X/}$ *whose objects are morphisms* $X \to Z$ *and whose morphisms are similar commutative diagrams.*

EXAMPLE 10.11. Let $\mathcal{C} = \mathrm{Ring}$ and k be a commutative ring. Then $\mathrm{Ring}_{k/}$ is equivalent to the category of k-algebras.

EXAMPLE 10.12. If X is terminal then $\mathcal{C}_{/X} \simeq \mathcal{C}$. In particular $\mathrm{Sch}_{/\operatorname{Spec}\mathbb{Z}} \simeq \mathrm{Sch}$.

DEFINITION 10.13. *Let* S *be a scheme. Then the category* $\mathrm{Sch}_{/S}$ *is the category of schemes over* S*. If* k *is a commutative ring then the category of schemes over* k *is the category of schemes over* $\mathrm{Spec}(k)$*, which we also write as* Sch_k*.*

Note that for a scheme X the structure of a scheme over $\mathrm{Spec}(k)$ is equivalent to a morphism

$$k \to \mathcal{O}_X(X) \ .$$

In particular for any k-algebra A the scheme $\mathrm{Spec}(A)$ is a scheme over k. Moreover using the restrictions $\mathcal{O}_X(X) \to \mathcal{O}_X(U)$ we see that a morphism $k \to \mathcal{O}_X(X)$ makes the sheaf $\mathcal{O}_X$ into a sheaf of k-algebras. In fact giving a morphism $X \to \mathrm{Spec}(k)$ is equivalent to lifting the structure sheaf $\mathcal{O}_X$ to a sheaf of k-algebras. Moreover we get that the category of affine schemes over k (as a full subcategory of Sch_k) is equivalent to $\mathrm{Alg}_k^{\mathrm{op}}$, the opposite of the category of k-algebras. Thus as a conclusion we could rephrase the definition of a scheme over k as follows:

> A scheme over k is a topological space X, with a sheaf $\mathcal{O}_X$ of k-algebras such that the stalks are local k-algebras and such that the pair $(X, \mathcal{O}_X)$ is locally isomorphic to $\mathrm{Spec}(A)$ for A a k-algebra.

We will often implicitly adopt this point of view.

PROPOSITION 10.14. *The category* AffVar_k *for* k *algebraically closed embeds fully faithfully into the category* $\mathrm{Sch}_{/\operatorname{Spec}(k)}$ *by the functor*

$$V \mapsto \mathrm{Spec}(\mathcal{O}(V)) \ .$$

PROOF. The claim follows since this functor is the composition

$$\mathrm{AffVar}_k \to \mathrm{Alg}_k^{\mathrm{op}} \to \mathrm{Sch}_k$$

of fully faithful functors. □

For a scheme X over $\mathrm{Spec}(k)$ we have that for any k-algebra A we can define the A-valued points $X(A)$ to be the morphisms $\mathrm{Spec}(A) \to X$ over $\mathrm{Spec}(k)$ (i.e. in the category $\mathrm{Sch}_k = \mathrm{Sch}_{/\operatorname{Spec}(k)}$). Similarly to the absolute case we obtain a functor

$$\mathrm{Sch}_k \to \mathrm{Fun}(\mathrm{Alg}_k, \mathrm{Set})$$

which is fully faithful.

WARNING 10.15. *If we consider a scheme over* $\mathrm{Spec}(k)$ *as a plain scheme (i.e. forgetting the morphism to* $\mathrm{Spec}(k)$*) then the two notions of* $X(A)$ *are different (one relative to* k *and one absolute). Therefore it is very important to always keep the base in mind.*

11. Tensor products and fibre products

The question in this section is whether the category Sch of schemes has (fibre) products. This is closely related to the question whether the category Alg_k has coproducts. We will see that the coproduct of A and B is the tensor product $A \otimes_k B$, equipped with its algebra structure. We use modules and their tensor products from [**AM**, Chapter 2]. Our notation is Mod_R for R-modules and $M \otimes_R N$ for the module representing R-bilinear maps:

$$\mathrm{Hom}_R(M \otimes_R N, P) \cong \mathrm{Bil}_R(M \times N, P).$$

The construction by generators and relations is part of the prerequisites. We also use symmetry, associativity, the unit R, and compatibility with finite direct sums [**AM**, Proposition 2.14], as well as right exactness [**AM**, Proposition 2.18] and compatibility with localization [**AM**, 3.5–3.7]. In particular

$$(R/I) \otimes_R (R/J) \cong R/(I+J), \qquad (M \otimes_R N)[1/f] \cong M[1/f] \otimes_{R[1/f]} N[1/f].$$

These formulas will describe intersections and restrictions of schemes.
The categorical meaning of the familiar tensor product is useful here. Currying a bilinear map gives the natural bijection

$$\mathrm{Hom}_R(M \otimes_R N, P) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P)).$$

Thus $- \otimes_R N$ is left adjoint to $\mathrm{Hom}_R(N, -)$, and consequently preserves colimits. We will now use the same universal-property viewpoint for *algebras* and then for fibre products of schemes.
Now let k be a commutative ring and assume that A and B are k-algebras (as usual, commutative). Note that the multiplication

$$A \times A \to A \qquad (a, b) \mapsto ab$$

is k-bilinear. Therefore we can consider it as a map $A \otimes_k A \to A$, and similarly for B.

PROPOSITION 11.1. *The k-module $A \otimes_k B$ admits the structure of a commutative k-algebra by the multiplication*

$$(a \otimes b) \cdot (a' \otimes b') = aa' \otimes bb' \ .$$

The maps

$$A \to A \otimes_k B \qquad a \mapsto a \otimes 1$$
$$B \to A \otimes_k B \qquad b \mapsto 1 \otimes b$$

exhibit $A \otimes_k B$ as the coproduct in the category of k-algebras.

PROOF. The k-module $A \otimes_k B$ has a k-bilinear multiplication

$$(A \otimes_k B) \times (A \otimes_k B) \longrightarrow A \otimes_k B, \qquad (a \otimes b, a' \otimes b') \longmapsto aa' \otimes bb'.$$

It is well-defined because it is induced by the quadrilinear map $A \times B \times A \times B$ sending (a, b, a', b') to $aa' \otimes bb'$. Bilinearity gives distributivity. Associativity and commutativity can be checked on elementary tensors, and the unit is $1 \otimes 1$. The structure map $k \to A \otimes_k B$ sends c to $c(1 \otimes 1)$.
The maps $A \to A \otimes_k B$ and $B \to A \otimes_k B$ are maps of k-algebras. Every element of $A \otimes_k B$ is a sum of elementary tensors $a \otimes b$, and each such tensor is the product $(a \otimes 1)(1 \otimes b)$. Thus every algebra map $A \otimes_k B \to C$ is uniquely determined by the induced maps $A \to C$ and $B \to C$.

Conversely, a pair of k-algebra maps $f_A : A \to C$ and $f_B : B \to C$ induces a k-linear map

$$f : A \otimes_k B \to C \qquad (a \otimes b) \mapsto f_A(a) \cdot f_B(b)$$

since the expression on the right is bilinear. This map preserves the unit and is multiplicative, since

$$\begin{aligned} f((a \otimes b) \cdot (a' \otimes b')) &= f(aa' \otimes bb') \\ &= f_A(aa') \cdot f_B(bb') \\ &= f_A(a) f_A(a') f_B(b) f_B(b') \\ &= f(a \otimes b) \cdot f(a' \otimes b') \ . \end{aligned}$$

Note that in the last step we have crucially used the commutativity of C. □

Example 11.2. We have $R[x] \otimes_R R[y] \cong R[x, y]$. Both sides are the coproduct of $R[x]$ and $R[y]$ as R-algebras, so this follows from their universal properties. We can also see it directly: the powers of x form an R-basis of $R[x]$, and similarly for $R[y]$. Thus the tensors $x^i \otimes y^j$ form a basis of the tensor product and correspond to the monomials $x^i y^j$.

Proposition 11.3. *For a diagram*

$$\begin{array}{ccc} C & \longrightarrow & A \\ \downarrow & & \\ B & & \end{array}$$

of commutative rings we have that $A \otimes_C B$ is the pushout, i.e. the colimit of this diagram in the category Ring *(here we forget that it is actually a C-algebra).*

Proof. A ring map $f : A \otimes_C B \to D$ gives D a C-algebra structure by composition with $C \to A \otimes_C B$, and f is then a C-algebra map. By the preceding proposition, such a map is equivalent to a pair of C-algebra maps $A \to D$ and $B \to D$. Equivalently, these are ring maps whose composites with the given maps from C agree. Conversely, any such pair equips D with their common C-algebra structure and gives a unique map $A \otimes_C B \to D$. This is the universal property of the pushout. □

Dually, a limit of a diagram

$$\begin{array}{ccc} & & X \\ & & \downarrow \\ Y & \longrightarrow & S \end{array}$$

is called a pullback. In fact, such a limit Z can also be depicted by a square

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & S \end{array}$$

which we then call a pullback square. We often also write $Z = X \times_S Y$.
If I is a finite category (with finitely many objects and morphisms), a functor $I \to \mathcal{C}$ is called a finite diagram, and its limit is called a finite limit. We say that $\mathcal{C}$ admits all finite limits if every finite diagram admits a limit.

THEOREM 11.4. *The category of schemes has all finite limits. For a diagram*

$$\begin{array}{ccc} & & \operatorname{Spec}(A) \\ & & \downarrow \\ \operatorname{Spec}(B) & \longrightarrow & \operatorname{Spec}(C) \end{array}$$

the fibre product is $\operatorname{Spec}(A \otimes_C B)$.

EXAMPLE 11.5. Let k be a field and put $R = k[x, y]$. Inside $\mathbb{A}^2_k = \operatorname{Spec}(R)$ consider the line and parabola

$$L = \operatorname{Spec}(R/(y)), \qquad C = \operatorname{Spec}(R/(y - x^2)),$$

with maps to $\mathbb{A}^2_k$ induced by the quotient maps. Their intersection as schemes is the fibre product. Its coordinate ring is

$$\begin{aligned} (R/(y)) \otimes_R (R/(y - x^2)) &\cong R/(y, y - x^2) \\ &\cong k[x]/(x^2). \end{aligned}$$

Here we used $(R/I) \otimes_R (R/J) \cong R/(I + J)$: the tensor product imposes both sets of relations. Thus

$$L \times_{\mathbb{A}^2_k} C \cong \operatorname{Spec}(k[x]/(x^2)),$$

the double point of Example 10.4. Its only field-valued point is the origin, but its coordinate ring contains the nonzero nilpotent class of x.
For comparison, intersect L with the vertical line $M = \operatorname{Spec}(R/(x))$. Then

$$(R/(y)) \otimes_R (R/(x)) \cong R/(x, y) \cong k,$$

so $L \times_{\mathbb{A}^2_k} M \cong \operatorname{Spec}(k)$. Both intersections give just the origin as a classical algebraic set. The scheme intersections are different: the first coordinate ring has k-basis $1, x$, while the second has basis 1.
We can also see the difference directly in the universal property. Over $A = k[\varepsilon]/(\varepsilon^2)$, the pair $(x, y) = (\varepsilon, 0)$ satisfies both $y = 0$ and $y = x^2$, and hence gives an A-valued point of the first fibre product. It does not satisfy $x = 0$, so it gives no point of the second intersection. Nilpotents are therefore necessary to describe intersections by fibre products, even when both curves being intersected are reduced.

Proving this theorem will take some time and additional lemmas. But let us first make a remark.

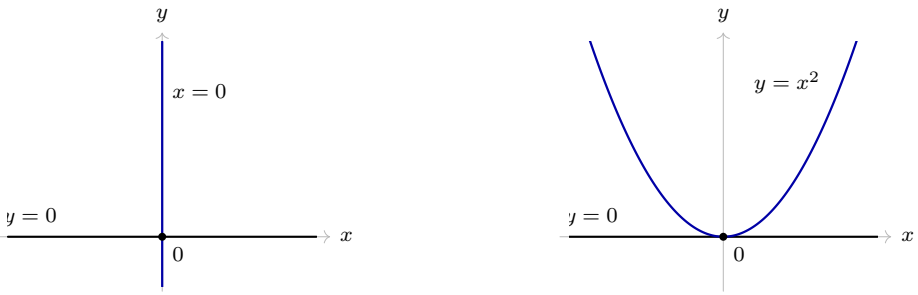
REMARK 11.6. Arbitrary limits of schemes need not exist. Note that by the universal property the functor of points of the pullback can be described as the pullback

$$(X \times_S Y)(R) = X(R) \times_{S(R)} Y(R) \ .$$

This is a convenient description that we will use. The main question is whether this pullback of functors is represented by a scheme.

We first need a categorical lemma.

LEMMA 11.7. *Assume that a category $\mathcal{C}$ has a terminal object and all pullbacks. Then it has all finite limits.*



Transverse: $R/(x,y) \cong k$ Tangent: $R/(y, y-x^2) \cong k[x]/(x^2)$

FIGURE 1. Both intersections have just the origin as their underlying point set. Their coordinate rings distinguish a transverse intersection from a tangency: the latter retains the nonzero nilpotent class of x. The curves are drawn over $\mathbb{R}$; the ring calculations hold over any field k.

PROOF. If $\mathcal{C}$ has pullbacks and a terminal object pt, then it also has finite products, since $A \times B$ is the pullback $A \times_{\mathrm{pt}} B$. It also has equalizers: the equalizer of $f, g : A \to B$ is the pullback

$$\begin{array}{ccc} & & B \\ & & \downarrow{\scriptstyle \Delta} \\ A & \xrightarrow{(f,g)} & B \times B \end{array}$$

For a finite diagram $X : I \to \mathcal{C}$, its limit is the equalizer of two maps

$$\prod_{i \in \mathrm{Ob}(I)} X_i \rightrightarrows \prod_{\alpha : i \to j} X_j.$$

For the component indexed by $\alpha : i \to j$, the first map is the projection to X_i followed by $X(\alpha)$, and the second is the projection to X_j. Thus the equalizer imposes exactly the compatibility conditions defining a cone. □

The category Sch already has a terminal object, namely $\mathrm{Spec}(\mathbb{Z})$. It therefore remains to construct pullbacks. We have shown that the square

$$\begin{array}{ccc} C & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & B \otimes_C A \end{array}$$

is a pushout in Ring. Thus the square

$$\begin{array}{ccc} \mathrm{Spec}(B \otimes_C A) & \longrightarrow & \mathrm{Spec}(A) \\ \downarrow & & \downarrow \\ \mathrm{Spec}(B) & \longrightarrow & \mathrm{Spec}(C) \end{array}$$

is a pullback in AffSch. The inclusion AffSch $\to$ Sch is a right adjoint, with left adjoint $X \mapsto \mathrm{Spec}(\Gamma(X, \mathcal{O}_X))$ (Corollary 9.6). It therefore preserves limits, so this square is also a pullback in Sch.

For a general diagram of schemes

$$\begin{array}{ccc} & & X \\ & & \downarrow p \\ Y & \xrightarrow{q} & S \end{array}$$

start with an affine open cover (S_α) of S. For each α, choose affine open covers $(X_{\alpha\beta})$ of $p^{-1}(S_\alpha)$ and $(Y_{\alpha\gamma})$ of $q^{-1}(S_\alpha)$. We must consider every pair of such charts lying over the same S_α, so the appropriate index set consists of the triples $i = (\alpha, \beta, \gamma)$. Write $X_i = X_{\alpha\beta}$, $Y_i = Y_{\alpha\gamma}$, and $S_i = S_\alpha$.
We get induced diagrams

$$\begin{array}{ccc} & & X_i \\ & & \downarrow p \\ Y_i & \xrightarrow{q} & S_i \end{array}$$

and can form the affine fibre products $Z_i := X_i \times_{S_i} Y_i$. The idea is to construct a scheme Z with these Z_i as an open cover. To carry this out we need to discuss the gluing of schemes.
This leads us to discuss colimits of schemes. Recall that for a family of topological spaces X_i we can form a new topological space

$$\coprod X_i$$

whose underlying set is the disjoint union of the X_i. A subset $U \subseteq \coprod X_i$ is open precisely if every $U \cap X_i$ is open in X_i. Equivalently, an open set is a disjoint union of open subsets $U_i \subseteq X_i$. It is straightforward to check that this is the coproduct in the category of topological spaces and continuous maps.

LEMMA 11.8. *Let $X_i = (X_i, \mathcal{O}_{X_i})$ be a family of schemes. Then the coproduct in* Sch *exists and is given by the disjoint union of topological spaces*

$$X = \coprod X_i$$

with the structure sheaf $\mathcal{O}_X$ given by $\mathcal{O}_X(\coprod_i U_i) = \prod_i \mathcal{O}_{X_i}(U_i)$.

PROOF. This is a sheaf: it is the product $\prod_i (i_i)_* \mathcal{O}_{X_i}$, where $i_i : X_i \to X$ is the inclusion. Its restriction to X_i is $\mathcal{O}_{X_i}$, so affine covers of the X_i together give an affine cover of $(X, \mathcal{O}_X)$. The universal property can be checked directly. A continuous map $f : X \to Y$ is a family of continuous maps $f_i : X_i \to Y$, and a morphism of sheaves

$$f^\sharp : \mathcal{O}_Y \to f_*(\mathcal{O}_X) = f_* \prod_I (i_i)_*(\mathcal{O}_{X_i}) = \prod (f_i)_*(\mathcal{O}_{X_i})$$

is a family of morphisms $f_i^\sharp$ as desired. This gives the coproduct in ringed spaces. The maps on stalks are local precisely when the corresponding maps for each f_i are local, so it is also the coproduct in locally ringed spaces and hence in schemes. □

LEMMA 11.9. *Assume that we have a diagram of schemes*

$$\begin{array}{ccc} U & \xrightarrow{i} & X \\ \downarrow j & & \\ Y & & \end{array}$$

where both maps are open immersions. Then the pushout $X \amalg_U Y$ in schemes exists, and the maps $X \to X \amalg_U Y \leftarrow Y$ are open immersions.

PROOF. Form the topological pushout Z as the quotient of $X \amalg Y$ by $i(u) \sim j(u)$ for $u \in U$, with the quotient topology. The maps $X \to Z$ and $Y \to Z$ are open embeddings: for an open subset $V \subseteq X$, its image has inverse image V in X and $j(i^{-1}(V))$ in Y, both open, and similarly for open subsets of Y.
Now denote the inclusions into Z by $i : X \to Z$, $j : Y \to Z$, and $k : U \to Z$. The pushforward structure sheaves have restriction maps

$$\begin{array}{ccc} & & i_*(\mathcal{O}_X) \\ & & \downarrow \\ j_*(\mathcal{O}_Y) & \longrightarrow & k_*(\mathcal{O}_U) \ . \end{array}$$

Define $\mathcal{O}_Z = i_*\mathcal{O}_X \times_{k_*\mathcal{O}_U} j_*\mathcal{O}_Y$. On the open subset $X \subseteq Z$, a section on the overlap with Y is determined by its restriction from X, so this sheaf restricts to $\mathcal{O}_X$; similarly it restricts to $\mathcal{O}_Y$. Thus $(Z, \mathcal{O}_Z)$ is a scheme. By the topological universal property, a continuous map $f : Z \to W$ is a pair of maps $f_X : X \to W$ and $f_Y : Y \to W$ agreeing on U. A morphism of ringed spaces

$$(f, f^\sharp) : (Z, \mathcal{O}_Z) \to (W, \mathcal{O}_W)$$

is then specified by a map of sheaves

$$\mathcal{O}_W \to f_*(\mathcal{O}_Z) \ .$$

Since f_* preserves limits, the pullback description of $\mathcal{O}_Z$ identifies this map with a pair of maps $\mathcal{O}_W \to (f_X)_*\mathcal{O}_X$ and $\mathcal{O}_W \to (f_Y)_*\mathcal{O}_Y$ whose restrictions to U agree. This proves the pushout property in ringed spaces. Locality of the stalk maps can be checked on the open cover X, Y of Z, so the same property holds in locally ringed spaces and in schemes. □

REMARK 11.10. For any small diagram $(X_i, \mathcal{O}_i)$ of ringed spaces indexed by I, we can form its colimit as

$$\left(\operatorname{colim}_I X_i, \ \lim_{I^{\mathrm{op}}} (j_i)_* \mathcal{O}_i \right),$$

where $j_i : X_i \to \operatorname{colim}_I X_i$ is the canonical map. The diagram of sheaves runs in the opposite direction because structure-sheaf maps are contravariant. The universal property follows as in the preceding proof. In the gluing diagrams considered here, the limit sheaf restricts to the given structure sheaves on the open pieces. If those pieces are locally ringed spaces, the glued space is locally ringed and is also the colimit in locally ringed spaces. This is what we have used so far.

Every scheme X has an affine open cover $X_i \subseteq X$. How can we reconstruct X from these pieces?

PROPOSITION 11.11. *For every scheme X and every open cover (X_i), the scheme X is the coequalizer of the diagram*

$$\coprod_{i,j} (X_i \cap X_j) \rightrightarrows \coprod X_i$$

where the first map is induced by the inclusion $X_i \cap X_j \to X_i$ and the second by the inclusion $X_i \cap X_j \to X_j$.

PROOF. A compatible family of morphisms $X_i \to Y$ agrees on every overlap. Such morphisms glue uniquely: their continuous maps glue on the open cover, and their maps of structure sheaves glue by the sheaf axiom. The stalk maps remain local because this can be checked on each X_i. Thus these families are precisely the morphisms $X \to Y$. □

Thus we can recover X from the X_i if we also remember the double intersections and their inclusions in the two pieces. Denote the copy of $X_i \cap X_j$ in X_i by $U_{ij} \subseteq X_i$, and its copy in X_j by U_{ji}. There is an isomorphism

$$\varphi_{ij} : U_{ij} \to U_{ji}$$

which identifies these two copies of the same open subset. The triple intersection is represented by

$$U_{ij} \cap U_{ik} \subseteq X_i, \qquad U_{ji} \cap U_{jk} \subseteq X_j, \qquad U_{ki} \cap U_{kj} \subseteq X_k.$$

On these subsets the two maps $\varphi_{jk}\varphi_{ij}$ and φ_{ik} agree.

THEOREM 11.12 (Gluing schemes). *Assume that we have a family of schemes X_i for $i \in I$, for each pair i, j an open subscheme $U_{ij} \subseteq X_i$, and isomorphisms $\varphi_{ij} : U_{ij} \to U_{ji}$. Assume that $U_{ii} = X_i$, that $\varphi_{ii} = \mathrm{id}$ and that the following* cocycle condition *is satisfied: for each triple i, j, k we have $\varphi_{ij}^{-1}(U_{ji} \cap U_{jk}) = U_{ij} \cap U_{ik}$ and the diagram*

$$\begin{array}{ccc} U_{ij} \cap U_{ik} & \xrightarrow{\quad\varphi_{ik}\quad} & U_{ki} \cap U_{kj} \\ & {\scriptstyle\varphi_{ij}} \searrow \quad \nearrow {\scriptstyle\varphi_{jk}} & \\ & U_{ji} \cap U_{jk} & \end{array}$$

commutes. Then there is a scheme X with open immersions $X_i \to X$ whose images cover X, such that U_{ij} is the inverse image of $X_i \cap X_j$ in X_i and the induced identifications are the φ_{ij}. It is unique up to a unique isomorphism compatible with these open immersions.

PROOF. Let X be the quotient of $\coprod_i X_i$ by the identifications $x \sim \varphi_{ij}(x)$ for $x \in U_{ij}$, with the quotient topology. The cocycle conditions show that two points $x \in X_i$ and $y \in X_j$ are identified precisely when $x \in U_{ij}$ and $y = \varphi_{ij}(x)$. Indeed, the conditions give symmetry by taking $k = i$, and transitivity by composing the indicated maps on the triple overlaps. Since $\varphi_{ii} = \mathrm{id}$, each map $X_i \to X$ is injective. For an open subset $V \subseteq X_i$, the inverse image of its image in X_j is $\varphi_{ij}(V \cap U_{ij})$, which is open. Thus $X_i \to X$ is an open embedding, with the stated intersections. As in Remark 11.10, take the limit of the pushforward structure sheaves of the pieces and overlaps. Equivalently, a section on an open subset of X is a family of sections on its intersections with the X_i agreeing via $\varphi_{ij}^\sharp$ on overlaps. This is a sheaf and restricts to $\mathcal{O}_{X_i}$ on each X_i, by the cocycle condition. The resulting ringed space is a scheme because each X_i is a scheme. Morphisms from it are exactly compatible families of morphisms from the X_i, by the preceding proposition. This also proves the asserted uniqueness. □

PROOF OF THEOREM 11.4. If $X \times_S Y$ exists and $U \subseteq X$ is open, then $U \times_S Y$ exists and is the open subscheme $p^{-1}(U)$, where $p : X \times_S Y \to X$ is the projection. Indeed, a morphism to $X \times_S Y$ factors through $p^{-1}(U)$ precisely when its component in X factors through U. This is the required universal property.

Now suppose $X = \bigcup_i X_i$ and each $Z_i = X_i \times_S Y$ exists. The preceding observation identifies the inverse image of $X_i \cap X_j$ in Z_i with $(X_i \cap X_j) \times_S Y$. This is canonically isomorphic to the corresponding open in Z_j. These isomorphisms satisfy the cocycle conditions by uniqueness in the universal property. Theorem 11.12 therefore glues the Z_i to a scheme Z. Their maps to X and Y also glue.
Given morphisms $g : K \to X$ and $h : K \to Y$ with the same composite to S, set $K_i = g^{-1}(X_i)$. The universal property of Z_i gives a unique morphism $K_i \to Z_i$ with these components. On $K_i \cap K_j$ the two morphisms factor through $(X_i \cap X_j) \times_S Y$ and agree by its universal property. They therefore glue to a unique morphism $K \to Z$. Conversely, its components recover g and h. Thus $Z = X \times_S Y$. The same argument applies to an open cover of Y.
If S is affine, first take X affine and cover Y by affine opens. The affine fibre products exist by the tensor-product construction, so gluing in the Y-variable gives $X \times_S Y$. Covering an arbitrary X by affine opens and gluing in the X-variable then proves existence over every affine base.
Finally, cover a general S by affine opens S_α, and put $X_\alpha = p^{-1}(S_\alpha)$ and $Y_\alpha = q^{-1}(S_\alpha)$. The preceding case constructs $Z_\alpha = X_\alpha \times_{S_\alpha} Y_\alpha$. Over $S_\alpha \cap S_\beta$, the corresponding open subschemes of Z_α and Z_β both have the universal property of the fibre product over that intersection. They are therefore canonically isomorphic, with the cocycle conditions again following from uniqueness. Glue them to Z. For compatible maps $K \to X$ and $K \to Y$, their common composite $K \to S$ pulls back the cover (S_α) to an open cover of K. On each member there is a unique map to Z_α; these maps agree on overlaps and glue uniquely. Hence Z is the desired fibre product over S. Together with the terminal object and the finite-limit lemma, this proves the theorem. □

12. Projective space and Proj

12.1. Projective space. We now introduce the scheme $\mathbb{P}^n$, called n-dimensional projective space. Its $\mathbb{R}$-valued and $\mathbb{C}$-valued points are the classical real and complex projective spaces $\mathbb{P}^n(\mathbb{R})$ and $\mathbb{P}^n(\mathbb{C})$, usually denoted $\mathbb{RP}^n$ and $\mathbb{CP}^n$. Let us first recall their definition.

Definition 12.1. *Let k be a field, for example $\mathbb{R}$ or $\mathbb{C}$. Set*

$$\mathbb{P}^n(k) := (k^{n+1} \setminus \{0\})/\sim,$$

where the equivalence relation is given by

$$(x_0, ..., x_n) \sim \lambda(x_0, ..., x_n) \qquad \text{for } \lambda \in k^\times$$

This is the orbit relation for the action of $k^\times$ on $k^{n+1} \setminus \{0\}$ by scalar multiplication. Thus

$$\mathbb{P}^n(k) = \frac{k^{n+1} \setminus \{0\}}{k^\times} = \{\text{one-dimensional subspaces of } k^{n+1}\}.$$

We write an element as $(x_0 : \ldots : x_n)$ and call this a representation in homogeneous coordinates.

Example 12.2. Let us understand $\mathbb{P}^1(\mathbb{R}) = (\mathbb{R}^2 \setminus \{0\})/\mathbb{R}^\times$. It contains the subset

$$U_0 = \{(x_0 : x_1) \in \mathbb{P}^1(\mathbb{R}) \mid x_0 \neq 0\}$$

The map $\mathbb{R} \to U_0$ given by $x \mapsto (1 : x)$ is a bijection with inverse $(x_0 : x_1) \mapsto x_1/x_0$. Then we have that

$$\mathbb{P}^1(\mathbb{R}) = U_0 \amalg \{(0 : 1)\} \cong \mathbb{R} \amalg \{\infty\}$$

as sets. Thus we can obtain $\mathbb{P}^1(\mathbb{R})$ by adding the point $(0:1)$, called ∞, to $\mathbb{R}$. With the quotient topology, $\mathbb{P}^1(\mathbb{R})$ is homeomorphic to S^1. The other standard open set $U_1 = \{(x_0 : x_1) \mid x_1 \neq 0\}$ is also homeomorphic to $\mathbb{R}$, and these two opens cover $\mathbb{P}^1(\mathbb{R})$. The displayed disjoint union is a decomposition of sets, not the disjoint-union topology.

EXAMPLE 12.3. Similarly, $\mathbb{P}^2(\mathbb{R})$ has an open set

$$U_0 = \{(x_0 : x_1 : x_2) \mid x_0 \neq 0\} \cong \mathbb{R}^2$$

and we have that

$$\mathbb{P}^2(\mathbb{R}) \cong \mathbb{R}^2 \amalg \mathbb{P}^1(\mathbb{R})$$

as sets. We obtain the real projective plane by adding one point at infinity for each line direction in $\mathbb{R}^2$. The three standard open sets U_0, U_1, U_2 are homeomorphic to $\mathbb{R}^2$ and cover $\mathbb{P}^2(\mathbb{R})$.

EXAMPLE 12.4. In general we have that

$$\mathbb{P}^n(k) \cong k^n \amalg \mathbb{P}^{n-1}(k) \cong k^n \amalg k^{n-1} \amalg \cdots \amalg k^0$$

as sets. We can cover $\mathbb{P}^n(k)$ by charts

$$U_i = \{(x_0 : ... : x_n) \mid x_i \neq 0\} \cong k^n$$

where $(x_0 : ... : x_n) \mapsto (x_0/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_n/x_i)$. We define the functions

$$X_{s,i} : U_i \to k^n \xrightarrow{\mathrm{pr}_s} k$$

for $s \neq i$, given by $(x_0 : \ldots : x_n) \mapsto x_s/x_i$. Here the projection selects the coordinate indexed by $s \neq i$. We also set $X_{i,i} = 1$. On double intersections $U_i \cap U_j$ we have the equality

$$X_{s,i} = x_s/x_i = x_s/x_j \cdot (x_i/x_j)^{-1} = X_{s,j} X_{i,j}^{-1}$$

for all i, j, s. This description works over every field k.

Our goal is to construct a scheme $\mathbb{P}^n$ such that for every field the k-valued points are given by $\mathbb{P}^n(k)$ as defined above. We do this by gluing the standard affine charts, using Theorem 11.12.

DEFINITION 12.5. *For $i = 0, .., n$ we set*

$$U_i := \mathrm{Spec}\left(\frac{\mathbb{Z}[X_{0,i}, \ldots, X_{n,i}]}{X_{i,i} - 1}\right) \cong \mathbb{A}^n \ .$$

We have open subsets

$$U_{i,j} = D(X_{j,i}) \subseteq U_i$$

given by $\mathrm{Spec}(\mathbb{Z}[X_{0,i}, \ldots, X_{n,i}, X_{j,i}^{-1}]/(X_{i,i} - 1))$. *We define isomorphisms*

$$\varphi_{i,j} : U_{j,i} \xrightarrow{\simeq} U_{i,j}$$

whose coordinate-ring maps run in the opposite direction and are defined by

$$\varphi_{i,j}^\sharp(X_{s,i}) = X_{s,j} X_{i,j}^{-1}.$$

As before, $X_{j,j} = 1$. On the appropriate triple overlaps, the coordinate-ring maps satisfy

$$\begin{aligned}\varphi^\sharp_{j,k} \circ \varphi^\sharp_{i,j}(X_{s,i}) &= \varphi^\sharp_{j,k}(X_{s,j}X_{i,j}^{-1}) \\ &= \varphi^\sharp_{j,k}(X_{s,j})\varphi^\sharp_{j,k}(X_{i,j})^{-1} \\ &= (X_{s,k}X_{j,k}^{-1}) \cdot (X_{i,k}X_{j,k}^{-1})^{-1} \\ &= \varphi^\sharp_{i,k}(X_{s,i}).\end{aligned}$$

The same coordinate changes identify the required overlap domains. Thus the maps $\varphi_{j,i} : U_{i,j} \to U_{j,i}$ give gluing data in the convention of Theorem 11.12. The resulting scheme is denoted $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{Z}}$.

By construction, $\mathbb{P}^n$ has an open cover by the $U_i \cong \mathbb{A}^n_{\mathbb{Z}}$, and the copies $U_{i,j}$ and $U_{j,i}$ of each overlap are identified by $\varphi_{i,j}$.

12.2. The homogeneous spectrum: the construction of Proj**.** The affine charts of projective space suggest a more general construction. The coordinates X_s/X_i have degree zero if all X_i have degree one. Thus we should first invert a homogeneous coordinate, and then take the degree-zero part of the resulting ring. We now carry this out for an arbitrary nonnegatively graded ring; no finite generation assumption is needed for the construction.

Definition 12.6. *A nonnegatively graded ring is a commutative ring*

$$S = \bigoplus_{m \geq 0} S_m, \qquad 1 \in S_0, \qquad S_m S_n \subseteq S_{m+n}.$$

An element of S_m is called homogeneous of degree m. An ideal I is homogeneous if it is generated by homogeneous elements, or equivalently if it contains every homogeneous component of each of its elements. Set

$$S_+ = \bigoplus_{m > 0} S_m.$$

This is an ideal, called the irrelevant ideal. The homogeneous spectrum of S is the set

$$\operatorname{Proj}(S) = \{\mathfrak{p} \subset S \mid \mathfrak{p} \text{ is a homogeneous prime ideal and } S_+ \not\subseteq \mathfrak{p}\}.$$

For a homogeneous ideal $I \subseteq S$, put

$$V_+(I) = \{\mathfrak{p} \in \operatorname{Proj}(S) \mid I \subseteq \mathfrak{p}\}.$$

For a homogeneous element f of positive degree, put

$$D_+(f) = \{\mathfrak{p} \in \operatorname{Proj}(S) \mid f \notin \mathfrak{p}\}.$$

We give Proj(S) *the topology whose closed sets are the $V_+(I)$.*

These sets do define a topology: $V_+(0) = \operatorname{Proj}(S)$, $V_+(S) = \varnothing$,

$$\bigcap_\alpha V_+(I_\alpha) = V_+\left(\sum_\alpha I_\alpha\right), \qquad V_+(I) \cup V_+(J) = V_+(IJ),$$

where the last equality uses primality. The standard opens $D_+(f)$, with f homogeneous of positive degree, form a basis, and

$$D_+(f) \cap D_+(g) = D_+(fg).$$

To check the basis assertion even when I contains degree-zero elements, let $\mathfrak{p} \notin V_+(I)$. There is a homogeneous $a \in I \setminus \mathfrak{p}$. If a has positive degree, use $D_+(a)$. If a has degree zero, choose a homogeneous $h \in S_+ \setminus \mathfrak{p}$ and use $D_+(ah)$. In both cases this is a standard open neighbourhood of $\mathfrak{p}$ contained in $\operatorname{Proj}(S) \setminus V_+(I)$.
Let $f \in S_d$ with $d > 0$. Its localization has a $\mathbb{Z}$-grading

$$S_f = S[f^{-1}], \qquad \deg(s/f^r) = \deg(s) - rd$$

for homogeneous s. We write

$$S_{(f)} := (S_f)_0 = \{s/f^r \mid r \geq 0,\ s \in S_{rd}\}.$$

Here the displayed fractions include all elements of degree zero: a finite sum of such fractions can be put over a common denominator. The parentheses in $S_{(f)}$ distinguish taking degree zero from the full localization S_f.

Proposition 12.7 (The affine charts of Proj). *There is a canonical scheme structure on* $\operatorname{Proj}(S)$ *with*

$$D_+(f) \cong \operatorname{Spec}(S_{(f)})$$

for every homogeneous f of positive degree. Under this isomorphism, the point $\mathfrak{p}$ corresponds to

$$(\mathfrak{p}S_f) \cap S_{(f)}.$$

In particular,

$$\mathcal{O}_{\operatorname{Proj}(S)}(D_+(f)) = S_{(f)}.$$

The structure is compatible with localization on overlaps. More explicitly, for $f \in S_d$ and $g \in S_e$, with $d, e > 0$, the overlap $D_+(fg)$ corresponds in $\operatorname{Spec}(S_{(f)})$ *to*

$$D\left(\frac{g^d}{f^e}\right), \qquad \left(S_{(f)}\right)_{g^d/f^e} \cong S_{(fg)}.$$

Proof. We first establish the correspondence of prime ideals, including when $d > 1$. We use two elementary facts about a $\mathbb{Z}$-graded ring T. First, the radical of a homogeneous ideal J is homogeneous. Indeed, if $a^N \in J$, the highest-degree homogeneous component a_m of a satisfies $a_m^N \in J$. Thus $a_m \in \sqrt{J}$; subtracting a_m and repeating proves that every component of a belongs to $\sqrt{J}$. Second, a proper homogeneous ideal P is prime if primality holds for products of homogeneous elements. To see this, pass to the graded quotient T/P. The highest nonzero components of two nonzero elements have nonzero product under this hypothesis. Hence their product is nonzero, so T/P is a domain.
Now take $T = S_f$ and $B = T_0 = S_{(f)}$. If f is nilpotent, then both $D_+(f)$ and $\operatorname{Spec}(B)$ are empty, so there is nothing to prove. Otherwise, localization gives a bijection between the homogeneous primes of S avoiding f and the homogeneous primes of T. Every homogeneous prime of S avoiding f lies in $\operatorname{Proj}(S)$.
Let $\mathfrak{q} \subset B$ be a prime ideal. Since $\mathfrak{q}T$ is homogeneous and its degree-zero part is $\mathfrak{q}$, the homogeneous ideal

$$P = \sqrt{\mathfrak{q}T}$$

satisfies $P \cap B = \mathfrak{q}$ and is proper. We claim that P is prime. For a homogeneous $t \in T_m$, where m may be negative, the element

$$c(t) := t^d f^{-m}$$

has degree zero. Here integer powers of f make sense because f is a unit in T. If homogeneous elements $u \in T_m$ and $v \in T_n$ satisfy $uv \in P$, then

$$c(u)c(v) = c(uv) \in P \cap B = \mathfrak{q}.$$

Consequently $c(u) \in \mathfrak{q}$ or $c(v) \in \mathfrak{q}$. In the first case $u^d = f^m c(u) \in P$, hence $u \in P$; in the second case $v \in P$. The homogeneous primality criterion proves the claim.
Conversely, if $P' \subset T$ is a homogeneous prime with $P' \cap B = \mathfrak{q}$, then for every homogeneous $t \in T_m$,

$$t \in P' \quad \Longleftrightarrow \quad t^d f^{-m} \in \mathfrak{q}.$$

The forward implication uses homogeneity of degrees and the fact that f is a unit; the reverse implication uses primality. This condition determines P' uniquely from $\mathfrak{q}$. Thus extension to T followed by contraction to B gives the claimed bijection

$$D_+(f) \longrightarrow \operatorname{Spec}(B).$$

We next check the topology. If $g \in S_e$ is homogeneous of positive degree, the displayed membership criterion identifies $D_+(f) \cap D_+(g)$ with $D(g^d/f^e)$ in $\operatorname{Spec}(B)$. Conversely, any $a \in B$ can be written s/f^r with $s \in S_{rd}$; the inverse image of $D(a)$ is $D_+(sf) \subseteq D_+(f)$. Since standard opens form bases on both sides, the bijection is a homeomorphism.
Finally, set $a = g^d/f^e \in T_0$. Inverting a in T is equivalent to inverting g: f is already a unit, and g is a unit exactly when g^d is a unit. Taking the degree-zero parts therefore gives canonical isomorphisms

$$B_a = (T_a)_0 \cong \left(S[f^{-1}, g^{-1}]\right)_0 = S_{(fg)}.$$

The first equality holds because a has degree zero. These are exactly the coordinate-ring identifications of the stated open subsets. The analogous identifications from the g-chart agree with them, and the cocycle condition on a triple overlap follows by localizing all three homogeneous elements in the same ring.
The gluing theorem for schemes now glues the affine schemes $\operatorname{Spec}(S_{(f)})$ along these open isomorphisms. Their underlying topological spaces glue to the already defined space $\operatorname{Proj}(S)$. Their affine structure sheaves glue to a sheaf $\mathcal{O}_{\operatorname{Proj}(S)}$, whose restriction to each $D_+(f)$ is its affine structure sheaf. This proves every asserted chart and section formula. It also proves uniqueness of the scheme structure compatible with these charts and their localization maps. □

PROPOSITION 12.8. *There is a canonical morphism*

$$\operatorname{Proj}(S) \longrightarrow \operatorname{Spec}(S_0), \qquad \mathfrak{p} \longmapsto \mathfrak{p} \cap S_0.$$

If S is generated over S_0 by finitely many elements $s_0, \dots, s_n$ of degree one, then the $D_+(s_i)$ form a finite affine cover. Each chart ring $S_{(s_i)}$ is generated over S_0 by the fractions s_j/s_i. In particular, $\operatorname{Proj}(S)$ *is quasi-compact in this case.*

PROOF. The maps $S_0 \to S_{(f)}$, $a \mapsto a/1$, are compatible with all the localization maps on overlaps. Their affine morphisms therefore glue to the asserted morphism. The formula on points follows by contracting a prime through $S_0 \to S_f \leftarrow S_{(f)}$.
Under the finite-generation hypothesis the ideal S_+ is generated by $s_0, \dots, s_n$. A point of $\operatorname{Proj}(S)$ therefore avoids at least one s_i, which proves the covering assertion. Any homogeneous element of degree r is a homogeneous polynomial of degree r in the s_j, with coefficients in S_0. Dividing by s_i^r expresses every element of $S_{(s_i)}$ as a polynomial in the s_j/s_i. Finally, each affine chart is quasi-compact, and a finite union of quasi-compact open subspaces is quasi-compact. □

PROPOSITION 12.9. *Let A be any commutative ring. Give $A[X_0, \dots, X_n]$ the grading with A in degree zero and all X_i in degree one. There is a canonical isomorphism over* $\operatorname{Spec}(A)$

$$\operatorname{Proj}\big(A[X_0, \dots, X_n]\big) \cong \mathbb{P}^n_A,$$

where $\mathbb{P}^n_A$ is obtained by the projective-space chart construction above with A in place of $\mathbb{Z}$. It identifies $D_+(X_i)$ with U_i.

PROOF. The opens $D_+(X_i)$ cover, and their coordinate rings are

$$\big(A[X_0, \dots, X_n, X_i^{-1}]\big)_0 = A\left[\frac{X_0}{X_i}, \dots, \frac{\widehat{X_i}}{X_i}, \dots, \frac{X_n}{X_i}\right].$$

This is a polynomial ring in n variables over A. To see both generation and independence, introduce variables z_j for $j \neq i$: the substitutions $z_j \mapsto X_j/X_i$ and $X_j \mapsto z_j X_i$ give inverse isomorphisms

$$A[z_j \mid j \neq i][X_i, X_i^{-1}] \cong A[X_0, \dots, X_n, X_i^{-1}],$$

where the z_j have degree zero. On an overlap the fractions satisfy

$$\frac{X_s}{X_i} = \frac{X_s}{X_j}\left(\frac{X_i}{X_j}\right)^{-1}.$$

These are precisely the transition maps used to construct $\mathbb{P}^n_A$. Hence the affine isomorphisms glue to the asserted isomorphism, and they preserve the coefficient maps from A, so it is an isomorphism over $\operatorname{Spec}(A)$. □

EXAMPLE 12.10 (Homogeneous equations and affine charts). Let I be a homogeneous ideal of $A[X_0, \dots, X_n]$, and let

$$S = A[X_0, \dots, X_n]/I.$$

Write x_i for the image of X_i in S. The standard opens $D_+(x_i)$ cover $\operatorname{Proj}(S)$. For a homogeneous polynomial F, let

$$F_i(z_j \mid j \neq i) = F(z_0, \dots, z_{i-1}, 1, z_{i+1}, \dots, z_n).$$

Then

$$D_+(x_i) \cong \operatorname{Spec}\left(\frac{A[z_j \mid j \neq i]}{(F_i \mid F \in I \text{ homogeneous})}\right).$$

Indeed, in the Laurent-polynomial description from the preceding proof, a homogeneous F of degree r becomes $X_i^r F_i$. Since X_i is a unit, the localized ideal is generated by the degree-zero polynomials F_i. Taking degree zero therefore gives exactly the displayed quotient. If I is generated by homogeneous $F_1, \dots, F_t$, it suffices to use their dehomogenizations.

Taking inverse images under the quotient map identifies homogeneous prime ideals of S with homogeneous prime ideals of the polynomial ring containing I. The condition of avoiding the irrelevant ideal is preserved. Since containment of homogeneous ideals is also preserved, this is a homeomorphism

$$\operatorname{Proj}(S) \cong V_+(I) \subseteq \mathbb{P}^n_A$$

of underlying topological spaces. The quotient chart rings describe the additional scheme structure carried by these homogeneous equations, including any nilpotent elements.

EXAMPLE 12.11 (A projective conic). Let k be a field, and set

$$C = \operatorname{Proj}\big(k[X,Y,Z]/(XZ - Y^2)\big), \qquad \deg X = \deg Y = \deg Z = 1.$$

We use the same letters for the images in the quotient. The opens $D_+(X)$ and $D_+(Z)$ cover C: if a homogeneous prime contained both X and Z, then it would contain $Y^2 = XZ$, hence Y, and therefore the entire irrelevant ideal.
On the X-chart put $u = Y/X$ and $v = Z/X$. The equation becomes $v = u^2$, so

$$D_+(X) = \operatorname{Spec}\big(k[u,v]/(v - u^2)\big) \cong \operatorname{Spec}(k[u]).$$

On the Z-chart put $w = Y/Z$ and $r = X/Z$. Here $r = w^2$, giving

$$D_+(Z) = \operatorname{Spec}\big(k[w,r]/(r - w^2)\big) \cong \operatorname{Spec}(k[w]).$$

The intersection is $D(v) = D(u^2) = D(u)$ on the first chart and $D(r) = D(w)$ on the second. On it,

$$w = \frac{Y}{Z} = \frac{Y/X}{Z/X} = \frac{u}{u^2} = u^{-1}.$$

Thus C is obtained by gluing two affine lines along their punctured lines using $w = u^{-1}$, exactly as in the construction of $\mathbb{P}^1_k$. The two chart identifications glue to an isomorphism

$$\mathbb{P}^1_k \cong C.$$

On field-valued points it is given by

$$(a : b) \longmapsto (a^2 : ab : b^2).$$

Indeed, on the chart $a \neq 0$ its coordinates are $u = b/a$ and $v = (b/a)^2$, and on the chart $b \neq 0$ they are $w = a/b$ and $r = (a/b)^2$. These are the chart maps just constructed. The argument works in every characteristic.

The construction of Proj gives projective space and homogeneous equations a common scheme-theoretic description. We next relate its points to homogeneous coordinates.
For a field k, the scheme just constructed has the expected points:

$$\operatorname{Hom}(\operatorname{Spec} k, \mathbb{P}^n) \cong (k^{n+1} \setminus \{0\})/k^\times.$$

Indeed $\operatorname{Spec} k$ has one point, so every morphism factors through a chart U_i. Its affine coordinates give the unique homogeneous representative with i-th coordinate 1. The chart transitions identify precisely the representatives that differ by a nonzero scalar. Over an arbitrary ring R, homogeneous coordinates require more care: a map $\operatorname{Spec} R \to \mathbb{P}^n$ need not factor through any single standard chart, and the naive quotient $(R^{n+1} \setminus \{0\})/R^\times$ does not describe its points. We introduce the required algebraic notion now, and prove the resulting description in the line-bundle section. Proposition 15.4 will also describe maps to $\operatorname{Proj}(A[X_0,\dots,X_n]/I)$ by generating sections of a line bundle satisfying the homogeneous relations in I.

12.3. An outlook on ring-valued points.

DEFINITION 12.12. *An R-module L is* invertible *if there is an R-module L' with $L \otimes_R L' \cong R$. Over a field this is equivalent to being one-dimensional. A submodule $L \subseteq R^{n+1}$ is* complementable *if its inclusion admits an R-linear retraction.*

EXAMPLE 12.13. Invertibility is preserved by extension of scalars: if $R \to B$, then $(L \otimes_R B) \otimes_B (L' \otimes_R B) \cong B$. A split inclusion remains split after extension of scalars, because one tensors its retraction as well.

The right replacement for a line in a vector space is a complementable invertible submodule. In Corollary 14.29 we will prove, for every ring R, the natural bijection

$$\mathbb{P}^n(R) \cong \{L \subseteq R^{n+1} \mid L \text{ invertible and complementable}\}.$$

We defer the proof until line bundles and descent for modules have been developed. This avoids using local triviality or descent before proving them.

13. Quasi-coherent sheaves and descent

13.1. Nakayama's lemma and generators near a point. Recall the Jacobson radical $\operatorname{rad}(R) = \bigcap_{\mathfrak{m} \text{ maximal}} \mathfrak{m}$. For a local ring $(R, \mathfrak{m})$ it is $\mathfrak{m}$. The following algebraic statements are prerequisites; see [**AM**, 2.6–2.8].

THEOREM 13.1 (Nakayama's lemma). *Let M be a finitely generated R-module and $I \subseteq \operatorname{rad}(R)$ an ideal. If $M/IM = 0$, then $M = 0$.*

COROLLARY 13.2. *Under the hypotheses of Nakayama's lemma, suppose $m_1, \dots, m_r \in M$. If their images generate M/IM over R/I, then $m_1, \dots, m_r$ generate M.*

PROOF. Apply Nakayama's lemma to the finitely generated quotient $M/(Rm_1 + \cdots + Rm_r)$. □

REMARK 13.3. Finite generation matters: $\mathbb{Q}$ is a nonzero $\mathbb{Z}_{(2)}$-module although $\mathbb{Q}/2\mathbb{Q} = 0$.

We now spell out the geometric consequence which we will use for vector bundles: generators in a fibre remain generators on a neighbourhood.

COROLLARY 13.4 (Generators near a point). *Let M be a finitely generated R-module, $\mathfrak{p} \in \operatorname{Spec} R$, and $m_1, \dots, m_r \in M$. If their images generate $M \otimes_R \kappa(\mathfrak{p})$, then there is $f \notin \mathfrak{p}$ such that they generate $M[1/f]$ over $R[1/f]$.*

PROOF. Put $Q = M/(Rm_1 + \cdots + Rm_r)$. Its localization $Q_\mathfrak{p}$ is a finitely generated module over the local ring $R_\mathfrak{p}$, and its quotient by the maximal ideal is zero. Nakayama gives $Q_\mathfrak{p} = 0$. Choose finitely many generators $q_1, \dots, q_s$ of Q. Each becomes zero after localization at $\mathfrak{p}$, so $f_i q_i = 0$ for some $f_i \notin \mathfrak{p}$. With $f = \prod_i f_i$ we have $Q[1/f] = 0$, which is the required assertion. □

13.2. Modules and quasi-coherent sheaves. Schemes are a global geometric version of rings. We now introduce the corresponding geometric generalization of modules. For an R-module M and $f \in R$, we can functorially form

$$M[1/f] = M \otimes_R R[1/f],$$

which is an $R[1/f]$-module (and will sometimes also be considered as an R-module).

PROPOSITION 13.5. *For every R-module M, there is a sheaf of abelian groups $\widetilde{M} \in \operatorname{Shv}(\operatorname{Spec}(R); \mathrm{Ab})$ given on principal opens by*

$$D(f) \mapsto M[1/f] = M \otimes_R R[1/f] \ .$$

PROOF. If $D(g) \subseteq D(f)$, then f is invertible in $R[1/g]$. The localization map $R[1/f] \to R[1/g]$ therefore induces the restriction $M[1/f] \to M[1/g]$. These maps compose correctly by the universal property of localization, and depend only on the principal opens.

For the sheaf property, localize first at f to reduce a cover of $D(f)$ to a cover of an affine spectrum. Principal opens are quasi-compact, so it suffices to consider a finite cover $D(f_1), \dots, D(f_n)$, where $(f_1, \dots, f_n) = R$. We must show that

$$M \to \prod M[1/f_i] \rightrightarrows \prod M[1/f_i, 1/f_j]$$

is an equalizer. The clearing-denominators argument in the proof of Theorem 6.11 applies verbatim to elements of M: injectivity follows by multiplying an element killed locally by a relation $1 = \sum_i g_i f_i^{n_i}$; compatible local elements are glued by the same formula $\sum_i g_i a_i$ after clearing denominators. This verifies the sheaf property on the basis and hence constructs $\widetilde{M}$. □

The stalks of $\widetilde{M}$ are given by

$$\widetilde{M}_x = \operatorname{colim}_{f \notin \mathfrak{p}} M[1/f] = M[(R \setminus \mathfrak{p})^{-1}] = M_{\mathfrak{p}},$$

where $\mathfrak{p}$ is the prime ideal corresponding to x.

Definition 13.6. *Let $(X, \mathcal{O}_X)$ be a ringed space. A sheaf of $\mathcal{O}_X$-modules is a sheaf of abelian groups $\mathcal{M}$ together with a morphism of sheaves of sets*

$$\mathcal{O}_X \times \mathcal{M} \to \mathcal{M}$$

such that for every open U the induced map $\mathcal{O}_X(U) \times \mathcal{M}(U) \to \mathcal{M}(U)$ makes $\mathcal{M}(U)$ into an $\mathcal{O}_X(U)$-module. Morphisms of $\mathcal{O}_X$-modules are morphisms of sheaves of abelian groups commuting with the action. Their set is denoted by

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{N}) \ .$$

We denote the category of sheaves of $\mathcal{O}_X$-modules by $\operatorname{Mod}(\mathcal{O}_X)$.

Thus each $\mathcal{M}(U)$ is an $\mathcal{O}_X(U)$-module, compatibly with restrictions.

Example 13.7. For readers familiar with vector bundles, let $p : \mathcal{V} \to X$ be a real topological vector bundle. Its total space $\mathcal{V}$ is a topological space, and each fibre $\mathcal{V}_x = p^{-1}(x)$ is an $\mathbb{R}$-vector space. Its sheaf of continuous sections is

$$U \longmapsto \{s : U \to \mathcal{V}|_U \mid s \text{ continuous and } p \circ s = \mathrm{id}_U\}.$$

This is a sheaf of modules over the sheaf C^0_X of continuous real-valued functions, by pointwise scalar multiplication. The vector bundle can be recovered up to isomorphism from this sheaf, but we will not pursue this here.

Proposition 13.8. *For any R-module M the sheaf $\widetilde{M}$ is canonically a sheaf of $\mathcal{O}_{\operatorname{Spec}(R)}$-modules.*

Proof. As usual by Lemma 6.10 it suffices to define everything on principal opens. The action on $U = D(f)$ is

$$R[1/f] \times M[1/f] \to M[1/f]$$

and commutes with restriction because localization respects scalar multiplication. It therefore extends to the required action on the sheaf. □

We now prove an analogue of Theorem 9.2, which recovers an affine scheme from its global sections. For an associated module sheaf, we already have $\Gamma(\operatorname{Spec}(R), \widetilde{M}) = M$.

PROPOSITION 13.9. *Let M be an R-module for some ring R and $\mathcal{N}$ be a sheaf of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules on* $\mathrm{Spec}(R)$. *Then the map*

$$\Gamma : \mathrm{Hom}_{\mathcal{O}_{\mathrm{Spec}(R)}}(\widetilde{M}, \mathcal{N}) \to \mathrm{Hom}_R\left(M, \mathcal{N}(\mathrm{Spec}(R))\right)$$

is a bijection.

PROOF. For injectivity, suppose $\varphi, \psi : \widetilde{M} \to \mathcal{N}$ agree on global sections. It suffices to show that they agree on principal opens, by Lemma 6.10. We have a commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & M[1/f] \\ \downarrow{\scriptstyle \varphi=\psi} & & \varphi \Big\downarrow\!\!\Big\downarrow \psi \\ \mathcal{N}(\mathrm{Spec}(R)) & \longrightarrow & \mathcal{N}(D(f)) \end{array}$$

The right vertical maps are $R[1/f]$-linear. For every $R[1/f]$-module N, the universal property of localization gives a bijection

$$\mathrm{Hom}_{R[1/f]}(M[1/f], N) \to \mathrm{Hom}_R(M, N) \tag{7}$$

whose inverse sends an R-linear map $a : M \to N$ to $m/f^r \mapsto f^{-r}a(m)$. Taking $N = \mathcal{N}(D(f))$ shows that the right vertical maps are determined by their composites with M, and hence agree. This proves injectivity.
For surjectivity, start with an R-linear map $\varphi : M \to \mathcal{N}(\mathrm{Spec}(R))$. For each $f \in R$ the same universal property gives the unique $R[1/f]$-linear dashed arrow

$$\begin{array}{ccc} M & \longrightarrow & M[1/f] \\ \downarrow{\scriptstyle \varphi} & & \vdots \\ \mathcal{N}(\mathrm{Spec}(R)) & \longrightarrow & \mathcal{N}(D(f)) \end{array}$$

making the square commute. Uniqueness in (7) makes these arrows compatible with restrictions. They therefore define a morphism $\widetilde{M} \to \mathcal{N}$ with the prescribed map on global sections. □

COROLLARY 13.10. *The functor*

$$\mathrm{Mod}_R \to \mathrm{Mod}(\mathcal{O}_{\mathrm{Spec}(R)}) \qquad M \mapsto \widetilde{M}$$

from R-modules to sheaves of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules is fully faithful and left adjoint to the global sections functor.

We now identify the essential image of this functor and globalize it.

DEFINITION 13.11. *Let X be a scheme. An $\mathcal{O}_X$-module $\mathcal{M}$ is* quasi-coherent *if there is an affine open cover $U_i = \mathrm{Spec}(A_i)$ of X such that $\mathcal{M}|_{U_i} \cong \widetilde{M_i}$ as $\mathcal{O}_{U_i}$-modules for some A_i-module M_i. We denote the category of quasi-coherent sheaves by*

$$\mathrm{QCoh}(X) \subseteq \mathrm{Mod}(\mathcal{O}_X)$$

defined as a full subcategory of $\mathrm{Mod}(\mathcal{O}_X)$.

EXAMPLE 13.12. For every R-module M, the sheaf $\widetilde{M}$ is quasi-coherent: take the one-member affine cover. Any cover by principal opens also works, since

$$(\widetilde{M})|_{D(f)} \cong \widetilde{M[1/f]}$$

as sheaves on $D(f) = \mathrm{Spec}(R[1/f])$, with $M[1/f]$ regarded as an $R[1/f]$-module.

PROPOSITION 13.13. *Taking global sections induces an equivalence of categories*

$$\mathrm{QCoh}(\mathrm{Spec}(R)) \xrightarrow{\simeq} \mathrm{Mod}_R$$

with inverse given by $M \mapsto \widetilde{M}$.

PROOF. The functor $M \mapsto \widetilde{M}$ is fully faithful by Corollary 13.10. It remains to show that every quasi-coherent $\mathcal{O}_{\mathrm{Spec}(R)}$-module $\mathcal{M}$ is associated to $M = \mathcal{M}(\mathrm{Spec}(R))$. The counit of the adjunction is a morphism

$$\widetilde{M} \to \mathcal{M}$$

of $\mathcal{O}_{\mathrm{Spec}(R)}$-modules. It suffices to show that on each principal open $D(g)$ the induced map

$$M[1/g] = \widetilde{M}(D(g)) \to \mathcal{M}(D(g))$$

is an isomorphism. Choose an affine cover on which $\mathcal{M}$ is associated to modules. Refine it by principal opens $U_i = D(f_i)$ of $\mathrm{Spec}(R)$, and take a finite subcover. Each such $D(f_i)$ is also principal in the affine chart containing it, defined by the restriction of f_i. Thus $\mathcal{M}|_{U_i}$ is still associated to a module, now over $R[1/f_i]$. Write $M_i = \mathcal{M}(U_i)$. The restrictions give canonical identifications $\mathcal{M}(U_i \cap U_j) \cong M_i[1/f_j] \cong M_j[1/f_i]$; denote this module by $M_{i,j}$. By the sheaf property, M is the equalizer

$$\mathrm{Eq}\left(\prod_i M_i \rightrightarrows \prod_{i,j} M_{i,j}\right)$$

Localization is exact, so it commutes with equalizers, and it also commutes with finite products of modules, which are finite direct sums. Hence

$$\begin{aligned} M[1/g] &\cong \mathrm{Eq}\left(\prod_i M_i[1/g] \rightrightarrows \prod_{i,j} M_{i,j}[1/g]\right) \\ &\cong \mathrm{Eq}\left(\prod_i \mathcal{M}(D(gf_i)) \rightrightarrows \prod_{i,j} \mathcal{M}(D(gf_if_j))\right) \\ &\cong \mathcal{M}(D(g)) \end{aligned}$$

The last isomorphism uses the sheaf property for the cover $(D(gf_i))$ of $D(g)$, whose pairwise intersections are $D(gf_if_j)$. All maps are induced by restriction and localization, so the resulting isomorphism is the map $M[1/g] \to \mathcal{M}(D(g))$ above. This proves the claim. □

13.3. Descent for modules. The sheaf condition glues sections. We now glue sheaves and module sheaves themselves. Let $X = \bigcup_i U_i$ and put $U_{ij} = U_i \cap U_j$.

DEFINITION 13.14. *A* descent datum *for sheaves with values in a category* $\mathcal{C}$ *consists of sheaves* F_i *on* U_i *and isomorphisms* $\varphi_{ij} : F_i|_{U_{ij}} \to F_j|_{U_{ij}}$ *such that* $\varphi_{ii} = \mathrm{id}$ *and* $\varphi_{jk}\varphi_{ij} = \varphi_{ik}$ *on every triple intersection. Morphisms are families* $f_i : F_i \to G_i$ *commuting with these isomorphisms. We denote this category by* $\mathrm{Desc}_{\mathcal{U}}(\mathrm{Shv}(-;\mathcal{C}))$.

PROPOSITION 13.15 (Descent for sheaves). *If* $\mathcal{C}$ *has small limits, restriction induces an equivalence*

$$\mathrm{Shv}(X;\mathcal{C}) \simeq \mathrm{Desc}_{\mathcal{U}}(\mathrm{Shv}(-;\mathcal{C})).$$

PROOF. Given a descent datum, define $F(V)$ as the equalizer of

$$\prod_i F_i(V \cap U_i) \rightrightarrows \prod_{i,j} F_j(V \cap U_{ij}),$$

where one arrow restricts the j-th component and the other restricts the i-th component and applies φ_{ij}. Restrictions in V make F a presheaf. It is a sheaf: gluing compatible sections in each F_i produces sections still satisfying the equalizer relations; in an arbitrary complete category the same argument is expressed by the fact that limits commute with limits.
On an open $V \subseteq U_i$, a section of $F_i(V)$ determines the compatible family on $V \cap U_j$ by φ_{ij}, and the cocycle condition makes these assignments inverse to projection onto the i-th component. Thus $F|_{U_i} \cong F_i$. Compatible morphisms glue componentwise by the same formula. Conversely, for a sheaf already on X, the sheaf axiom identifies it with this equalizer. These identifications prove both full faithfulness and essential surjectivity. For an arbitrary complete category $\mathcal{C}$, apply $\mathrm{Hom}_{\mathcal{C}}(C, -)$ for each $C \in \mathcal{C}$ to these constructions. These functors preserve limits, so the elementwise arguments apply to the resulting set-valued sheaves. Yoneda then gives the stated identifications in $\mathcal{C}$. □

PROPOSITION 13.16. *For a scheme X with an open cover $\mathcal{U}$, restriction gives equivalences*

$$\mathrm{Mod}(\mathcal{O}_X) \simeq \mathrm{Desc}_{\mathcal{U}}(\mathrm{Mod}(\mathcal{O}_-)), \qquad \mathrm{QCoh}(X) \simeq \mathrm{Desc}_{\mathcal{U}}(\mathrm{QCoh}(-)).$$

PROOF. Glue the underlying sheaves of abelian groups by Proposition 13.15. The $\mathcal{O}_X$-action is defined componentwise and is compatible with the transition maps, so it descends to the glued sheaf. Module morphisms glue as well. Quasi-coherence is local by definition, which gives the second equivalence. □

COROLLARY 13.17 (Descent for modules). *Let* $\operatorname{Spec} R = \bigcup_i D(f_i)$, $R_i = R[1/f_i]$, *and* $R_{ij} = R[1/(f_i f_j)]$. *An R-module is equivalent to modules M_i over R_i and isomorphisms*

$$M_i \otimes_{R_i} R_{ij} \xrightarrow{\sim} M_j \otimes_{R_j} R_{ij}$$

satisfying the cocycle condition. Morphisms are compatible families of module homomorphisms. The cocycle condition is imposed after further extension of scalars to $R[1/(f_i f_j f_k)]$, and the isomorphism for $i = j$ is the identity.

PROOF. Apply the quasi-coherent descent equivalence and the equivalence between modules and quasi-coherent sheaves on each affine scheme. Restriction to $D(f_j) \subseteq \operatorname{Spec} R_i$ corresponds to localization at f_j, so the displayed algebraic datum is exactly the sheaf-theoretic descent datum. □

Tensor products are compatible with these transition maps: the descent datum for $M \otimes_R N$ is obtained from the canonical isomorphisms

$$(M \otimes_R N) \otimes_R R_i \cong (M \otimes_R R_i) \otimes_{R_i} (N \otimes_R R_i)$$

by tensoring the local transition maps over R_{ij} on overlaps. We will apply this principle to line bundles and subbundles in the next section.

14. Line bundles and vector bundles

We have seen that quasi-coherent sheaves are the geometric counterparts of modules. In this section we study the geometric counterparts of invertible modules and finitely generated projective modules. We also complete our description of maps to projective space. Twisting sheaves and the Picard group of projective space will be studied in Section 15. We use the universal property, canonical isomorphisms, and right exactness of tensor products from [**AM**, Propositions 2.12, 2.14, and 2.18].

14.1. Tensor products and invertible sheaves.

DEFINITION 14.1. *Let X be a scheme and $\mathcal{M}, \mathcal{N}$ be $\mathcal{O}_X$-modules. Their tensor product $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N}$ is the sheafification of the presheaf*

$$U \longmapsto \mathcal{M}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{N}(U).$$

PROPOSITION 14.2. *For R-modules M, N there is a natural isomorphism*

$$\widetilde{M} \otimes_{\mathcal{O}_{\mathrm{Spec}(R)}} \widetilde{N} \cong \widetilde{M \otimes_R N}.$$

Consequently the tensor product of two quasi-coherent sheaves on a scheme is quasi-coherent. On stalks the tensor product is given by

$$(\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{N})_x \cong \mathcal{M}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{N}_x.$$

PROOF. On a principal open $D(f) \subseteq \mathrm{Spec}(R)$ we have

$$M[1/f] \otimes_{R[1/f]} N[1/f] \cong (M \otimes_R N)[1/f].$$

This localization identity is [**AM**, Proposition 3.7]. These isomorphisms commute with restrictions and identify the presheaf on the principal-open basis with the sheaf associated to $M \otimes_R N$. They therefore give the asserted sheaf isomorphism. Quasi-coherence can be checked on an affine open cover, so the second assertion follows. For the stalk formula, sheafification leaves stalks unchanged and filtered colimits commute with tensor products. Taking the colimit over neighbourhoods of x thus gives the displayed formula. □

DEFINITION 14.3. *An $\mathcal{O}_X$-module $\mathcal{L}$ is called invertible, or a line bundle, if there exists an $\mathcal{O}_X$-module $\mathcal{L}'$ with*

$$\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}' \cong \mathcal{O}_X.$$

We write $\mathcal{L}^{-1}$ for an inverse. The line bundle $\mathcal{O}_X$ is called trivial.

An inverse is unique up to isomorphism: if both $\mathcal{L}'$ and $\mathcal{L}''$ are inverses, then

$$\mathcal{L}' \cong \mathcal{L}' \otimes (\mathcal{L} \otimes \mathcal{L}'') \cong (\mathcal{L}' \otimes \mathcal{L}) \otimes \mathcal{L}'' \cong \mathcal{L}''.$$

Here and below unlabelled tensor products of sheaves are over the structure sheaf. Tensoring with an invertible sheaf is an equivalence, with inverse given by tensoring with its inverse. In particular this equivalence preserves and detects epimorphisms.

14.2. Local triviality of line bundles.

DEFINITION 14.4. *A morphism $\pi : M \to N$ in a category is an epimorphism if $f\pi = g\pi$ implies $f = g$ for every pair of morphisms $f, g : N \to P$.*

EXAMPLE 14.5. Every morphism admitting a section is an epimorphism. In the category of sets the epimorphisms are precisely the surjections: if a point $n \in N$ is not in the image, the constant zero map $N \to \{0,1\}$ and the characteristic function of $\{n\}$ agree after composition with π but are different. For modules, surjections are epimorphisms, and if $M \to N$ is not surjective then the quotient map and zero map

$$N \longrightarrow N/\operatorname{im}(M)$$

witness that it is not an epimorphism. In the category of rings epimorphisms need not be surjective: $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism because a ring map out of $\mathbb{Q}$ is uniquely determined by its restriction to $\mathbb{Z}$.

LEMMA 14.6. *A morphism $\pi : M \to N$ is an epimorphism if and only if*

$$\begin{array}{ccc} M & \xrightarrow{\pi} & N \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\mathrm{id}} \\ N & \xrightarrow[\mathrm{id}]{} & N \end{array}$$

is a pushout. Thus left adjoints preserve epimorphisms, and equivalences preserve and detect them.

PROOF. The pushout property says that whenever $f, g : N \to P$ agree after composition with π, there is a unique $h : N \to P$ with $h = f$ and $h = g$. This is precisely the epimorphism condition. Left adjoints preserve pushouts, giving the next assertion. Apply it to an equivalence and its inverse for the final assertion. □

LEMMA 14.7. *A morphism of $\mathcal{O}_X$-module sheaves is an epimorphism if and only if it is surjective on every stalk. On an affine scheme a morphism $\widetilde{M} \to \widetilde{N}$ is an epimorphism if and only if $M \to N$ is surjective. In particular taking global sections of quasi-coherent sheaves on an affine scheme preserves short exact sequences.*

PROOF. The cokernel of a morphism of module sheaves is the sheafification of its sectionwise cokernel. A morphism is an epimorphism precisely when this cokernel is zero. Its stalk at x is the cokernel of the map on stalks, so vanishing is equivalent to surjectivity on every stalk. For quasi-coherent sheaves on $\operatorname{Spec}(R)$, exactness of localization [**AM**, Proposition 3.3] identifies the cokernel with $\widetilde{\operatorname{coker}(M \to N)}$. This is zero precisely when its module of global sections, $\operatorname{coker}(M \to N)$, is zero. Kernels are likewise computed by localizing the module kernel. The affine equivalence between modules and quasi-coherent sheaves therefore preserves exact sequences in both directions. The corresponding local criteria for module maps are [**AM**, Propositions 3.8 and 3.9]. □

WARNING 14.8. *An epimorphism of sheaves need not be surjective on sections over every open set. For example, the exponential map from the sheaf of continuous $\mathbb{C}$-valued functions to the sheaf of continuous $\mathbb{C}^\times$-valued functions is surjective on stalks because local logarithms exist. On $\mathbb{C}^\times$ the identity function has no continuous logarithm, so the map on global sections is not surjective. The additional quasi-coherence and affineness hypotheses in the preceding lemma are essential.*

DEFINITION 14.9. *An $\mathcal{O}_X$-module $\mathcal{M}$ is of finite type if every point has an open neighbourhood U admitting an epimorphism $\mathcal{O}_U^n \to \mathcal{M}|_U$ for some finite n.*

A map $\mathcal{O}_U \to \mathcal{M}|_U$ is the same as an element of $\mathcal{M}(U)$, namely the image of 1. Thus a map $\mathcal{O}_U^n \to \mathcal{M}|_U$ is an n-tuple of sections, and it is an epimorphism precisely when their germs generate each stalk over U.

PROPOSITION 14.10. *For a quasi-coherent sheaf $\mathcal{M}$ on $\operatorname{Spec}(R)$, being of finite type is equivalent to finite generation of $\mathcal{M}(\operatorname{Spec}(R))$ as an R-module. On an arbitrary scheme, a quasi-coherent sheaf is of finite type if and only if every point has an affine neighbourhood U on which $\mathcal{M}(U)$ is finitely generated over $\mathcal{O}_X(U)$.*

PROOF. Write $M = \mathcal{M}(\operatorname{Spec}(R))$. If M is generated by $m_1, \ldots, m_n$, the corresponding map $\mathcal{O}^n \to \widetilde{M}$ is an epimorphism by Lemma 14.7. Conversely, choose a finite principal-open cover $D(f_i)$ on which $\mathcal{M}$ has finitely many generating sections. Each such section lies in $M[1/f_i]$. Clearing its denominator, without changing the module it generates after localization, replaces it by an element $m_{ij} \in M$. Let $M' \subseteq M$ be generated by the finitely many m_{ij}. Then $(M/M')[1/f_i] = 0$ for every i, so the sheaf $\widetilde{M/M'}$ vanishes on a cover and is zero. Taking global sections gives $M/M' = 0$. This proves the affine assertion. Restricting it to affine neighbourhoods proves the second assertion. □

PROPOSITION 14.11. *Every line bundle is of finite type.*

PROOF. Choose an isomorphism $\psi : \mathcal{L} \otimes \mathcal{L}^{-1} \to \mathcal{O}_X$. Around any point, the inverse image of 1 is represented by a finite sum $\sum_i s_i \otimes t_i$, with $s_i \in \mathcal{L}(U)$ and $t_i \in \mathcal{L}^{-1}(U)$. After shrinking U this sum maps to 1. The map $\mathcal{O}_U^n \to \mathcal{L}|_U$ determined by the s_i becomes an epimorphism after tensoring with $\mathcal{L}^{-1}$, since its image then contains 1. Tensoring with the inverse is an equivalence, so the original map is an epimorphism too. □

LEMMA 14.12. *Every epimorphism of $\mathcal{O}_X$-modules $\mathcal{M} \to \mathcal{O}_X$ splits locally on X.*

PROOF. Surjectivity on the stalk at x gives a germ mapping to 1. Represent it by a section s on a neighbourhood of x and shrink that neighbourhood so that the section maps to 1 there. The map $\mathcal{O}_U \to \mathcal{M}|_U$ sending 1 to s is a section of the epimorphism. □

PROPOSITION 14.13. *A line bundle is locally a direct summand of $\mathcal{O}_X^n$ for some finite n. In particular it is quasi-coherent.*

PROOF. Locally choose an epimorphism $\mathcal{O}_U^n \to \mathcal{L}|_U$. After tensoring with $\mathcal{L}^{-1}$ its target is $\mathcal{O}_U$, so it splits after a further restriction by Lemma 14.12. Tensoring the splitting with $\mathcal{L}$ gives a splitting of the original epimorphism. To verify quasi-coherence, restrict further to an affine $U = \operatorname{Spec}(R)$. The direct summand is the image of an idempotent endomorphism of $\mathcal{O}_U^n$. This endomorphism is given by an idempotent matrix over R, and exactness of localization identifies its image sheaf with the sheaf associated to the image of that matrix on R^n. Hence the summand is quasi-coherent. □

DEFINITION 14.14. *A vector bundle on X is an $\mathcal{O}_X$-module that is locally free of finite rank: each point has a neighbourhood U on which it is isomorphic to $\mathcal{O}_U^n$ for some finite n. The rank may vary between connected components.*

PROPOSITION 14.15. *An $\mathcal{O}_X$-module is a line bundle if and only if it is locally free of rank 1. Consequently:*

(1) *Every line bundle is quasi-coherent and of finite type.*
(2) *On* $\operatorname{Spec}(R)$*, line bundles correspond under global sections to invertible* R*-modules.*
(3) *Invertibility can be checked on an open cover.*

PROOF. Let $\psi : \mathcal{L}\otimes\mathcal{N} \xrightarrow{\sim} \mathcal{O}_X$ exhibit an inverse. At x, write 1 as a finite sum of elements $\psi(s_i \otimes t_i)$ in the local ring $\mathcal{O}_{X,x}$. At least one summand is a unit: otherwise all summands would belong to the maximal ideal. Lift the corresponding germs to sections s, t on a neighbourhood and shrink it so that $\psi(s \otimes t)$ is a unit. Rescaling t, assume $\psi(s \otimes t) = 1$ on this neighbourhood U.
The map $a : \mathcal{O}_U \to \mathcal{L}|_U$, $1 \mapsto s$, has a left inverse $b(\ell) = \psi(\ell \otimes t)$. Moreover

$$\mathcal{N}|_U \cong \mathcal{O}_U \otimes \mathcal{N}|_U \xrightarrow{a\otimes \mathrm{id}} \mathcal{L}|_U \otimes \mathcal{N}|_U \xrightarrow{\psi} \mathcal{O}_U$$

has a right inverse $1 \mapsto t$ and is an epimorphism. Tensoring $a \otimes \mathrm{id}_{\mathcal{N}}$ with $\mathcal{L}$ therefore gives an epimorphism. Using the same isomorphism $\mathcal{N} \otimes \mathcal{L} \cong \mathcal{O}_U$ in source and target, and associating the target as $\mathcal{L} \otimes (\mathcal{N} \otimes \mathcal{L})$, this map identifies with a. Thus a is an epimorphism and has a left inverse. Since $ba = \mathrm{id}$, we have $(ab)a = a$, and cancellation of the epimorphism a gives $ab = \mathrm{id}$. Hence a is an isomorphism.
Conversely, suppose $\mathcal{L}$ is locally free of rank 1. Define its dual by

$$\mathcal{L}^\vee(U) = \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{L}|_U, \mathcal{O}_U).$$

Morphisms of sheaves glue uniquely, so this is a sheaf. The evaluation map $\mathcal{L}\otimes\mathcal{L}^\vee \to \mathcal{O}_X$ is an isomorphism wherever $\mathcal{L}$ is trivial, hence is an isomorphism globally. This proves the equivalence. Quasi-coherence and finite type have already been proved (and also follow directly from local freeness). The affine assertion follows from the affine equivalence and Proposition 14.2. Finally, if a sheaf is invertible on an open cover it is locally free of rank 1 on that cover, and the dual construction gives a global inverse. □

For a vector bundle $\mathcal{E}$, the same definition of $\mathcal{E}^\vee$ gives another vector bundle: on an open where $\mathcal{E} \cong \mathcal{O}^n$ its dual is $\mathcal{O}^n$. Evaluation gives a canonical isomorphism $\mathcal{E} \cong (\mathcal{E}^\vee)^\vee$. These assertions follow by checking the usual finite free-module maps in local trivializations. For line bundles the dual is the tensor inverse.
The constructions above extend algebraic objects and operations to schemes:

Algebraic object	**Global geometric object**
Ring	Scheme
Module	Quasi-coherent sheaf
$- \otimes_R -$	$- \otimes_{\mathcal{O}_X} -$
Invertible module	Line bundle

On affine schemes, rings and modules are recovered by taking global sections. General schemes and their quasi-coherent sheaves are assembled by gluing these affine objects.

14.3. Projective space and its tautological line bundle. We will use pullbacks of vector bundles. For a morphism $f : X \to Y$ and an $\mathcal{O}_Y$-module $\mathcal{E}$, set

$$f^*\mathcal{E} = f^{-1}\mathcal{E} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

For a vector bundle this construction is particularly concrete: pull back a trivializing cover and apply $f^\sharp$ to the entries of its transition matrices. It is therefore again a vector bundle of the same rank locally. The same local calculation shows that

pullback commutes with tensor products and duals of vector bundles and preserves locally split inclusions. On affine schemes it is ordinary base change of modules.

THEOREM 14.16. *For any scheme X over a ring R there is a natural bijection*

$$\mathrm{Hom}_R(X, \mathbb{P}^n_R) \cong \{\mathcal{L} \subseteq \mathcal{O}_X^{n+1} \mid \mathcal{L} \text{ is a line bundle and the inclusion splits locally}\}.$$

On the right we record the submodule sheaf itself, or equivalently an inclusion up to isomorphism commuting with that inclusion.

PROOF. Write $U_i \subseteq \mathbb{P}^n_R$ for its standard affine charts, with coordinates $X_{j,i} = x_j/x_i$ and $X_{i,i} = 1$. On U_i the vector

$$v_i = (X_{0,i}, \ldots, X_{n,i})$$

generates a free rank-1 submodule of $\mathcal{O}_{U_i}^{n+1}$. Projection to the i-th coordinate splits its inclusion. On $U_i \cap U_j$, the equality

$$v_i = X_{i,j}^{-1} v_j$$

shows that these submodule sheaves agree. They glue to a line subbundle $\mathcal{T} \subseteq \mathcal{O}_{\mathbb{P}^n_R}^{n+1}$. A morphism $f : X \to \mathbb{P}^n_R$ gives the required subbundle by pulling back this locally split inclusion.

Conversely, let $\mathcal{L} \subseteq \mathcal{O}_X^{n+1}$ be such a subbundle. For each i let V_i be the locus where the projection $\mathcal{L} \to \mathcal{O}_X$ to the i-th coordinate is nonzero on the fibre. In a trivialization of $\mathcal{L}$ this projection is multiplication by a function a_i, and the indicated locus is where a_i is a unit in the local ring. It is therefore open, and on V_i the projection is an isomorphism. The V_i cover X: locally a generator $(a_0, \ldots, a_n)$ of $\mathcal{L}$ has a retraction $(b_0, \ldots, b_n)$ with $\sum_i b_i a_i = 1$, so at every point at least one a_i is a unit.

On V_i there is a unique generator whose i-th coordinate is 1. Write its other coordinates as $a_{j,i} \in \mathcal{O}_X(V_i)$. These sections define a unique morphism $V_i \to U_i$ by the universal property of affine space. On $V_i \cap V_j$, $a_{j,i}$ is a unit and normalization gives

$$a_{s,j} = a_{s,i}/a_{j,i}.$$

These are exactly the transition equations of the standard charts of projective space. Thus the morphisms glue to $f : X \to \mathbb{P}^n_R$.

Pulling back $\mathcal{T}$ along this f recovers the original normalized generators on every V_i, hence recovers the original subbundle. Conversely, for a morphism f the loci V_i just constructed are $f^{-1}(U_i)$, and their coordinate sections are the pullbacks of $X_{j,i}$. The reconstructed morphism therefore equals f on a cover and hence globally. The construction commutes with pullback, because normalized coordinates and locally split subbundles commute with base change. This proves the natural bijection. □

COROLLARY 14.17. *For an R-scheme X, an R-morphism $X \to \mathbb{P}^n_R$ is equivalently a line bundle $\mathcal{A}$ together with an epimorphism*

$$\mathcal{O}_X^{n+1} \twoheadrightarrow \mathcal{A},$$

up to isomorphism of $\mathcal{A}$ commuting with this map. Equivalently, it is a line bundle with $n+1$ global sections generating every stalk.

PROOF. Dualizing a locally split line subbundle gives an epimorphism to its dual. Conversely, an epimorphism to a line bundle splits locally: trivialize the target and apply Lemma 14.12. Dualizing therefore gives a locally split line subbundle of $\mathcal{O}_X^{n+1}$. The bidual identification shows that these constructions are inverse. Maps

out of $\mathcal{O}_X^{n+1}$ are precisely tuples of global sections, and Lemma 14.7 identifies the epimorphism condition with generation of stalks. □

DEFINITION 14.18. *The tautological line subbundle $\mathcal{T}$ is denoted $\mathcal{O}_{\mathbb{P}^n_R}(-1)$. For $d \in \mathbb{Z}$ set*

$$\mathcal{O}_{\mathbb{P}^n_R}(d) = \mathcal{O}_{\mathbb{P}^n_R}(-1)^{\otimes(-d)},$$

where negative tensor powers use the dual and the zeroth tensor power is $\mathcal{O}_{\mathbb{P}^n_R}$. We often abbreviate this to $\mathcal{O}(d)$.

PROPOSITION 14.19. *The line bundle $\mathcal{O}(d)$ is trivial on each U_i. If $e_i^{(d)}$ denotes its standard frame, then on overlaps*

$$e_j^{(d)} = X_{j,i}^d e_i^{(d)} = X_{i,j}^{-d} e_i^{(d)}.$$

Thus, with the transition map from the j-th trivialization to the i-th trivialization, its coefficient is $X_{i,j}^{-d}$.

PROOF. The standard frame of $\mathcal{O}(-1)$ on U_i is v_i. The equality $v_j = X_{i,j} v_i$ proves the formula for $d = -1$. Taking duals gives the formula for $d = 1$, and tensor powers give it for every integer d. □

EXAMPLE 14.20. The projection $\mathcal{O}(-1) \to \mathcal{O}$ to the k-th coordinate is a section x_k of $\mathcal{O}(1)$. On U_i it is given by

$$x_k = X_{k,i} e_i^{(1)}.$$

Indeed the projection evaluates v_i to $X_{k,i}$. The $n+1$ sections $x_0, \ldots, x_n$ generate $\mathcal{O}(1)$, since x_i is its standard frame on U_i.

DEFINITION 14.21. *For a line bundle $\mathcal{L}$ on X and $s \in \mathcal{L}(X)$, let*

$$D(s) = \{x \in X \mid s_x \otimes 1 \neq 0 \text{ in } \mathcal{L}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)\}.$$

LEMMA 14.22. *The set $D(s)$ is open, s trivializes $\mathcal{L}$ there, and there is a unique section $s^{-1} \in \mathcal{L}^{-1}(D(s))$ with $s \otimes s^{-1} = 1$.*

PROOF. Trivialize $\mathcal{L}$ on an affine neighbourhood $\operatorname{Spec}(A)$. The section becomes an element $a \in A$, and this locus is the principal open $D(a)$. This proves openness. On this locus multiplication by a is invertible, so $s : \mathcal{O}_{D(s)} \to \mathcal{L}|_{D(s)}$ is an isomorphism. Its inverse, viewed as a section of the dual, gives s^{-1}. Uniqueness follows by applying the same argument in each trivialization. □

Sections of line bundles can be multiplied: tensoring the maps $s : \mathcal{O}_X \to \mathcal{L}$ and $t : \mathcal{O}_X \to \mathcal{N}$ gives a section st of $\mathcal{L} \otimes \mathcal{N}$. This agrees with multiplication of functions after local trivialization.

PROPOSITION 14.23. *On $\mathbb{P}^n_R$, $D(x_i) = U_i$, and on this open*

$$x_j x_i^{-1} = X_{j,i}.$$

PROOF. On U_k the section x_i is represented by $X_{i,k}$, whose nonvanishing locus is $U_k \cap U_i$. These loci cover $D(x_i)$ and identify it with U_i. In the frame $e_i^{(1)}$, the sections x_j and x_i have coefficients $X_{j,i}$ and 1; their quotient is therefore $X_{j,i}$. □

14.4. Vector bundles on affine schemes. For a vector bundle $\mathcal{E}$, define

$$\operatorname{rk}_{\mathcal{E}}(x) = \dim_{\kappa(x)}\bigl(\mathcal{E}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)\bigr).$$

On a trivializing open where $\mathcal{E} \cong \mathcal{O}^n$ this function has constant value n, so it is locally constant. Vector bundles are quasi-coherent and of finite type, since these properties can be checked on a trivializing open cover.

Definition 14.24. *An R-module P is projective if for every surjection $q : M \twoheadrightarrow N$ and every map $a : P \to N$ there is a map $b : P \to M$ with $qb = a$.*

Lemma 14.25. *For an R-module P, the following are equivalent:*

1. *P is projective.*
2. *Every surjection onto P splits.*
3. *P is a direct summand of a free module.*

If P is finitely generated, the free module in (3) can be chosen finitely generated.

Proof. The lifting property applied to id_P proves (1)⇒(2). Choose a free module F surjecting onto P; it can be chosen finite free if P is finitely generated. A splitting exhibits P as a summand, proving (2)⇒(3). Finally write P as a summand of F, with inclusion $i : P \to F$ and projection $r : F \to P$. For a surjection $M \to N$ and $a : P \to N$, lift $ar : F \to N$ to $F \to M$ by choosing preimages for the images of a basis of F. Composing this lift with i gives the required lift of a. This proves (3)⇒(1). The elementary splitting argument for a finite free quotient also appears in [**AM**, Chapter 2, Exercise 12]. □

Example 14.26. Free modules, and in particular vector spaces, are projective. The module $\mathbb{Z}/2$ is not projective because $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2$ does not split. For a product ring $R \times S$, the module R is projective as the summand $(R \times S)(1, 0)$ of $R \times S$. Geometrically

$$\operatorname{Spec}(R \times S) = \operatorname{Spec}(R) \amalg \operatorname{Spec}(S),$$

and its associated vector bundle has rank 1 on $\operatorname{Spec}(R)$ and rank 0 on $\operatorname{Spec}(S)$. This illustrates why the rank need only be locally constant.

Theorem 14.27. *A quasi-coherent sheaf $\mathcal{M}$ on $\operatorname{Spec}(R)$ is a vector bundle if and only if $\mathcal{M}(\operatorname{Spec}(R))$ is a finitely generated projective R-module. Thus the affine equivalence restricts to an equivalence between finite-rank vector bundles and finitely generated projective modules.*

Proof. Let M be finitely generated projective. Its base change to any R-algebra is again finitely generated projective, by the direct-summand characterization. Fix $\mathfrak{p} \in \operatorname{Spec}(R)$ and choose elements of M whose images form a basis of $M \otimes_R \kappa(\mathfrak{p})$. This is possible because the images of elements of M span that vector space. They give a map $R^r \to M$. Its cokernel is finitely generated and vanishes after tensoring with $\kappa(\mathfrak{p})$. Nakayama's lemma [**AM**, Proposition 2.6; see also Proposition 2.8] implies that the localized cokernel at $\mathfrak{p}$ vanishes. A finite generating set then shows that the cokernel vanishes on some principal neighbourhood $D(f)$ of $\mathfrak{p}$. Thus

$$R[1/f]^r \twoheadrightarrow M[1/f]$$

is surjective and splits. Write $R[1/f]^r = M[1/f] \oplus K$. The summand K is finitely generated, and its fibre at $\mathfrak{p}$ is zero because the chosen map is an isomorphism on that fibre. Another application of Nakayama, followed by clearing annihilators of

finitely many generators, makes K zero on a smaller principal neighbourhood. On that neighbourhood M is free. Hence $\widetilde{M}$ is a vector bundle.
Conversely let $\widetilde{M}$ be a vector bundle. Proposition 14.10 makes M finitely generated. Choose an epimorphism $q : \mathcal{O}_X^r \twoheadrightarrow \widetilde{M}$, corresponding to $R^r \twoheadrightarrow M$. Define module sheaves

$$\mathcal{H}(U) = \operatorname{Hom}_{\mathcal{O}_U}(\widetilde{M}|_U, \mathcal{O}_U^r), \qquad \mathcal{K}(U) = \operatorname{Hom}_{\mathcal{O}_U}(\widetilde{M}|_U, \widetilde{M}|_U).$$

They are vector bundles: on a neighbourhood where $\widetilde{M}$ is free, they are finite free sheaves. Composition with q gives $\mathcal{H} \to \mathcal{K}$, which is surjective on stalks since maps from a finite free module lift through surjections. By Lemma 14.7, taking global sections on the affine scheme gives a surjection

$$\operatorname{Hom}_R(M, R^r) \twoheadrightarrow \operatorname{Hom}_R(M, M).$$

A preimage of id_M is a splitting of $R^r \twoheadrightarrow M$. Lemma 14.25 therefore proves that M is projective. □

Lemma 14.28. *An R-module M is invertible if and only if the associated sheaf $\widetilde{M}$ is a line bundle. In particular, invertibility of modules can be checked on a principal-open cover: if the $D(f_i)$ cover $\operatorname{Spec}(R)$, then M is invertible if and only if every $M[1/f_i]$ is invertible over $R[1/f_i]$.*

Proof. An inverse module gives an inverse sheaf by Proposition 14.2. Conversely, if $\widetilde{M}$ is a line bundle, its inverse is quasi-coherent by Proposition 14.15, hence is $\widetilde{N}$ for an R-module N. The isomorphism $\widetilde{M} \otimes \widetilde{N} \cong \mathcal{O}$ gives $M \otimes_R N \cong R$ on global sections. Since $\widetilde{M}|_{D(f_i)}$ is associated to $M[1/f_i]$, the local assertion now follows from the fact that invertibility of sheaves can be checked on an open cover. □

Corollary 14.29. *For every ring R there is a natural bijection*

$$\begin{aligned}\mathbb{P}^n(R) &= \operatorname{Hom}(\operatorname{Spec}(R), \mathbb{P}^n_{\mathbb{Z}}) \\ &\cong \{L \subseteq R^{n+1} \mid L \text{ is invertible and is a direct summand}\}.\end{aligned}$$

For an invertible submodule of R^{n+1}, being a direct summand can be checked on a principal-open cover.

Proof. Apply Theorem 14.16 over $\mathbb{Z}$ to $X = \operatorname{Spec}(R)$. An invertible submodule L gives a line subbundle $\widetilde{L}$ whenever its inclusion splits locally, by Lemma 14.28. Its quotient sheaf $\mathcal{Q} = \mathcal{O}_X^{n+1}/\widetilde{L}$ is then a vector bundle: locally the inclusion of a free rank-1 summand of a finite free module can be made a coordinate inclusion by extending its generator to a basis. To justify this last assertion over a neighbourhood, the split inclusion is a vector with a coordinate that is a unit at the point; on the corresponding principal neighbourhood elementary changes of basis make it the first basis vector. The quotient is consequently free of rank n there.
By Theorem 14.27, $Q = \Gamma(X, \mathcal{Q})$ is a finitely generated projective module. Lemma 14.7 gives an exact sequence

$$0 \longrightarrow L \longrightarrow R^{n+1} \longrightarrow Q \longrightarrow 0.$$

Projectivity of Q splits this sequence, proving that the inclusion of L splits globally. Conversely, a direct-summand inclusion certainly splits locally and gives the subbundle required by Theorem 14.16. This proves both assertions. Naturality follows from base change of the subbundle; global split inclusions remain split after tensoring with any R-algebra. □

LEMMA 14.30. *The functor assigning to R the set of invertible direct-summand submodules $L \subseteq R^{n+1}$ satisfies descent for principal-open covers.*

PROOF. Let the $D(f_i)$ cover $\operatorname{Spec}(R)$. On each open choose an invertible direct summand $L_i \subseteq R[1/f_i]^{n+1}$, and assume that their further localizations agree as submodules over every $R[1/f_i, 1/f_j]$. Their associated sheaves are line subbundles of the restrictions of $\mathcal{O}^{n+1}$. The stated equalities give canonical identifications on overlaps, and these satisfy the cocycle condition because they are equalities inside the same restricted sheaf $\mathcal{O}^{n+1}$. Descent for sheaves glues them to a line subbundle $\mathcal{L} \subseteq \mathcal{O}^{n+1}$ with locally split inclusion. Since $\mathcal{L}$ is quasi-coherent, it is associated to $L = \Gamma(\operatorname{Spec}(R), \mathcal{L}) \subseteq R^{n+1}$; its localizations are exactly the L_i. The preceding corollary makes L an invertible direct summand. Any other such submodule has the same associated submodule sheaf on the given cover, hence the same sheaf globally by uniqueness of gluing, and therefore the same global sections. This proves existence and uniqueness of descent. □

EXAMPLE 14.31. Let $A = \mathbb{Z}[\sqrt{-5}]$ and $I = (2, 1+\sqrt{-5}) \subseteq A$. This gives a nontrivial line bundle on $\operatorname{Spec}(A)$. Indeed $D(2), D(3)$ cover $\operatorname{Spec}(A)$, since $(2, 3) = A$. After inverting 2 the ideal I is all of $A[1/2]$. After inverting 3 it is generated by $1 + \sqrt{-5}$, because

$$2 = (1 + \sqrt{-5})(1 - \sqrt{-5})/3.$$

These are free rank-1 modules, so Lemma 14.28 makes I invertible.
It is not a principal ideal. The quotient A/I is $\mathbb{F}_2$, with $\sqrt{-5}$ mapping to 1, so I is proper. If $I = (a + b\sqrt{-5})$, the generator divides both 2 and $1 + \sqrt{-5}$. Its norm $a^2 + 5b^2$ therefore divides both 4 and 6, and must be 1 or 2. Norm 2 is impossible for integers a, b, whereas norm 1 makes the generator ± 1, contradicting properness. Hence the associated line bundle is nontrivial. By Theorem 14.27, I is a finitely generated projective module that is not free.

15. Twisting sheaves and the Picard group of projective space

We now construct quasi-coherent sheaves on Proj from graded modules and study their twists. We then compute the Picard group of projective space over a unique factorisation domain and calculate sections of line bundles on the projective line.

15.1. Graded modules and the twisting sheaves on Proj. The construction of $\operatorname{Proj}(S)$ in Definition 12.6 has a corresponding construction for graded modules. For the algebraic definitions in nonnegative degrees, see [**AM**, Chapter 10, "Graded rings and modules"]; here we also allow modules graded by $\mathbb{Z}$. Let $S = \bigoplus_{d \geq 0} S_d$ and let $M = \bigoplus_{d \in \mathbb{Z}} M_d$ be a graded S-module. For a homogeneous $f \in S$ of positive degree, write

$$S_{(f)} = (S[1/f])_0, \qquad M_{(f)} = (M[1/f])_0.$$

The subscript 0 means the degree-zero part, with $\deg(f^{-1}) = -\deg(f)$.

PROPOSITION 15.1. *There is a quasi-coherent sheaf $\widetilde{M}$ on $\operatorname{Proj}(S)$ whose restriction to the affine chart $D_+(f) = \operatorname{Spec}(S_{(f)})$ is the sheaf associated to the $S_{(f)}$-module $M_{(f)}$. In particular*

$$\widetilde{M}(D_+(f)) = M_{(f)}.$$

PROOF. We verify that the specified affine sheaves glue. If $\deg f = a > 0$ and $\deg g = b > 0$, then $D_+(f) \cap D_+(g)$ corresponds in $\operatorname{Spec}(S_{(f)})$ to the principal open defined by

$$h = g^a/f^b \in S_{(f)}.$$

This follows from the prime correspondence in Proposition 12.7: a homogeneous prime not containing f contains g if and only if the corresponding degree-zero prime contains h. There is a canonical localization isomorphism

$$(M_{(f)})[1/h] \xrightarrow{\sim} (M[1/f, 1/g])_0.$$

Indeed, localizing the entire graded module $M[1/f]$ at the degree-zero element h commutes with taking its degree-zero part. Moreover adjoining h^{-1} makes g invertible, since $g^a = f^b h$ and f is already invertible. Conversely inverting g inverts h. Thus the displayed isomorphism is induced by the usual localization isomorphism of graded modules. Since inverting fg is equivalent to inverting both factors, its target is $M_{(fg)}$.

It follows that the affine sheaves on $D_+(f)$ and $D_+(g)$ have canonically isomorphic restrictions to their intersection. On triple intersections these identifications agree because they are all induced by the same maps into $M[1/f, 1/g, 1/\ell]$. Descent for sheaves therefore glues them uniquely. The result is quasi-coherent because each of its indicated affine restrictions is associated to a module. □

For $d \in \mathbb{Z}$ let $M(d)$ denote the shifted graded module $M(d)_r = M_{r+d}$.

PROPOSITION 15.2. *Suppose S is generated over S_0 by finitely many elements $z_0, \dots, z_m$ of degree 1. Then $\widetilde{S(d)}$ is a line bundle for every $d \in \mathbb{Z}$, and there are natural isomorphisms*

$$\widetilde{S(d)} \otimes \widetilde{S(e)} \cong \widetilde{S(d+e)}, \qquad \widetilde{M(d)} \cong \widetilde{M} \otimes \widetilde{S(d)}.$$

For $S = R[x_0, \dots, x_n]$ with the variables in degree 1, the identification $\operatorname{Proj}(S) = \mathbb{P}^n_R$ identifies $\widetilde{S(d)}$ with the previously constructed $\mathcal{O}(d)$.

PROOF. The $D_+(z_i)$ cover $\operatorname{Proj}(S)$, since a homogeneous prime containing all the z_i contains S_+. On $D_+(z_i)$ the module $(S(d)[1/z_i])_0 = (S[1/z_i])_d$ is free of rank 1 over $S_{(z_i)}$, with basis z_i^d. Multiplication of these homogeneous fractions identifies the tensor product of the bases z_i^d, z_i^e with z_i^{d+e}, and identifies $M_{(z_i)} \otimes S_{(z_i)} z_i^d$ with $(M[1/z_i])_d$. These multiplication maps commute with all restrictions, proving the natural isomorphisms. For the polynomial ring the frame x_i^d has transition $x_j^d = (x_j/x_i)^d x_i^d$, exactly the transition in Proposition 14.19. This proves the final identification, using Proposition 12.9. □

WARNING 15.3. *For an arbitrary positive grading, $\widetilde{S(1)}$ need not be invertible, and the preceding tensor identities require additional hypotheses. Even in the degree-1 situation, one cannot recover the graded module simply as $M_d = \Gamma(\operatorname{Proj}(S), \widetilde{M(d)})$. For example, if $S = k[x_0, \dots, x_n]$ and $M = S/S_+$, then $M_0 = k$ but $\widetilde{M} = 0$: localizing at any x_i kills M. Global sections must be computed using the sheaf and its gluing conditions.*

We write

$$\mathcal{O}_{\operatorname{Proj}(S)}(d) := \widetilde{S(d)}.$$

Under the hypotheses of Proposition 15.2, these are line bundles. This lets us return to the homogeneous equations used to construct Proj: projective coordinates become generating sections of a line bundle, and homogeneous equations become relations between their tensor products. The following description also appears in [**Stacks**, Tag 01NA].

PROPOSITION 15.4. *Let A be a ring, let $I \subseteq A[X_0, \dots, X_n]$ be a homogeneous ideal, and set $S = A[X_0, \dots, X_n]/I$, with the standard grading $\deg X_i = 1$. For every A-scheme T, the A-morphisms $T \to \operatorname{Proj}(S)$ correspond naturally to isomorphism classes of line bundles $\mathcal{L}$ on T with global sections $s_0, \dots, s_n$ satisfying:*

1. *The germs of $s_0, \dots, s_n$ generate $\mathcal{L}_t$ for every $t \in T$.*
2. *For every homogeneous $F \in I$ of degree d,*

$$F(s_0, \dots, s_n) = 0 \quad \text{in } \Gamma(T, \mathcal{L}^{\otimes d}).$$

The coefficients of F act through the given A-scheme structure, and $\mathcal{L}^{\otimes 0} = \mathcal{O}_T$. An isomorphism of these data is an isomorphism of line bundles taking every s_i to the corresponding section. For $f : T \to \operatorname{Proj}(S)$ the corresponding line bundle is $\mathcal{L} = f^*\mathcal{O}_{\operatorname{Proj}(S)}(1)$, with the pullbacks of the coordinate sections. The relations in (2) are equalities of sheaf sections, so they retain nilpotent information.

PROOF. Write x_i for the image of X_i in S and $U_i = D_+(x_i)$. On U_i, the line bundle $\mathcal{O}_{\operatorname{Proj}(S)}(1)$ has frame x_i. The local sections

$$\xi_j|_{U_i} = (x_j/x_i)x_i$$

agree on overlaps and therefore define global sections ξ_j. In particular $\xi_i|_{U_i} = x_i$, so these sections generate every stalk. For homogeneous $F \in I$ of degree d, their evaluation on U_i is $\big(F(x_0, \dots, x_n)/x_i^d\big)x_i^{\otimes d} = 0$. Thus $f : T \to \operatorname{Proj}(S)$ gives the stated data by pullback.

Conversely, suppose $(\mathcal{L}, s_0, \dots, s_n)$ satisfies (1) and (2). Set $V_i = D(s_i) \subseteq T$. These opens cover T: in a local trivialization the sections are functions generating the unit ideal, so at every point at least one is a unit in the local ring. By Lemma 14.22, s_i trivializes $\mathcal{L}$ on V_i. We can therefore form the functions

$$a_{j,i} = s_j/s_i \in \Gamma(V_i, \mathcal{O}_T), \qquad a_{i,i} = 1.$$

For homogeneous $F \in I$ of degree d, let F_i be its dehomogenization with $X_i = 1$, as in Example 12.10. On V_i the relation (2) reads

$$0 = F(s_0, \dots, s_n) = s_i^{\otimes d} F_i(a_{j,i} \mid j \neq i).$$

Since $s_i^{\otimes d}$ is a frame, $F_i(a_{j,i}) = 0$. Consequently the coefficient map from A and the assignments $X_j/X_i \mapsto a_{j,i}$ define an A-algebra homomorphism

$$S_{(x_i)} \longrightarrow \Gamma(V_i, \mathcal{O}_T).$$

The affine dictionary gives a morphism $f_i : V_i \to U_i$.

On $V_i \cap V_j$, the function $a_{j,i}$ is invertible, so f_i lands in $U_i \cap U_j$ there. Moreover

$$a_{k,j} = a_{k,i}/a_{j,i},$$

which is exactly the coordinate change on the Proj charts. Hence f_i and f_j agree on their overlap, and the f_i glue to an A-morphism $f : T \to \operatorname{Proj}(S)$. Isomorphic data give the same ratios and therefore the same morphism.

On V_i send the pulled-back frame f^*x_i of $f^*\mathcal{O}_{\operatorname{Proj}(S)}(1)$ to s_i. On overlaps both frames change by $a_{j,i}$, so these maps glue to an isomorphism $f^*\mathcal{O}_{\operatorname{Proj}(S)}(1) \cong \mathcal{L}$

taking every $f^*\xi_j$ to s_j. Thus starting with the data and then pulling back recovers their isomorphism class. If we start with a morphism f, the opens $D(f^*\xi_i)$ are exactly $f^{-1}(U_i)$: on U_j the section ξ_i has coefficient x_i/x_j, whose nonvanishing locus is $U_i \cap U_j$. The functions used in the reconstruction are $f^*(x_j/x_i)$, so the reconstructed morphism agrees with f on this cover. The two constructions are therefore inverse. Finally, pulling back along an A-morphism $T' \to T$ pulls back the line bundle and its sections, their nonvanishing opens, and all the coordinate ratios. The construction is consequently natural in T. □

Equivalently, the data in the proposition give a graded A-algebra map

$$S \longrightarrow \bigoplus_{d\geq 0} \Gamma(T, \mathcal{L}^{\otimes d})$$

whose degree-1 image generates $\mathcal{L}$ at every stalk. Multiplication on the right is tensor multiplication of sections. This is the algebra–geometry dictionary for Proj in the standard grading.

15.2. The Picard group of projective space.

Definition 15.5. *The Picard group* $\mathrm{Pic}(X)$ *is the group of isomorphism classes of line bundles on* X*, with tensor product as the group operation.*

Tensor products of line bundles are line bundles, the identity is $[\mathcal{O}_X]$, and the inverse of $[\mathcal{L}]$ is $[\mathcal{L}^\vee]$. Associativity and commutativity come from those of tensor products. The isomorphism classes form a set: every line bundle is specified by a trivializing open cover and invertible transition functions satisfying a cocycle condition. There is a set of open covers of X and, for each cover, a set of such functions.

Lemma 15.6. *If* A *is a unique factorisation domain, then every line bundle on* $\mathrm{Spec}(A)$ *is trivial. Equivalently, every invertible* A*-module is isomorphic to* A*.*

Proof. Let P be an invertible A-module and let Q be an inverse, with isomorphism $\psi : P \otimes_A Q \to A$. By Theorem 14.27 both modules are finitely generated projective and therefore torsion-free. Write $K = \mathrm{Frac}(A)$. Their base changes to K are one-dimensional vector spaces. Choose an identification $P \otimes_A K \cong K$ and then the unique identification $Q \otimes_A K \cong K$ for which $\psi \otimes K$ becomes multiplication. The injections into these base changes identify P, Q with finitely generated fractional ideals $I, J \subseteq K$, and surjectivity of ψ says $IJ = A$.
Multiplying I by a nonzero element of A and J by its inverse, arrange $I \subseteq A$. Let $a_1, \dots, a_r$ generate I, and let g be a greatest common divisor of its nonzero generators. Put $I_0 = g^{-1}I \subseteq A$ and $J_0 = gJ$. Then $I_0 J_0 = A$, and no prime element divides every generator of I_0.
We claim that $\{q \in K : qI_0 \subseteq A\} = A$. Write $q = u/v$ with $u, v \in A$ coprime and $v \neq 0$. If $qI_0 \subseteq A$, unique factorisation implies that v divides every generator of I_0. Hence v is a unit, proving the claim. Since J_0 is contained in this set, $J_0 \subseteq A$. Therefore

$$A = I_0 J_0 \subseteq I_0 \subseteq A,$$

so $I_0 = A$ and $I = gA$. Thus P is free of rank 1. □

Example 15.7 (Gluing line bundles on the projective line). Consider the standard cover of $\mathbb{P}^1_{\mathbb{Z}}$ by

$$U_0 = \mathrm{Spec}(\mathbb{Z}[t]), \qquad U_1 = \mathrm{Spec}(\mathbb{Z}[u]),$$

where $t = x_1/x_0$ and $u = x_0/x_1$. Their intersection is

$$U_0 \cap U_1 = \operatorname{Spec}(\mathbb{Z}[t, t^{-1}]), \qquad u = t^{-1}.$$

By descent, a line bundle is specified by line bundles $\mathcal{L}_0$ on U_0 and $\mathcal{L}_1$ on U_1, together with an isomorphism between their restrictions to the intersection. For two charts there is no additional cocycle condition beyond the identity and inverse conditions. Since $\mathbb{Z}[t]$ and $\mathbb{Z}[u]$ are unique factorisation domains, Lemma 15.6 shows that both $\mathcal{L}_0$ and $\mathcal{L}_1$ are trivial. Choose frames e_0 and e_1. The gluing isomorphism is then encoded by a unit $g(t)$ through

$$e_1 = g(t)e_0 \quad \text{on } U_0 \cap U_1.$$

Conversely, every such unit defines a line bundle by gluing.
The units of $\mathbb{Z}[t, t^{-1}]$ are exactly $\pm t^d$, with $d \in \mathbb{Z}$. Indeed, the lowest and highest exponents add under multiplication, so a unit has just one nonzero term, whose coefficient must be a unit of $\mathbb{Z}$. Replacing e_1 by $-e_1$ removes a negative sign. Thus the gluing can always be written as

$$e_1 = t^d e_0.$$

By Proposition 14.19, this is precisely the transition function of $\mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}(d)$.
The exponent d is determined by the isomorphism class. Changing a frame on either chart multiplies $g(t)$ only by a sign, since

$$\mathbb{Z}[t]^\times = \mathbb{Z}[u]^\times = \{1, -1\}.$$

Hence t^d and t^e define isomorphic line bundles only when $d = e$. This gives a concrete classification of line bundles on $\mathbb{P}^1_{\mathbb{Z}}$ by integers. Tensoring bundles multiplies their transition functions, so $\mathcal{O}(d) \otimes \mathcal{O}(e) \cong \mathcal{O}(d + e)$.

We now apply this method to $\mathbb{P}^n_R$ over any unique factorisation domain.

THEOREM 15.8. *Let R be a unique factorisation domain and $n \geq 1$. Then*

$$\mathbb{Z} \xrightarrow{\sim} \operatorname{Pic}(\mathbb{P}^n_R), \qquad d \longmapsto [\mathcal{O}(d)].$$

In particular every line bundle on $\mathbb{P}^n_{\mathbb{Z}}$ is isomorphic to exactly one $\mathcal{O}(d)$.

PROOF. Let $\mathcal{L}$ be a line bundle. Each standard chart has coordinate ring a polynomial ring over the UFD R, hence is a UFD. Lemma 15.6 allows us to choose a frame e_i of $\mathcal{L}$ on each U_i. For distinct i, j, the units of

$$\mathcal{O}(U_i \cap U_j) = R[X_{s,i} \mid s \neq i, j][X_{j,i}, X_{j,i}^{-1}]$$

are precisely $cX_{j,i}^d$ with $c \in R^\times$ and $d \in \mathbb{Z}$. Indeed a polynomial unit over a domain is constant in every non-inverted variable, and in a Laurent polynomial ring the lowest and highest exponents of a product add, forcing an invertible Laurent polynomial to be a single monomial. Consequently there are unique $c_{ij} \in R^\times$ and $d_{ij} \in \mathbb{Z}$ such that

$$e_j = c_{ij} X_{j,i}^{d_{ij}} e_i.$$

Reversing i, j gives $c_{ji} = c_{ij}^{-1}$ and $d_{ji} = d_{ij}$.
For three distinct indices, the cocycle identity on $U_i \cap U_j \cap U_k$ gives

$$c_{ik} X_{k,i}^{d_{ik}} = c_{ij} c_{jk} X_{j,i}^{d_{ij} - d_{jk}} X_{k,i}^{d_{jk}}.$$

Uniqueness of Laurent monomials implies $d_{ij} = d_{jk} = d_{ik}$ and $c_{ik} = c_{ij}c_{jk}$. For $n \geq 2$ these identities imply that all pairwise exponents have a common value d. For $n = 1$ there is only one unordered pair, and we take $d = d_{01}$.
Put $a_0 = 1$ and $a_i = c_{0i}$ for $i \neq 0$. The cocycle identities, including the inverse identities, imply $c_{ij} = a_j/a_i$; this also holds directly when $n = 1$. Replacing e_i by e_i/a_i removes all the constants. The remaining transition equations are

$$e_j = X_{j,i}^d e_i,$$

which are precisely those of $\mathcal{O}(d)$ by Proposition 14.19. This proves surjectivity.
If $\mathcal{O}(d) \cong \mathcal{O}(e)$, the isomorphism on each chart is multiplication in the chosen frames by a unit of its polynomial coordinate ring, hence by an element of $R^\times$. On $U_0 \cap U_1$, compatibility would therefore express $X_{1,0}^{d-e}$ as an element of $R^\times$. The Laurent monomials are distinct, so $d = e$. Finally $\mathcal{O}(d) \otimes \mathcal{O}(e) \cong \mathcal{O}(d+e)$, so the obtained bijection is a group isomorphism. □

15.3. Sections on the projective line.

PROPOSITION 15.9. *For a field k and $d \in \mathbb{Z}$,*

$$\Gamma(\mathbb{P}^1_k, \mathcal{O}(d)) = \begin{cases} \displaystyle\bigoplus_{j=0}^{d} k\, x_0^{d-j} x_1^j, & d \geq 0, \\ 0, & d < 0. \end{cases}$$

Thus its dimension is $d+1$ for $d \geq 0$, and 0 for $d < 0$.

PROOF. Write $t = x_1/x_0$ on U_0 and $u = x_0/x_1$ on U_1. A global section is a pair

$$p(t)e_0^{(d)}, \qquad q(u)e_1^{(d)}$$

with $p \in k[t]$, $q \in k[u]$, and the gluing equation

$$p(t) = t^d q(t^{-1}) \quad \text{in } k[t, t^{-1}].$$

If $d < 0$, the right side has only strictly negative powers of t, whereas the left side has only nonnegative powers, so both vanish. If $d \geq 0$, the equation says exactly that p is a linear combination of $1, t, \ldots, t^d$ and determines q uniquely. The term t^j corresponds to the global section $x_0^{d-j}x_1^j$, giving the stated basis. □

EXAMPLE 15.10. The three sections x_0^2, x_0x_1, x_1^2 generate $\mathcal{O}_{\mathbb{P}^1_k}(2)$ and hence define

$$\mathbb{P}^1_k \longrightarrow \mathbb{P}^2_k, \qquad [x_0 : x_1] \longmapsto [x_0^2 : x_0x_1 : x_1^2].$$

Generation follows because x_0^2 is a frame on U_0 and x_1^2 is a frame on U_1. This map identifies $\mathbb{P}^1_k$ with the conic

$$C = \operatorname{Proj}\big(k[X_0, X_1, X_2]/(X_0X_2 - X_1^2)\big)$$

of Example 12.11. Indeed, the relation

$$x_0^2x_1^2 - (x_0x_1)^2 = 0 \quad \text{in } \Gamma(\mathbb{P}^1_k, \mathcal{O}(4))$$

shows by Proposition 15.4 that the generating sections define a morphism directly to C, with pullback of $\mathcal{O}_C(1)$ equal to $\mathcal{O}_{\mathbb{P}^1_k}(2)$. To verify that it is an isomorphism, on U_0 the map is $[1 : t : t^2]$. The corresponding chart of C has coordinate ring

$$k[a, b]/(b - a^2) \cong k[a], \qquad a = X_1/X_0, \quad b = X_2/X_0,$$

and the map sends a to t, so it is an isomorphism on this chart. On U_1 the same calculation with $X_2 \neq 0$ sends X_1/X_2 to u. These two charts cover C: a homogeneous prime containing X_0, X_2 would also contain X_1 and thus the irrelevant ideal. Their coordinate transition is $u = t^{-1}$, exactly as on $\mathbb{P}^1_k$. The chart isomorphisms therefore glue to the asserted isomorphism. This example shows how a line bundle together with generating sections produces concrete projective coordinates.

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