Algebraic Topology

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Contents

Chapter 1

Topology I, Winter term 2024/ 2025

Organizational matters:

- Exams at the end of the semester. Allowance criteria:
  1. 50 percent of the points for the weekly excercises
  2. present a solution

  Exam Date: January 28th, 3-5 pm.
- Exercices: One exercise sheet per week (starting today, always uploaded by monday evening). Turn in in pairs of two before the monday lecture (submit electronically).
- Here are the dates for the exercises:

1	Wednesday 14-16	David Eigendorf
2	Thursday 14-16	Yash Jolly

  Subscribe in the Learnweb tonight.
- Literature:
  - Bredon: Topology and Geometry
  - Lück: Algebraische Topologie
  - Hatcher: Algebraic Topology
  - tom Dieck: Algebraic Topology
  - Weibel: Homological Algebraic
  - Bott, Tu: Differential forms in algebraic topology
- We assume knowledge of the basic material from "Grundlagen der Analysis, Topologie, Geometrie" such as
  - Notion of a topological space
  - Examples of spaces (projective spaces, spheres etc.)
  - (path) connected components
  - fumdamental groups
  - techniques to compute π_1 (Covering theory,...)

1.1 Preface: What is algebraic topology?

Goal of Topology: understand topological spaces. This can mean:

- classify them up to homeomorphism (picture circle, square)
- classify them up to homotopy equivalence (picture: thickened circle)
- Understand geometric properties of the spaces (e.g. how many covering spaces does it admit).

Example 1.1.1. Given two spaces X, Y, decide whether they are homemorphic or homotopy equivalent. Take $\mathbb{R}^n$ and $\mathbb{R}^m$ for $n \neq m$ or S^n and S^m for $n \neq m$. Clearly $\mathbb{R}^n \simeq \mathbb{R}^m$ for all n, m. Need: Invariants to distinguish.

Definition 1.1.2 (Preliminary)**.** An invariant is an assignment which assigns some algebraic object $F(X)$ to a topological space, such that for homotopy equivalent (resp. homeomorphic) spaces $X \simeq Y$ we have $F(X) \cong F(Y)$.

Example 1.1.3.

- The assignment $X \mapsto \pi_0(X)$ is an invariant (under homotopy equivalence). We have $\pi_0(S^0) = \{1, 2\}$ and $\pi_0(S^n) = *$ for $n > 1$. Thus we can conclude that S^0 can not be homotopy equivalent to S^n for $n > 0$.
- The assignment $(X, x) \mapsto \pi_1(X, x)$ is an invariant. We have $\pi_1(S^1, 1) = \mathbb{Z}$ and $\pi_1(S^n) = 0$ for $n \neq 1$ [1]. Thus we can conclude that S^1 can not be homotopy equivalent to S^n for $n \neq 1$.

What makes these good invariants:

1. They can distinguish spaces (and help us understand geometric properties). Unfortunately they see only low dimensional phenomena.
2. They are computable

If we want to know whether a space X is the n-sphere consider the invariant

$$F_n(X) = \begin{cases} 1 & \text{if } X \simeq S^n \\ 0 & \text{else} \end{cases}$$

Completely useless. For $\pi_1(X, x)$ we have a lot of tools to compute:

- Covering theory
- long exact sequences
- Seifert van Kampen
- ...

Recall that $\pi_1(X, x) = [S^1, 1; X, x]$.

[1] This was not shown last term and will be shown on an exericise sheet

Definition 1.1.4. The higher homotopy groups of a pointed space (X, x) are defined as

$$\pi_n(X, x) := [S^n, 1; X, x]$$

It turns out these are abelian groups. But very hard to compute.

Example 1.1.5. For $X = S^2$ it is known that:

- There are infinitely many $\pi_n(S^2, 1) \neq 0$.
- All groups $\pi_n(S^2, 1)$ for $n > 3$ are torsion (Serre).
- Here is a short list

n	1	2	3	4	5	6	7	8	9	10
$\pi_n(S^2)$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbb{Z}/15$

- unknown for large n
- no satisfactory pattern.

The invariants π_n are very good from perspective (1), i.e. the can in principle distinguish many spaces. In fact, one can show that up to homotopy, every reasonable space can be glued together from spheres ($\rightarrow$ CW complexes).

But from perspective (2) they are not as good, since computing them is practically impossible apart from specific cases.

We will get to know other invariants in this course called homology and cohomology, denoted $H_n(X)$ and $H^n(X)$ which can be computed quite efficiently and which can be also be used to distinguish a lot of spaces. Unfortunately not as easy to define...

1.2 Singular homology

Definition 1.2.1. The *standard n-simplex* is defined as

$$\Delta^n := \{(x_0, ..., x_n) \mid \sum x_i = 1, x_i \geq 0\} \subseteq \mathbb{R}^{n+1}$$

with the subspace topology.

Examples 1.2.2.

1. $\Delta^0 = \{1\} \subseteq \mathbb{R}$
2. Δ^1: Bild. We have a Homeomorphism

$$[0,1] \to \Delta^1 \qquad t \mapsto (t, 1-t)$$

since it is clearly bijective, continuous and inverse is also continuous (given by $(x_0, x_1) \mapsto x_0$).

3. Bild vom 2-simplex.
4. In general the n-simplex Δ^n is homeomorphic to the subspace

$$\{(t_1, ..., t_n) \mid \sum t_i \leq 1, t_i \geq 0\} \subseteq \mathbb{R}^n$$

We prefer the parametrization as a subspace of $\mathbb{R}^{n+1}$, which is called barycentric parametrisation/barycentric coordinates.

Definition 1.2.3. Let V be a real vector space. Given $n+1$ elements $v_0, ..., v_n \in V$ define the affine n-simplex spanned by the v_i to be the map

$$[v_0, ..., v_n] : \Delta^n \to V \qquad (x_0, ..., x_n) \mapsto \sum_{i=1}^{n} x_i v_i$$

Note that the image of $[v_0, ..., v_n]$ is the convex span of the v_i's. Also note that they need not be linearly independent, for example $[e_0, e_1, e_1] \subseteq \mathbb{R}^2$ spans a line. We write a hat over one of v_i's if it is omited.

Definition 1.2.4. For $0 \leq i \leq n$ the $i-th$-face map of the n-simplex is defined to be the map

$$d_i^n : \Delta^{n-1} \to \Delta^n$$

induced by $[e_0, ..., \hat{e}_i, ..., e_n] : \Delta^{n-1} \to \mathbb{R}^{n+1}$, i.e. sends

$$(x_0, ..., x_{n-1}) \mapsto (x_0, ..., x_{i-1}, 0, x_i, ..., x_n)$$

Example 1.2.5. 2-simplex draw the picture.

Lemma 1.2.6. *We have the equalities*

$$\begin{aligned} d_j^{n+1} d_i^n &= [e_0, ..., \hat{e}_i, ..., \hat{e}_j, ..., e_{n+1}] : \Delta^{n-1} \to \Delta^{n+1} && \textit{for } j > i \\ d_j^{n+1} d_i^n &= [e_0, ..., \hat{e}_j, ..., \hat{e}_{i+1}, ..., e_{n+1}] : \Delta^{n-1} \to \Delta^{n+1} && \textit{for } j \leq i \\ d_j^{n+1} d_i^n &= d_{i+1}^{n+1} d_j^n && \textit{for } j \leq i \end{aligned}$$

Proof. Let $(x_0, ..., x_{n-1}) \in \Delta^{n-1}$. Then we have

$$\begin{aligned} d_j d_i (x_0, ..., x_{n-1}) &= d_j (x_0, ..., x_{i-1}, 0, x_i, ..., x_{n-1}) \\ &= (x_0, ..., x_{i-1}, 0, x_i, ..., x_{j-2}, 0, x_{j-1}, ..., x_{n-1}) \end{aligned}$$

This shows the first assertion. The second works similar. Finally the third follows from the second and the first. □

Definition 1.2.7. 1. Let X be a topological space. A (singular) n-simplex in X is a (continuous) map $\sigma : \Delta^n \to X$.[2] The group of singular n-chains $C_n(X)$ is defined to be the free abelian group generated by the n-chains.
This means elements of $C_n(X)$ are formal finite sums $\sum_\sigma n_\sigma \sigma$ where $n_\sigma \in \mathbb{Z}$ (i.e. only finitely many $n\sigma \neq 0$). Addition is defined componentwise.

2. The i-th face of an n-simplex σ in X is the $n-1$ simplex defined by the composition

$$\Delta^{n-1} \xrightarrow{d_i^n} \Delta^n \xrightarrow{\sigma} X \ .$$

We denote it by $\sigma^{(i)}$. The boundary of and n-simplex σ is defined to be the element

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \sigma^{(i)} \in C_{n-1}(X)$$

This extends to a group homomorphism (is this clear?)

$$\partial_n : C_n(X) \to C_{n-1}(X)$$

[2]We agree on the convention that all maps are automatically continuous.

Diagramatically

$$\dots \to C_3(X) \xrightarrow{\partial_3} C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X)$$

Lemma 1.2.8. *The composition $\partial_n \partial_{n+1} = 0$ for every n.*

Proof. We calculate for $\sigma : \Delta^{n+1} \to X$:

$$\begin{aligned}
\partial_n \partial_{n+1} \sigma &= \partial_n \sum_{j=0}^{n+1} (-1)^j \sigma^{(j)} \\
&= \sum_{i=0}^{n} \sum_{j=0}^{n+1} (-1)^i (-1)^j (\sigma^{(j)})^{(i)} \\
&= \sum_{i=0}^{n} \sum_{j=0}^{n+1} (-1)^{i+j} (\sigma \circ d_j \circ d_i) \\
&= \sum_{0 \le i < j \le n+1} (-1)^{i+j} (\sigma \circ d_j \circ d_i) + \sum_{0 \le j \le i \le n} (-1)^{i+j} (\sigma \circ d_j \circ d_i)
\end{aligned}$$

By Lemma 1.2.6 the second summand is equal to

$$\sum_{0 \le j \le i \le n} (-1)^{i+j} (\sigma \circ d_{i+1} \circ d_j) = \sum_{0 \le j < i \le n+1} (-1)^{i+j-1} (\sigma \circ d_i \circ d_j)$$

which is the negative of the first. □

Example 1.2.9. Draw a picture for the 2-simplex.

Definition 1.2.10. The group of n-cycles in a space X is defined to be the group

$$Z_n(X) := \ker \partial_n \subseteq C_n(X)$$

for $n > 0$ and $Z_0(X) := C_0(X)$. The group of n-boundaries

$$B_n(X) := \operatorname{im} \partial_{n+1} \subseteq C_n(X)$$

By Lemma 1.2.8 we have $B_n(X) \subseteq Z_n(X)$ and we can define the n-th singular homology group of X as the quotient

$$H_n(X) := Z_n(X)/B_n(X)$$

Remark 1.2.11. Intuitively this measures holes.

Picture of $\mathbb{R}^2$ with simplex and hole.

Example 1.2.12. Consider $X = \mathrm{pt}$. Then $C_n(X) = \mathbb{Z}$ for every n and we have

$$\partial_n(\mathrm{pt}) = \sum_{i=0}^{n} (-1)^n$$

Thus we have $\partial_n = 0$ for n odd and $\partial_n = \mathrm{id}$ for n even. We have the complex

$$\dots \xrightarrow{\mathrm{id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

Thus

$$H_n(\mathrm{pt}) = \begin{cases} \mathbb{Z} & \text{n=0} \\ 0 & \text{sonst} \end{cases}$$

Before doing first computations we discuss functorial properties

Definition 1.2.13. Let $f : X \to Y$ be a map of topological spaces. Then we get an induced group homomorphism $C_n(f) : C_n(X) \to C_n(Y)$ uniquely determined by

$$(\sigma : \Delta^n \to X) \mapsto (f\sigma : \Delta^n \to Y)$$

Lemma 1.2.14. *For a continuous map $f : X \to Y$ we have the equality $\partial_n \circ C_n(f) = C_{n-1}(f) \circ \partial_n$ for all n. In other words, the ladder diagram*

$$\begin{tikzcd}
\dots \arrow[r, "\partial"] & C_3(X) \arrow[r, "\partial"] \arrow[d, "C(f)"] & C_2(X) \arrow[r, "\partial"] \arrow[d, "C(f)"] & C_1(X) \arrow[r, "\partial"] \arrow[d, "C(f)"] & C_0(X) \arrow[d, "C(f)"] \\
\dots \arrow[r, "\partial"] & C_3(Y) \arrow[r, "\partial"] & C_2(Y) \arrow[r, "\partial"] & C_1(Y) \arrow[r, "\partial"] & C_0(Y)
\end{tikzcd}$$

commutes.

Proof. We have

$$\begin{aligned}
C(f)(\partial\sigma) &= \sum_{i=0}^{n} (-1)^i f\sigma^{(i)} \\
&= \sum_{i=0}^{n} (-1)^i f\sigma d_i \\
&= \partial(C(f)\sigma)
\end{aligned}$$

□

Corollary 1.2.15. *For every map $X \to Y$ the induced map $C(f)$ sends cycles to cycles and boundaries to boundaries.*

Proof. Let $x \in C_n(X)$ be a cycle, i.e. $\partial(x) = 0$. Then we have $\partial(C(f)x) = C(f)\partial x = 0$. Thus $C(f)x$ is also a cycle.

Consider a boundary $\partial y \in C_n(X)$. Then $C(f)\partial y = \partial(C(f)y)$ is also a boundary. □

Definition 1.2.16. Let $f : X \to Y$ be a continuous map. According to the last corollary $C(f)$ induces a map on homology

$$H_n(X) \to H_n(Y)$$

which we denote by f_*.

Proposition 1.2.17. *We have* $\mathrm{id}_* = \mathrm{id}_{H_n(X)}$ *and* $(f \circ g)_* = f_* \circ g_*$.

Corollary 1.2.18. *We have that for homeomorphic spaces X and Y the homology groups $H_n(X)$ and $H_n(Y)$ are isomorphic.*

Proof. Let $f : X \to Y$ be a homeomorphism with inverse $g : Y \to X$. Then we have

$$f_* \circ g_* = (f \circ g)_* = id_* = \mathrm{id}$$

and the same for $g_* \circ f_*$. Thus we can concude that f_* and g_* are also inverse to each other. □

Thus clearly $H_n(X)$ are invariants in the sense of definition 1.1.2. But can they be computed and can they distinguish spaces? (Remark: we will see later, that they are even invariant under homotopy equivalence, but that is an actual theorem).

1.3 The groups H_0 and H_1

We have $C_0(X)$ is the free group generated by points of X. Thus an element is a formal sum $\sum n_x x$ where $n_x \in \mathbb{Z}$ and $x \in X$ and only finitely many $n_x \neq 0$. Define a group homomorphism

$$\epsilon : C_0(X) \to \mathbb{Z} \qquad \sum n_x x \mapsto \sum_x n_x$$

Proposition 1.3.1. *The morphism ϵ induces a homomorphism $\epsilon_* : H_0(X) \to \mathbb{Z}$ which is an isomorphism for X path-connected. Moreover every point $x \in X$ generates $H_0(X)$.*

Proof. First we have to show that for a boundary $z \in C_0(X)$ we have $\epsilon(x) = 0$. It is enough to check this for boundaries of the form $z = \partial\sigma$. But we compute

$$\epsilon(x) = \epsilon\big(\sigma^{(0)} - \sigma^{(1)}\big) = 1 - 1 = 0.$$

As a second we want to show that the morphism $\epsilon_* : H_0(X) \to \mathbb{Z}$ is an isomorphism. For an arbitrary point $x \in X$ we have that $\epsilon([nx]) = n$, thus ϵ_* is surjective.

Let $z = \sum n_x x$ be an element in the kernel of ϵ, i.e. $\sum n_x = 0$. Choose a point $y \in X$ and for every point $x \in X$ a path γ_x from y to x. Then we find

$$\partial(\sum n_x \gamma_x) = \sum n_x(x - y) = \sum n_x x - (\sum n_x) y = z$$

Thus $[z] \in H_0(X) = 0$ and therefore ϵ_* injective as well. □

Remark 1.3.2. For us the empty space is <u>not</u> path-connected. Path-connected means exactly one path-connected component. For the empty space we have $H_0(\emptyset) = 0$.

Proposition 1.3.3. *For a not necessarily connected space X the group $H_0(X)$ is the free abelian group on the set of path components $\pi_0(X)$, i.e. there is a canonical isomorphism*

$$H_0(X) \cong \mathbb{Z}\langle \pi_0(X) \rangle$$

Proof. Follows directly from Proposition 1.3.1 using that for a space X with path-components X_α we have

$$H_n(X) = \bigoplus_{\alpha \in \pi_0(X)} H_n(X_\alpha)$$

which follows since the simplices Δ^n are path connected. □

We now come to $H_1(X)$.

Definition 1.3.4. Let G be a (not necessarily abelian) group. Then define its abelianization to be the group $G^{ab} := G/[G,G]$ where $[G,G]$ is the subgroup generated by commutators $ghg^{-1}h^{-1}$.

For this definition to make sense we use that the commutator subgrop is actually normal. This straighforward and left to the reader.

Examples 1.3.5. 1. For G abelian the canonical morphism $G \to G^{ab}$ is an isomorphism.

2. For $G = \Sigma_3$ we have $[G, G] = \langle(123)\rangle \cong \mathbb{Z}/3$ because we can write the generator
$$(123) = (23)(12)(23)(12) \in [G, G]$$
and $[G, G] \subseteq \ker \mathrm{sign} \cong \mathbb{Z}/3$. Where $\mathrm{sign} : \Sigma_3 \to \mathbb{Z}/2$ is the signum of a permutation. Thus $\Sigma_3^{ab} \cong \mathbb{Z}/2$.

Proposition 1.3.6. *Let G be a group. Then we have*

1. *G^{ab} is abelian.*
2. *G^{ab} satisifies the following universal property: for every group homomorphism $G \to A$ with A abelian there exists a unique group homomorphism $G^{ab} \to A$ which makes the following diagram commutative:*

$$\begin{tikzcd} G \arrow[rr] \arrow[dr, "\pi"'] & & A \\ & G^{ab} \arrow[ur, dashed] & \end{tikzcd}$$

Proof. Ad 1) Let $[g]$ and $[h]$ be elements in G^{ab}. Then we have
$$[g] \cdot [h] = [gh] = [ghg^{-1}h^{-1}] \cdot [hg] = [hg].$$
Ad 2): Let $\varphi : G \to A$ be a group homomorphism with A abelian. Then we have
$$\varphi(ghg^{-1}h^{-1}) = \varphi(g)\varphi(h)\varphi(g)^{-1}\varphi(h)^{-1} = 0$$
Thus we have $[G, G] \subseteq \ker \varphi$. Thus we get a well-defined and unique group homomorphism
$$[\varphi] : G/[G, G] \to A \qquad [g] \mapsto \varphi(g).$$
□

For the rest of the section let X be path-connected topological space and fix a basepoint $x \in X$. Recall that a based loop is a map $S^1 \to X$ which sends 1 to x. We can consider it also as a map $\Delta^1 \to X$ which sends the boundary of Δ^1 to x. In particular every (based) loop in X gives an element in $Z_1(X)$ which we will abusively identify with the loop.

Lemma 1.3.7. *Let f and g be paths in X. Then*

1. *the 1-chain $f * g - f - g \in C_1(X)$ is a boundary (here we assume the composition makes sense).*
2. *The chain $f + f^{-1}$ and constant paths are boundaries.*
3. *If f and g are based homotopic, then $f - g \in C_1(X)$ is a boundary.*

Proof. ad 1) Consider the map $p : \Delta^2 \to \Delta^1$ given by the perpendicular projection depicted as follows (1 goes to the middle between 0 and 2): Consider the map
$$\sigma : \Delta^2 \xrightarrow{p} \Delta^1 \xrightarrow{f*g} X$$
Then we have $\sigma^{(2)} = f$, $\sigma^{(0)} = g$ and $\sigma^{(1)} = f * g$. Thus $\partial\sigma = g + f * g - f$.

ad 2) Consider the 2-simplex $\sigma : \Delta^2 \to X$ constant at $x \in X$. Then $\partial\sigma = c_x - c_x + c_x = c_x$ is constant at x.

Now Consider the projection $p' : \Delta^2 \to \Delta^1$ to the side on the right. Then we define

$$\sigma' : \Delta^2 \xrightarrow{p'} \Delta^1 \xrightarrow{f^{-1}} X$$

and have that

$$\partial\sigma' = f^{-1} + c_x + f$$

ad 3) Consider a based homotopy $h : I \times I \to X$ between f and g. Then it factors through $I \times I \to \Delta^2$ given by the map depicted as follows

and we obtain a map $\sigma : \Delta^2 \to X$. Then we have

$$\partial\sigma = f - g + const$$ □

Corollary 1.3.8. *The assignment*

$$\pi_1(X, x) \to H_1(X) \qquad [\gamma] \mapsto [\Delta^1 \to S^1 \xrightarrow{\gamma} X]$$

is a well-defined group homomorphism. Therefore aplying the universal property 1.3.6 we find a well-defined homomorphism

$$\phi : \pi_1(X, x)^{ab} \to H_1(X)$$

Theorem 1.3.9 (Hurewicz)**.** *For a pathconnected space X with basepoint x the homomorphism $\phi : \pi_1(X, x)^{ab} \to H_1(X)$ is an isomorphism.*

Proof. Pick for every $y \in X$ a path λ_x from x to y. We choose the path λ_x to be constant.

For a 1-simplex $\gamma : \Delta^1 \to X$ in X we set $\psi'(\gamma) := \lambda_{\gamma(0)} * \gamma * \lambda^{-1}_{\gamma(1)} \in \pi_1(X, x)$. and extend to a group homomorphism

$$\psi' : C_1(X) \to \pi_1(X, x)^{ab}$$

Now we claim that ψ' sends boundaries in $C_1(X)$ to the unit in $\pi_1(X, x)^{ab}$. To see this compute:

$$\begin{aligned}
\psi'(\partial\sigma) &= \psi'(\sigma^{(0)} - \sigma^{(1)} + \sigma^{(2)}) \\
&= [\lambda_{\sigma(0,1,0)} * \sigma^{(0)} * \lambda^{-1}_{\sigma(0,0,1)} * \left(\lambda_{\sigma(1,0,0)} * \sigma^{(1)} * \lambda^{-1}_{\sigma(0,0,1)}\right)^{-1} * \lambda_{\sigma(1,0,0)} * \sigma^{(2)} * \lambda^{-1}_{\sigma(0,1,0)}] \\
&= 0
\end{aligned}$$

Thus we get an induced homomorphism

$$\psi : H_1(X) \to \pi_1(X, x)^{ab}.$$

and we want to show that its inverse to ϕ. To this end we compute

$$\psi \circ \phi([\gamma]) = \psi'(\gamma) = \lambda_{\gamma(0)} * \gamma * \lambda^{-1}_{\gamma(1)} = [\gamma].$$

and conversely

$$\begin{aligned}
\phi \circ \psi \left(\sum_{\sigma:\Delta^1 \to X} n_\sigma \sigma \right) &= \sum n_\sigma \phi\psi(\sigma) \\
&= \sum n_\sigma \phi(\lambda_{\sigma(0)} * \sigma * \lambda^{-1}_{\sigma(1)}) \\
&= \sum n_\sigma (\lambda_{\sigma(0)} + \sigma - \lambda_{\sigma(1)}) \\
&= \sum n_\sigma \sigma + \sum n_\sigma (\lambda_{\sigma(0)} - \lambda_{\sigma(1)})
\end{aligned}$$

Now consider the homomorphism

$$\lambda : C_0X \to C_1X \qquad x \mapsto \lambda_x \ .$$

For $c = \sum n_\sigma \sigma$ we have $\lambda_{\partial c} = \sum n_\sigma(\lambda_{\sigma(1)} - \lambda_{\sigma(0)})$ on the one hand, and on the other hand $\partial c = 0$, thus $\lambda_{\partial c} = 0$. Thus the additional term is zero. □

Corollary 1.3.10. *We have*

$$\begin{aligned} H_1(S^1) &\cong \mathbb{Z} \\ H_1(S^k) &\cong 0 \textit{ for } k > 1 \\ H_1(\mathbb{R}P^n) &\cong \mathbb{Z}/2 \textit{ for } n \geq 2 \end{aligned}$$

For X contractible we have $H_0(X) \cong H_1(X) \cong 0$. More generally the groups H_0 and H_1 are invariants under homotopy equivalence in the sense that for a map $X \to Y$ that is a homotopy equivalence the induced map

$$H_i(X) \to H_i(Y)$$

is an isomorphism for $i = 0.1$.

Again we note that this invariance will later be shown for every i.

1.4 Categories and functors

In this section we want to formalise a bit more systematically what an invariant is.

Definition 1.4.1. A category $\mathcal{C}$ consists of

- a large set (aka. class) of objects
- for every pair of objects a, b a set $\mathrm{Hom}_{\mathcal{C}}(a, b)$.
- for every triple of objects a, b, c a composition map

$$\circ : \mathrm{Hom}_{\mathcal{C}}(b, c) \times \mathrm{Hom}_{\mathcal{C}}(a, b) \to \mathrm{Hom}_{\mathcal{C}}(a, c)$$

such that the following axioms are satisifed:

- composition is unital, that is for every object a there is a morphism $\mathrm{id}_a \in \mathrm{Hom}(a, a)$ such that $f \circ \mathrm{id}_a = f$ and $\mathrm{id}_a \circ g = g$ for each pair of morphisms f and g such that the composition is defined.
- composition is associative, that $(f \circ g) \circ h = f \circ (g \circ h)$.

We will also write $\mathrm{Hom}(a, b)$ for $\mathrm{Hom}_{\mathcal{C}}(a, b)$ if the ambient category is clear and fg for $f \circ g$.

Example 1.4.2. Consider the category Set whose objects are sets. For two sets A, B the set $\mathrm{Hom}_{\mathrm{Set}}(A, B)$ is given by the set of maps from A to B. The operations $\circ$ is composition of maps and the units are the identities.

Remark 1.4.3. What is a large set? It is a like a set, but it is too large to be a set. There are two equivalent ways of implementing this, the easier one is to fix two nested models of ZFC, called (small) sets and large sets such that the the set of all (small) sets is a large set.

Formally in order to speak of large sets, we have to add an axiom to the usual ZFC (Zermelo–Frankel set theory plus axiom of choice), a so-called large cardinal axiom (the obtained set theory is called ZFCU):

For every cardinal τ there exists a inaccessible cardinal κ with $\tau < \kappa$.

Here a cardinal κ is called inaccessible if the collection V_κ of all subset of V_κ of cardinality $< \kappa$ (this has to be defined correctly!) satisify the axioms of ZFC set theory (power sets, unions, contains $\aleph_0$). Then V_κ is called a universe. This axiom can not be proven from ZFC and is in fact logically independent. In particular every uncountable inaccessible cardinal is larger then all $\aleph_k$'s.

We now fix (once and for all) a chain of inaccessible cardinals $\kappa_0 < \kappa_1 < \kappa_2 < ...$ and thus a nested sequence of universes

$$V_{\kappa_0} \subsetneq V_{\kappa_1} \subsetneq V_{\kappa_2} \subsetneq ...$$

It is easy to see that $|V_{\kappa_i}| < \kappa_{i+1}$. Finally we agree on the notation

A (small) set means a set of cardinality $< \kappa_0$.
A large set (aka class) means a set of cardinality $< \kappa_1$.
A very large set means a set of cardinality $< \kappa_2$.

In particular we have that the set of all small sets is a large set. The set of all large sets is a very large set etc...

Examples 1.4.4.
1. The category Ab of abelian groups, whose objects are abelian groups and morphisms are group homomoprhism.
2. The category Top of topological spaces, objects topological spaces and maps are continuous maps.
3. The category Top^2 of pairs of spaces, whose objects consist of a topological space X together with a subspace $A \subseteq X$. Morphisms $(X, A) \to (Y, B)$ are given by continuous maps $f : X \to Y$ such that $f(A) \subseteq B$ (note that we get an induced continuous map $A \to B$.
4. The category Top_* of pointed topological spaces.
5. The category $\mathcal{H}\mathrm{Top}$, called homotopy category of topological spaces. Objects are topological spaces. Morphisms are homotopy classes of continuous maps .
6. The empty category
7. Let G be a group, then there is a category BG which is a single object denoted $*$. Then the set of morphisms $\mathrm{Hom}_{BG}(*, *) = G$. Composition is given by the group multiplication of G.

We depict morphisms $f \in \mathrm{Hom}_\mathcal{C}(c, d)$ in a category $\mathcal{C}$ as arrows $c \xrightarrow{f} d$ and call c its source and d its target.

Definition 1.4.5. An isomorphism in a category is a morphism $f : c \to d$ such that there exists a morphism $g : d \to c$ such that $gf = \mathrm{id}$ and $fg = \mathrm{id}$.

Examples 1.4.6. In Set and isomorphism is a bijection, in Ab it is a group isomorphism, in Top it is a homeomorphism.

Definition 1.4.7. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A Functor $F : \mathcal{C} \to \mathcal{D}$ is given by an assignment of an object $F(C)$ for every object C of $\mathcal{C}$ and a morphism $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(c), F(d))$ for every morphism $f : c \to d$ in $\mathcal{C}$ such that $F(f \circ g) = F(f) \circ F(g)$ and $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$.

Examples 1.4.8.

- For every category $\mathcal{C}$ there is the identity functor $\mathcal{C} \to \mathcal{C}$.
- There is a 'forgetful functor' Top $\to$ Set which assigns to every topological spaces X the underlying set of X (i.e. forgets the topology). A continuous map is sent to the underlying map of sets.
- There is a similar forgetful functor Ab $\to$ Set.
- Abelianization is a functor from the category of groups to the category of abelian groups.
- Homology is a functor $H_n : \mathrm{Top} \to \mathrm{Ab}$ (by Proposition 1.2.17).
- The fundamental group is a functor $\pi_1 : \mathrm{Top}_* \to \mathrm{Grp}$

Proposition 1.4.9. *Functors preserve isomorphisms, that is for an isomorphism $f : c \xrightarrow{\simeq} d$ the induced map $F(f) : Fc \to Fd$ is also an isomorphism. Composition of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ is a again a functor $G \circ F : \mathcal{C} \to \mathcal{E}$.*

Proof. Clear □

Definition 1.4.10 (Improvement of Definition 1.1.2)**.** A homotopy *invariant* is a functor $F : \mathrm{Top} \to \mathcal{C}$ where $\mathcal{C}$ is some algebraic category, which factors through the homotopy category, i.e. we have an induced functor $F : \mathcal{H}\mathrm{Top} \to \mathcal{C}$. Equivalently $F(f) = F(g)$ for f and g homotopic.

So if we have a homotopy invariant functor F and we find that $F(X) \neq F(Y)$ then we can conclude that X cannot be homotopy equivalent to Y. That is how we will use invariants.

Examples 1.4.11.

- $\pi_0 : \mathrm{Top} \to \mathrm{Set}$ is an invariant.
- $\pi_1 : \mathrm{Top} \to \mathrm{Gr}$ is an invariant.
- $H_0 : \mathrm{Top} \to \mathrm{Ab}$ is an invariant.
- $H_1 : \mathrm{Top} \to \mathrm{Ab}$ is an invariant.
- $H_n : \mathrm{Top} \to \mathrm{Ab}$ are functors, but we do not yet know whether they are homotopy invariant.

Definition 1.4.12. For a category $\mathcal{C}$ we define another category $\mathcal{C}^{op}$, called the opposite category. Objects of $\mathcal{C}^{op}$ are the same as objects of $\mathcal{C}$ and we set

$$\mathrm{Hom}_{\mathcal{C}^{op}}(a,b) := \mathrm{Hom}_{\mathcal{C}}(b,a) \ .$$

Composition is defined using the composition in $\mathcal{C}$:

$$\begin{aligned} &\mathrm{Hom}_{\mathcal{C}^{op}}(b,c) \times \mathrm{Hom}_{\mathcal{C}^{op}}(a,b) \\ &= \mathrm{Hom}_{\mathcal{C}}(c,b) \times \mathrm{Hom}_{\mathcal{C}}(b,a) \\ &\cong \mathrm{Hom}_{\mathcal{C}}(b,a) \times \mathrm{Hom}_{\mathcal{C}}(c,b) \xrightarrow{\circ} \mathrm{Hom}_{\mathcal{C}}(c,a) = \mathrm{Hom}_{\mathcal{C}^{op}}(c,a). \end{aligned}$$

It is straightforward to verify that this is indeed a category.

Definition 1.4.13. A contravariant functor from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F : \mathcal{C}^{op} \to \mathcal{D}$. That is explicitly: an assignment that assigns to every object of $c \in \mathcal{C}$ an object $F(c) \in \mathcal{D}$ and to every morphism $f : c \to c'$ in $\mathcal{C}$ a morphism $F(f) : f(c') \to f(c)$ such that $F(f \circ g) = F(g) \circ F(f)$ and $F(\mathrm{id}_c) = \mathrm{id}_{F(c)}$. If we want to explicitly distinguish functors in the sense of Definition 1.4.7 from contravariant functors we refer to the former as covariant functors.

Example 1.4.14. Consider the assignment which takes a vector space V to its dual space V^*. This is canonically a contravariant functor from the category of vector spaces to itself.

Definition 1.4.15. A natural transformation $\eta : F \to G$ between functors $F, G : \mathcal{C} \to \mathcal{D}$ is a map $\eta_c : F(c) \to G(c)$ for every object $c \in \mathcal{C}$ such that for every morphism $f : c \to d$ the diagram

$$\begin{tikzcd} F(c) \arrow[r, "\eta_c"] \arrow[d, "F(f)"'] & G(c) \arrow[d, "G(f)"] \\ F(d) \arrow[r, "\eta_d"] & G(d) \end{tikzcd}$$

commutes, i.e. the clockwise composition and the counterclockwise composition agrees. We will often abusively say: there is a morphism $F(C) \to G(C)$ which is natural in C.

Example 1.4.16. Consider the functor $F : \mathrm{Top}_* \xrightarrow{\pi_1} \mathrm{Grp}$ and the functor $G : \mathrm{Top}_* \to \mathrm{Top} \xrightarrow{H_1} \mathrm{Ab}$. Then we have a natural transformation $F \to G$ which is given by the morphisms

$$\pi_1(X,x) \to H_1(X) \qquad [\gamma] \mapsto [\gamma] \ .$$

This is clear, since for a given map by ...

Definition 1.4.17. A natural transformation $\eta : F \to G$ is called natural isomorphism if for every $c \in \mathcal{C}$ the morphism $F(c) \to G(c)$ is an isomorphism. In this case we write $F \simeq G$. A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F \simeq \mathrm{id}$ and $F \circ G \simeq \mathrm{id}$.

Example 1.4.18 (Covering theory)**.** Let (X, x) be a pointed, connected space which admits a universal cover $\tilde{X} \to X$ (e.g. locally simply connected). We consider the category $\mathrm{Cov}_{/X}$ of coverings of X. The objects are given by coverings $Y \to X$ (we do not require Y to be connected itself). A morphism is a covering transformation, i.e. a map $Y \to Y'$ over X.

By $\mathrm{Set}^{\pi_1(X,x)}$ we denote the category of sets equipped with a left action by the group $\pi_1(X, x)$ and morphisms the equivariant maps.

Now we have the functor

$$\mathrm{Cov}_{/X} \to \mathrm{Set}^{\pi_1(X,x)} \qquad (Y \xrightarrow{p} X) \mapsto (Y_x = p^{-1}(y))$$

and the main result of covering theory is that this functor is an equivalence of categories, The inverse is given by the functor

$$\mathrm{Set}^{\pi_1(X,x)} \to \mathrm{Cov}_{/X} \qquad S \mapsto \tilde{X} \times_{\pi_1(X,x)} S = (X \times S)/\pi_1(X,x)\ .$$

Proposition 1.4.19. *A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if an only if it satisfies the following conditions:*

1. *It is fully faithful, that is for each pair of objects $c, c' \in \mathcal{C}$ the map $\mathrm{Hom}_{\mathcal{C}}(c, c') \to \mathrm{Hom}_{\mathcal{D}}(Fc, Fc')$ is a bijection.*
2. *It is essentially surjective, that is for each object $d \in \mathcal{D}$ there exists an object $c \in \mathcal{C}$ such that $Fc \cong d$.*

Proof. Assume that F is an equivalence with inverse G. We first want to prove that (1) and (2) hold. For (1) we consider the composition map

$$\mathrm{Hom}_{\mathcal{C}}(c, c') \to \mathrm{Hom}_{\mathcal{D}}(Fc, Fc') \to \mathrm{Hom}_{\mathcal{C}}(GFc, GFc') \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{C}}(c, c') \tag{1.1}$$

where the last map is obtained using the natural isomorphisms $GFc \to c$ and $GFc' \to c'$ which are part of the natural equivalence $GF \to \mathrm{id}$. We claim that this composite map is the identity. To see this we have to argue that for each $f : c \to c'$ the diagram

$$\begin{tikzcd} GFc \arrow[r, "GFf"] \arrow[d] & GFc' \arrow[d] \\ c \arrow[r, "f"] & c' \end{tikzcd}$$

commutes. But this is part of the naturality of the transformation. Thus (1.1) is a bijection which shows that the map $\mathrm{Hom}_{\mathcal{C}}(c, c') \to \mathrm{Hom}_{\mathcal{D}}(Fc, Fc')$ is injective and $\mathrm{Hom}_{\mathcal{D}}(Fc, Fc') \to \mathrm{Hom}_{\mathcal{C}}(GFc, GFc')$ is surjective. Applying the same reasoning with interchanged roles of G and G we deduce that the map $\mathrm{Hom}_{\mathcal{D}}(Fc, Fc') \to \mathrm{Hom}_{\mathcal{C}}(GFc, GFc')$ is also injective, thus both are bijective. This shows that F is fully faithful. To see that F is essentially surjective, we simply note that for $d \in \mathcal{D}$, the object $c := Gd$ does the job since $Fc = GFd \cong d$.

Now for the converse: assume that $F : \mathcal{C} \to \mathcal{D}$ satisfies (1) and (2). We want to construct an inverse $G : \mathcal{D} \to \mathcal{C}$. First by essential surjectivity of F we can choose for every $d \in \mathcal{D}$ an object $c \in \mathcal{C}$ together with an isomorphism $\eta_c : Fc \xrightarrow{\simeq} d$ (this required the axiom of choice). Now we set

$$G(d) := c\ .$$

This defined G for objects. For a morphism $f : d \to d'$ we let Gf be the unique morphism such that FGf makes the diagram

$$\begin{tikzcd} FGd \arrow[r, "FGf"] \arrow[d] & FGd' \arrow[d] \\ d \arrow[r, "f"] & d' \end{tikzcd}$$

commutative (this defines Gf by fully faithfulness). Now it is easy to see that G defines a functor and that η defines a natural isomorphism $GF \xrightarrow{\simeq}$ id. It remains to show that $GF \cong \mathrm{id}$. But we have a natural isomorphism

$$\eta_{Fc} : FGF(c) \to F(c) \ .$$

By fully faithfulness of F this gives rise to a morphism

$$\mu_c : GF(c) \to c \ .$$

Now it is straightforward to see that μ_c is also an isomorphism and that it is natural in c (by using the uniqueness and reducing to the respective properties of η_{Fc}). □

Example 1.4.20. Consider the category Vect of finite dimensional $\mathbb{R}$-vector spaces. We consider the 'sub-category' $\mathrm{Vect}' \subseteq \mathrm{Vect}$ consisting of the vector spaces $\mathbb{R}^n$ for each n and all linear maps between them. Then the inclusion is an equivalence of categories.

1.5 Chain complexes and relative homology

Definition 1.5.1. A graded abelian group is a collection of abelian group C_i indexed by the integers. We denote the graded group by C_*. A morphism of graded groups $C_* \to D_*$ is a sequence of morphisms $f_i : C_i \to D_i$. This defines the category grAb of graded abelian groups.

Example 1.5.2. The homology groups $H_*(X)$ of a space X form a graded abelian group. It follows from Proposition 1.2.17 that the assignment $X \mapsto H_*(X)$ together with the morphisms f_* for continuous maps $f : X \to Y$ form a functor

$$H_* : \mathrm{Top} \to \mathrm{grAb}$$

where we set $H_k(X) := 0$ for $k < 0$.

Definition 1.5.3. A chain complexes is a graded abelian group C_* together with a sequence of morphisms $\partial : C_i \to C_{i-1}$ such that $\partial^2 = 0$, depticted as

$$\ldots \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\partial} C_{-1} \xrightarrow{\partial} C_{-2} \xrightarrow{\partial} ...$$

A morphism of chain complexes is a sequence of morphisms $f_i : C_i \to D_i$ such that $\partial f = f\partial$ i.e. the ladder diagram commutes. The category of chain complexes is denoted Ch.

Example 1.5.4.

- The assigment $X \mapsto C_*(X)$ defines a functor

  $$C_* : \mathrm{Top} \to \mathrm{Ch}$$

  For that we let the chain complex $C_*(X)$ be defined as

  $$\to C_3(X) \xrightarrow{\partial} C_2(X) \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_0(X) \to 0 \to 0 \to$$

  This is a chain complex by Lemma 1.2.8 and the assignment C_* is functorial by Definition 1.2.13 and Lemma 1.2.

- Let C_* be a chain complex. Then we define its homology $H_n(C_*)$ as $Z_n(C_*)/B_n(C_*)$. This defines a functor
$$H_* : \mathrm{Ch} \to \mathrm{grAb}$$
as shown in Corollary 1.2.15 and Proposition 1.2.17.

- The composition $H_* \circ C_*$ is the singular homology functor from example 1.5.2

Recall that a sequence $A \xrightarrow{i} B \xrightarrow{p} C$ of abelian groups is called exact (at B) if $\operatorname{im} i = \ker p$. It is called short exact if the sequence $0 \to A \to B \to C \to 0$ is exact at A, B and C. Concretely this means that i is injective, p is surjective, $p \circ i = 0$ and the induced map $B/A \to C$ is an isomorphism.

Definition 1.5.5. A sequence of chain complexes $0 \to A_* \to B_* \to C_* \to 0$ is called short exact, if for each index i the induced sequence $0 \to A_i \to B_i \to C_i \to 0$ is short exact. A morphism of chain complexes $A_* \to B_*$ is called monomorphism (resp. epimorphism) if the morphisms $A_i \to B_i$ are injective (surjective).

Lemma 1.5.6. *For every monomorphism of chain complexes $0 \to A_* \to B_*$ the quotient complex defined as $C_* := B_*/A_*$ is a well defined chain complex and the sequence $0 \to A_* \to B_* \to C_* \to 0$ is short exact.*

Proof. We only have to check that the differential of B descends to a well defined map $\partial : B_n/A_n \to B_{n-1}/A_{n-1}$. But this is clear as the differential of B takes A_n to A_{n-1} by the fact that $A \to B$ is a chain map. $\square$

Example 1.5.7. Let $A \subseteq X$ be a subspace. Then $C_*(A) \to C_*(X)$ is injective, thus we get the exact sequence
$$0 \to C_*(A) \to C_*(X) \to C_*(X, A) \to 0$$
with $C_*(X, A) := C_*(X)/C_*(A)$. We define relative Homology groups as
$$H_n(X, A) := H_n(C_*(X, A))$$
and have $H_n(X, \emptyset) = H_n(X)$ and $H_n(X, X) = 0$.

Theorem 1.5.8. *For every short exact sequence of chain complexes $0 \to A_* \to B_* \to C_* \to 0$ there is an induced long exact sequence*
$$\ldots \xrightarrow{\delta_*} H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{p_*} H_n(C) \xrightarrow{\delta_*} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{p_*} H_{n-1}(C) \xrightarrow{\delta_*} \ldots$$
This sequence is natural in the sequence $0 \to A_* \to B_* \to C_* \to 0$.

Here naturallity in $0 \to A_* \to B_* \to C_* \to 0$ means that for a diagram
$$\begin{tikzcd} A_* \arrow[r] \arrow[d] & B_* \arrow[r] \arrow[d] & C_* \arrow[d] \\ A'_* \arrow[r] & B'_* \arrow[r] & C'_* \end{tikzcd}$$
commutative diagram of short exact sequences the induced ladder diagram involving the long exact sequences is commutative as well (How is this a natural transformation?).

Proof. We assume that $A_n \to B_n$ is actually a subset inclusion for every n for simplicity of notation. We look at the diagram

$$\begin{tikzcd} A_{n+1} \arrow[r, "i"] \arrow[d, "\partial"] & B_{n+1} \arrow[r, "p"] \arrow[d, "\partial"] & C_{n+1} \arrow[d, "\partial"] \\ A_n \arrow[r, "i"] \arrow[d, "\partial"] & B_n \arrow[r, "p"] \arrow[d, "\partial"] & C_n \arrow[d, "\partial"] \\ A_{n-1} \arrow[r, "i"] & B_{n-1} \arrow[r, "p"] & C_{n-1} \end{tikzcd}$$

We first want to define the natural map $\delta_n : H_n(C) \to H_{n-1}(A)$. Given a class $[c] \in H_n(C)$ with $c \in Z_n(C)$ pick a preimage $b \in B_n$ with $pb = c$. Then $p\partial b = \partial pb = \partial c = 0$. Thus $\partial b \in A_{n-1}$. Moreover we have that $\partial(\partial c) = 0$ (here the ∂ is the differential of the complex A but this 'agrees' with the one of the complex B under the inclusion by assumption). Thus we get a class $[\partial b] \in H_{n-1}(A)$. We claim that the assignment

$$[c] \in H_n(C) \mapsto [\partial b] \in H_{n-1}(C)$$

is well-defined. We have made two choices: the choice of c representing the class and the choice of lift b. First assume we had chosen a different lift $b' \in B$. Then $b-b'$ lies in A_n and we have that $\partial(b-b') = \partial b - \partial b'$. Thus $[\partial b] = [\partial b'] \in H_{n-1}(A)$. This shows independence of the lift b. Now assume we had picked another representative of the class c' so that $c' - c = \partial c_{n+1}$. Then pick a preimage $b_{n+1} \in B_{n+1}$ and consider the element $b' := b + \partial b_{n+1} \in B_n$. We find that $p(b + \partial b_{n+1}) = c + p\partial b_{n+1} = c + (c' - c) = c$". Thus we can use b' as our choice of lift to compute the image of δ on c' to get the class $[\partial b'] \in H_{n-1}(A)$. But $\partial b' = \partial b$ which show that we get the same class as for the representative c.

This establishes the map δ. By construction it is clear that this map δ is natural in the initial sequence since for a map f of sequences $A \to B \to C$ to $A' \to B' \to C'$ and an element in the image of the map $H_nC \to H_nC'$ we can pick the image of the choice of c and b as the choices of b' and c' and then the diagram

$$\begin{tikzcd} H_n(C) \arrow[r] \arrow[d] & H_n(C') \arrow[d] \\ H_{n-1}(A) \arrow[r] & H_{n-1}(A') \end{tikzcd}$$

commutes.

Now it remains to show that the resulting long sequence

$$... \xrightarrow{\delta_*} H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{p_*} H_n(C) \xrightarrow{\delta_*} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{p_*} H_{n-1}(C) \xrightarrow{\delta_*} ...$$

is exact. We distinguish three cases:

$\underline{\mathrm{Im}(i_*) = \ker(p_*)}$ Clearly $p_* i_* = 0$ thus $\mathrm{Im}(i_*) \subseteq \ker(p_*)$. For the converse assume that $[b] \in \ker(p_*)$, i.e. there is $c \in C_{n+1}$ with $\partial c = pb$. Choose a preimage $b_{n+1} \in B_{n+1}$. Then $[b] = [b - \partial b_{n+1})] \in H_n(B)$ and $p(b - \partial b_{n+1}) = 0$ thus $b - \partial b_{n+1} \in A_n$. Since it is also a cycle this shows that $[b] in \mathrm{Im}(i_*)$.

Exactness at the other spots works similar. □

Corollary 1.5.9. *For a subspace $A \subseteq X$ we obtain a long exact sequence*

$$\dots \xrightarrow{\delta_*} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{p_*} H_n(X,A) \xrightarrow{\delta_*} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(X) \xrightarrow{p_*} H_{n-1}(X,A) \xrightarrow{\delta_*} \dots$$

This sequence is natural in the pair (X,A).

Definition 1.5.10 (Homology with coefficients)**.** Let X be a space and G an abelian group. Define the abelian group of G-valued n-chains on X the be the group

$$\begin{aligned} C_n(X;G) &:= \left\{ \sum_{\sigma:\Delta^n \to X} n_\sigma \sigma \mid n_\sigma \in G, \text{ almost all } n_\sigma = 0 \right\} \\ &= \bigoplus_{\Delta^n \to X} G \end{aligned}$$

Then with the same definition as ... this becomes a chain complex $C_*(X;G)$ and we define

$$H_n(X;G) := H_n(C_*(X;G))$$

Example 1.5.11. $H_n(X) = H_n(X,\mathbb{Z})$. If R is a ring, then $H_n(X;R)$ is an R-module. Most important $R = \mathbb{F}_2 = \mathbb{Z}/2$. Then $H_n(X;\mathbb{Z}/2)$ is a vector space over $\mathbb{Z}/2$. Similarly $H_n(X;\mathbb{Q})$ is a rational vector space.

Corollary 1.5.12 (Bockstein sequence)**.**

- *Let $0 \to G \to H \to K \to 0$ be a short exact sequence of abelian groups and X be a space. Then we obtain a long exact sequence*

$$\dots \xrightarrow{\delta_*} H_n(X;G) \xrightarrow{i_*} H_n(X;L) \xrightarrow{p_*} H_n(X;K) \xrightarrow{\delta_*} H_{n-1}(X;G) \xrightarrow{i_*} H_{n-1}(X;L) \xrightarrow{p_*} \dots$$

- *For the sequence $\mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/p$ we get the induced long exact sequence*

$$\dots \xrightarrow{\beta_*} H_n(X;\mathbb{Z}) \xrightarrow{\cdot p} H_n(X;\mathbb{Z}) \xrightarrow{\pi_*} H_n(X;\mathbb{Z}/p) \xrightarrow{\beta} H_{n-1}(X;\mathbb{Z}) \xrightarrow{\cdot p} H_{n-1}(X;\mathbb{Z}) \xrightarrow{\pi_*} \dots$$

Proof. We have to argue that the induced sequence $0 \to C_n(X,G) \to C_n(X,H) \to C_n(X,L) \to 0$ is exact. But this is just a direct sum of the short exact sequences $0 \to G \to H \to K \to 0$, summed over all simplices $\sigma : \Delta^n \to X$. Clearly, direct sums of short exact sequences are short exact since kernel (resp. image) of a direct sum of morphisms is the direct sum of the kernels (resp. images). □

Example 1.5.13. Compute $H^1(S^1,\mathbb{Z}/p) = \mathbb{Z}/p$. Conclude that $H^1(S^1,G) \cong G$ for every finitely generated abelian group.

Lemma 1.5.14 (Fivelemma)**.** *Consider a diagram of the form*

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "f_1"] & A_2 \arrow[r] \arrow[d, "f_2"] & A_3 \arrow[r] \arrow[d, "f_3"] & A_4 \arrow[r] \arrow[d, "f_4"] & A_5 \arrow[d, "f_5"] \\ B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5 \end{tikzcd}$$

in which the horizontal morphisms are exact and f_1, f_2, f_3 and f_4 are isomorphisms. Then also f_3 is an iso.

1.6 Axioms for homology

In this section we want to isolate the important properties of the functors H_n.

Definition 1.6.1 (Eilenberg-Steenrod)**.** A *homology theory* is a functor

$$E_* : \mathrm{Top}^2 \to \mathrm{grAb}$$

together with morphisms

$$\delta : E_n(X, A) \to E_{n-1}(A) \qquad \big(\text{here we set } E_{n-1}(Y) := E_{n-1}(Y, \emptyset)\big)$$

called connecting homomorphisms, which are natural in the pair (A, X) such that the following axioms are satisfied:

1. (Homotopy invariance) If $f \simeq g : (X, A) \to (Y, B)$ then $f_* = g_* : E_n(X, B) \to E_n(Y, B)$
2. (Long exact sequence) For a pair $i : A \subseteq X$ (with morphism $j : (X, \emptyset) \to (X, A)$ the sequence
$$... \xrightarrow{\delta} E_n(A) \to E_n(X) \to E_n(X, A) \xrightarrow{\delta} E_{n-1}(A) \to ...$$
is exact.
3. (Excision) Given (X, A) and $U \subseteq A$ with $\overline{U} \subseteq \mathrm{int}(A)$ then the morphisms
$$H_n(X \setminus U, A \setminus U) \to H_n(X, A)$$
induced by the inclusion is an isomorphism.
4. (Additivity) For a topological space $X \cong \coprod_{\alpha \in A} X_\alpha$ the canonical morphism
$$\bigoplus E_*(X_\alpha) \to E_*(X)$$
is an isomorphism.
5. (Dimension axiom) We have $E_n(\mathrm{pt}) = 0$ for all $n \neq 0$.

Theorem 1.6.2. *Singular homology $H_n(, -; G)$ is a homology theory for every abelian group G. Conversely every homology theory E agrees on a huge class of spaces (CW-complexes) with singular homology with coefficients in $G := E(\mathrm{pt})$.*

Proof. The (long exact sequence) axiom is Corollary 1.5.9 and the additivity follows (as in the proof of Proposition 1.3.3). We will proof the other axioms later but use this theorem for now (the proof is independent from everything that follows). Uniqueness will follow from the comparison with cellular homology. □

Remark 1.6.3.

- If all axioms except for the dimension axiom are satisfied, then the theory is called *generalized homology theory.* There are much more homology theories than singular homology.
- The Additivity axiom for a finite index set A follows from the other axioms (exercise). So its really only an axiom for infinite disjoint unions.

- It is essential for the Excision axiom that the closure is contained in the interior, otherwise there are counterexamples (exercises).
- The group $E_0(*) = G$ is called the coefficient group of the homology theory. So singular homology with coefficients in G has coefficients group G. If not otherwise specified and we use singular homology, we implicitly assume $G = \mathbb{Z}$.

In what follows let E be a (generalized) homology theory and define the reduced homology groups

$$\widetilde{E}_n(X) := \ker(E_n(X) \xrightarrow{p_*} E_n(*))$$

where $p : X \to *$ is the unique map.

Proposition 1.6.4 (Easy properties)**.** *1. If $(X, A) \simeq (Y, B)$ then $E_*(X, A) \cong E_*(Y, B)$.*

2. If $(X, A) \xrightarrow{f} (Y, B)$ has the property that $f : X \to Y$ and $f|_A : A \to B$ are homotopy equivalences, then $f_* : E_*(X, A) \to E_*(Y, B)$ is an isomorphism.

3. If X is contractible, then $\widetilde{E}_n(X) = 0$ for all n.

4. If $A \to X$ is a homotopy equivalence, then $E_n(X, A) = 0$ for all n.

5. If X is contractible, then $E_{n+1}(X, A) \cong \widetilde{E}_n(A)$.

6. Every point $x \in X$ induces an isomorphism $E_n(X) \cong \widetilde{E}_n(X) \oplus E_n(\{x\})$ which is natural in (X, x). Moreover we have $\widetilde{E}_n(X) \cong E_n(X, \{x\})$.

Proof. 1. Easy from homotopy invariance and functoriality.

2. Use the long exact sequence and the 5-Lemma
3. The map $E_*(X) \to E_*(*)$ is an isomorphism.
4. Use the long exact sequence.
5. Use the long exact sequence plus the observation that $E_*(X) \to E_*(X, A)$ is zero since it factors over $E_*(X, \{a\})$ for some $a \in A$ and the latter is contractible by the previous point.
6. A point x exhibits a retract of $X \to \mathrm{pt}$ and thus a decomposition of $E_*(X)$ into $\tilde{E}_*(X)$ and $E_*(\mathrm{pt})$. This also show that the maps $E_*(\{x\}) \to E_*(X)$ are injective, thus the connecting homomorphisms in the long exact sequence vanish showing the last claim. □

Recall that an inclusion $A \subseteq V$ is said to be a deformation retract, if there is a homotopy $H : V \times I \to V$ with $H(x, 0) = x$, $H(x, 1) \in A$ for all x and $H(a, t) = a$ for all a. In particular the inclusion $A \simeq V$ is a homotopy equivalence..

Proposition 1.6.5. *Let $A \to X$ be a closed subspace which has a open neighborhood that deformation retracts onto A. Then the canonical morphism*

$$E_n(X, A) \to \widetilde{E}_n(X/A)$$

is an isomorphism for all n.

Proof. First observe that $E_n(X/A, A/A) \cong \widetilde{E}_n(X/A)$ by proposition 1.6.4.

Let $A \subseteq V \subseteq X$. Consider

$$\begin{tikzcd}
(X \setminus A, V \setminus A) \arrow[r] \arrow[d, "q|_{X\setminus A}"] & (X, V) \arrow[d, "q"] \\
(X/A \setminus A/A,\ V/A \setminus A/A) \arrow[r] & (X/A, A/A)
\end{tikzcd}$$

Then the assumptions imply that $q|_{X\setminus A}$ is a homeomorphism (here we use that A is closed). In particular it induces an isomorphism in homology. Now by excision the two horizontal maps induce isomorphisms in homology. Thus we can conlude that q also induces an isomorphism in homology

$$q_* : E_n(X, V) \to E_n(X/A, A/A)$$

Now we use that V deformation retracts onto A. Thus we have that the inclusion $(X, A) \to (X, V)$ induces an isomorphism in homology (by Proposition 1.6.4). Similarly we have that $(X/A, A/A) \to (X/A, V/A)$ induces an isomorphism in homology. Thus we consider the square

$$\begin{tikzcd}
(X, A) \arrow[r] \arrow[d] & (X, V) \arrow[d, "q"] \\
(X/A, A/A) \arrow[r] & (X/A, V/A)
\end{tikzcd}$$

Since the two horizontal maps and the right vertical map induce isomorphisms in homology, also the left hand morphism has to induce an isomorphism. □

Definition 1.6.6. Let X be a topological space. The (unreduced) suspension is

$$SX := (X \times [0, 1]/(X \times \{0\})/(X \times \{1\}).$$

This can also be seen as the quotient of $X \times [0, 1]$ by a single equivalence relation. Draw a picture. If X has basepoint $x \in X$, then the reduced suspension ΣX is defined as the space

$$\Sigma X := \frac{X \times [0, 1]}{X \times \{0\} \cup X \times \{1\} \cup \{x\} \times [0, 1]} = SX/(\{x\} \times [0, 1])$$

Example 1.6.7. $S(S^n) \cong S^{n+1}$ through the isomorphism $(x, t) \mapsto (\sin(\pi t) \cdot x, \cos(\pi t))$.

Remark 1.6.8. Note that we have if X is well pointed (that is the inclusion of the point is neighborhood deformation retract), then

$$\widetilde{E_*}(SX) \cong E_*(SX, (x, 0)) \cong E_*(SX, x \times I) \cong \widetilde{E}_*(\Sigma X)$$

Otherwise we are lost...

Theorem 1.6.9 (Suspension isomorphism)**.** *There are canonical isomorphisms*

$$\widetilde{E}_n(X) \cong \widetilde{E}_{n+1}(SX)$$

which are natural in X.

Proof. We compute

$$\widetilde{E}_{n+1}(SX) \cong E_{n+1}(SX, \{x\} \times [0,1]) \cong E_{n+1}(SX, C_+X)$$

Similarly we have

$$E_{n+1}(C_-X, X) \cong E_{n+1}\left(SX \setminus \frac{1}{2}C_+X, C_+X \setminus \frac{1}{2}SX_+\right) \cong E_{n+1}(SX, C_+X)$$

and then $E_{n+1}(C_-X, X) \cong \widetilde{E}_n(X)$ since C_-X is contractible. □

1.7 Applications

For this chapter we let H_* be a homology theory. For these applications we really only have to assume the existence of a homology theory.

Theorem 1.7.1. *For $k \geq 0$ we have*

$$H_n(S^k) = \begin{cases} \mathbb{Z} & \text{for } k = 0 \text{ and } k = n \\ 0 & \text{else} \end{cases}$$

Proof. We use that $S^n = \Sigma S^{n-1}$ and compute

$$\widetilde{H}^k(S^n) \cong \widetilde{H}^{k-1}(S^{n-1}) \cong \ldots \cong \widetilde{H}^{n-k}(S^0) \cong H^{n-k}(*).$$ □

Corollary 1.7.2. *The spheres S^n and S^m can not be homotopyequivalent (or homeomorphic). The sphere S^{n-1} is not a retract of the disk D^n (i.e. there is no continuous left inverse to the inclusion $i : S^{n-1} \to D^n$).*

Proof. First is clear since homology groups differ. The second follows since for a retract $r : D^n \to S^{n-1}$ we would obtain a factorization

$$H_{n-1}(S^{n-1}) \xrightarrow{i_*} H_{n-1}(D^n) \xrightarrow{r_*} H_{n-1}(S^{n-1})$$

but $H_{n-1}(D^n) = 0$ and $r_*i_* = (ri)_* = \mathrm{id}$. □

Corollary 1.7.3 (Brouwer Fixed Point theorem)**.** *Any map $D^n \to D^n$ has a fixed point.*

Proof. Assume $f : D^n \to D^n$ had no fixed point. Then define a new map $r : D^n \to S^{n-1}$ by $r(x)$ is the intersection point between the line $L_{x,r(x)}$ originating in $r(x)$ and ranging through x and $S^{n-1} \to D^n$.

Picture

Then clearly r is continuous and defines a retraction of D^n to S^{n-1} which is impossible by Corollary 1.7.2. □

Definition 1.7.4. A topological manifold of dimension n is a second countable Hausdorff space M such that every point in M has a neighborhood homeomorphic to $\mathbb{R}^n$.

Examples 1.7.5.

- Spheres S^n and $\mathbb{R}^n$ for $n \leq 0$ are manifolds of dimension n.
- Projective spaces $\mathbb{R}P^n$ and $\mathbb{C}P^n$ are manifolds of dimension n and $2n$.

- Genus g surfaces Σ_g.
- The figure 8 as a subset of $\mathbb{R}^2$ is not a manifold.
- Knot complements in S^3 (or $\mathbb{R}^3$).

Theorem 1.7.6 (Invariance of dimension)**.** *Let M^n and N^m be manifolds of dimensions $m \neq n$. Then M^n and N^m cannot be homeomorphic. In particular $\mathbb{R}^n$ and $\mathbb{R}^m$ cannot be homeomorphic.*

Proof. We embed a closed disk D^n around m into M and compute:

$$H_k(M, M \setminus \{m\}) \cong H_k(D^n, D^n \setminus \{0\}) \cong \widetilde{H}_{k-1}(D^m \setminus 0) \cong \widetilde{H}_{k-1}(S^{m-1})$$

Thus we obtain

$$H_k(M, M \setminus \{m\}) = \begin{cases} \mathbb{Z} & \text{for } k = n \\ 0 & \text{else} \end{cases}$$

This implies the claim since every homemorphism $f : M^n \to N^n$ would induces isomorphisms

$$H_n(M, M \setminus \{m\}) \cong H_n(M, M \setminus \{f(m)\})$$ □

Remark 1.7.7. Note that the last argument fails for homotopy equivalences. In fact manifolds of different dimension can be homotopy equivalent as $\mathbb{R}^n$ and $\mathbb{R}^m$ shows. But for compact manifolds this cannot happen as we will see later using Poincaré-Duality.

Definition 1.7.8. Let $f : S^n \to S^n$ be a continuous map. Then

$$f_* : H_n(S^n) \to H_n(S^n)$$

is given by multiplication with an integer, which we denote by $\deg(f)$.

Clearly homotopic maps have the same degree.

Example 1.7.9. For $n = 1$ this is the winding number. Consider the map $f : S^1 \to S^1, z \mapsto z^k$. Then we have that the induced map $\pi_1(S^1) \to \pi_1(S^1)$ is multiplication with k. Thus by naturality also the map $H_1(S^1) \to H_1(S^1)$ is multiplication by k. Thus $\deg(f) = k$. We see that

$$\deg : [S^1, S^1] \to \mathbb{Z}$$

is bijective.

Lemma 1.7.10. *1. Let $f, g : S^n \to S^n$. Then $\deg(fg) = \deg(f)\deg(g)$.*

2. Consider the map

$$f : S^n \to S^n \qquad (x_1, ..., x_n) \mapsto (-x_1, x_2, ..., x_n)$$

Then $\deg(f) = -1$.

Proof. The first follows from functoriality and the second assertion we prove inductively. First for S^0 we have

$$H_0(S^0) \cong a\mathbb{Z} \oplus b\mathbb{Z}$$

and the map induced by f is given by $(a, b) \mapsto (b, a)$. Then we get

$$\widetilde{H}_0(S^0) = (a - b)\mathbb{Z}$$

and the induced map is thus given by

$$\mathbb{Z} \to \mathbb{Z}, \qquad n \mapsto -n$$

Thus we have $\deg(f) = -1$. Since the suspension isomorphism is natural in X we get a commutative diagram

$$\begin{tikzcd} \widetilde{H}_{n+1}(S^{n+1}) \arrow[r, "\Phi"] \arrow[d, "\deg \Sigma(f)"] & \widetilde{H}_n(S^n) \arrow[d, "\deg f"] \\ \widetilde{H}_{n+1}(S^{n+1}) \arrow[r, "\Phi"] & \widetilde{H}_n(S^n) \end{tikzcd}$$

and thus we get $\Phi \circ \deg(\Sigma(f)) = \deg(f) \circ \Phi$, i.e. $\deg(f) = \deg(\Sigma(f))$. But its easy to see that Σf is the map f for $n + 1$. □

Corollary 1.7.11. *1. The antipodal map $A : S^n \to S^n$ defined by $A(x_0, ..., x_n) = (-x_0, ..., -x_n)$ has degree $(-1)^{n+1}$.*

2. For n even, A is not homotopic to the identity.

3. For n even and $f : S^n \to S^n$ there is a point $x \in S^n$ such that $f(x) = \pm x$.

4. For n even: if a map $f : S^n \to S^n$ has degree $\neq -1$, then f has a fixed point.

Proof. 1. Clear by Lemma 1.7.10 since its the composition of n-maps of degree -1.

2. Clear, since homotopic maps have the same degree.

3. If there is no point x with $f(x) = x$, then we have $f \simeq A$ since we can take the ray between $f(x)$ and x and let it run around the sphere. Formula:

$$H(x, t) = \frac{tf(x) + (t - 1)(-x)}{|tf(x) + (t - 1)(-x)|}$$

Similarly, if there is no point with $f(x) = -x$ then $f \simeq \mathrm{id}$.

Thus if there is not point x with $f(x) = \pm x$ then $\mathrm{id} \simeq f \simeq A$ which contradicts the last statement.

4. As before, if f had no fixed point, then $f \simeq A$ which shows degree equals -1.

□

Remark 1.7.12. The theorem of Hopf shows that if $f, g : S^n \to S^n$ have the same degree, then they are homotopic.

Definition 1.7.13. A tangent vector field for the sphere S^n is a continuous map $V : S^n \to \mathbb{R}^{n+1}$ such that $\langle V(x), x\rangle = 0$ for every $x \in S^n$ (this is the same as a comtinuous section of the tanget bundle $TS^n \to S^n$).

Example 1.7.14. Consider the map $V : S^1 \to \mathbb{R}^2$ defined by $V(x, y) = (-y, x)$ (picture).

This is a nowhere vanishing vector field. More generally we have $V : S^{2n-1} \to \mathbb{R}^{2n}$ given by $V(x_1, ..., x_n, y_1, ..., y_n) = (-y_1, .., -y_n, x_1, ..., x_n)$.

Theorem 1.7.15 (Hairy ball theorem / Satz vom Igel)**.** *Every vector field on S^{2n} has a zero.*

Proof. Assume $V : S^{2n} \to \mathbb{R}^{n+1}$ is a vectorfield. Norm it to get a map $\widetilde{V} : S^{2n} \to S^{2n}$. We have $V(X) \neq \pm x$ since $\langle x, \widetilde{V}(x)\rangle = 0$. This contradicts Corollary 1.7.11. □

1.8 Colimits

Recall the definition of $\mathbb{C}P^n$ as a topological manifold as the quotient $\mathbb{C}P^n = S^{2n+1}/U(1)$. In the exercises we showed that

$$\mathbb{C}P^{n+1} = \mathbb{C}P^n \cup_f D^{2n+2}$$

where $f : S^{2n+1} \to \mathbb{C}P^n$. In particular the square

$$\begin{tikzcd}
S^{2n+1} \arrow[r, "q"] \arrow[d] & \mathbb{C}P^n \arrow[d] \\
D^{2n+2} \arrow[r, "Q"] & \mathbb{C}P^{n+1}
\end{tikzcd}$$

is a pushout.

Definition 1.8.1. A square

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}$$

in a category is called a pushout, if it satisfies the following universal property: for every other square (consisting of A, B, C, E)

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d] \arrow[ddr] & \\
C \arrow[r] \arrow[drr] & D \arrow[dr, dashed] & \\
& & E
\end{tikzcd}$$

there is a unique morphism $D \to D'$ making the whole diagram commutative. In this case D is called the pushout of the diagram $A \to B$ and $B \to C$.

Remark 1.8.2. The pushout is, if it exists, uniquely determined. Namely if D and D' are both pushouts, then using the universal property of D we get a map $D \to D'$ and using the universal property of D' we get a map $D' \to D$ such that all the relevant diagrams commute. Now both composite have to be the identities again using universal properties.

Example 1.8.3. In topological spaces the pushout of a diagram $A \to B$ and $A \to C$ is given by the 'glueing' construction $B \cup_A C$. This can be seen by verifying the universal property: for another space E with maps $B \to E$ and $C \to E$ such that the composites $A \to B \to E$ and $A \to C \to E$ agree we first get an induced map

$$B \amalg C \to E$$

and this factors over the quotient

$$B \cup_A C = B \amalg C/\sim$$

precisely because the maps from A agree. Then the resulting map is continuous by definiton of the quotient topology.

Example 1.8.4. In abelian groups the pushout is given by the quotient

$$B \oplus_A C = \frac{B \oplus C}{\operatorname{im}(A \xrightarrow{(f,-g)} B \oplus C)} .$$

In particular for any morphism $f : A \to B$ of abelian groups the square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ 0 \arrow[r] & B/\operatorname{im}(f) \end{tikzcd}$$

is a pushout.

Definition 1.8.5. Let $\mathcal{C}$ be a category and $c_0 \to c_1 \to c_2 \to c_3 \to ...$ be a sequence of objects. A cone (under c_i) is an object d in $\mathcal{C}$ together with morphisms $j_n : c_n \to d$ such that the triangles all commute

$$\begin{tikzcd} c_0 \arrow[r] \arrow[drrr] & c_1 \arrow[r] \arrow[drr] & c_2 \arrow[r] \arrow[dr] & c_3 \arrow[r] \arrow[d] & c_4 \arrow[r] \arrow[dl] & ... \arrow[dll] \\ & & & d & & \end{tikzcd}$$

A cone (c, i_n) is called limit of the sequence c_i, if for every other cone (d, j_n) there is a unique morphism $f : c \to d$ such that $fi_n = j_n$ for all n. In this case we write $c = \varinjlim c_i$.

Proposition 1.8.6.

- *If the limit of the sequence c_i exists, then it is unique up to unique isomorphisms. More precisely: if (c, i_n) and (c', i'_n) are limits, then there is a unique isomorphism $f : c \to c'$ that commutes with the structure maps.*
- *In the category* Top *let X be a space together with an open sequence $U_0 \subseteq U_1 \subseteq U_2 \subseteq U_3 \subseteq ...$ in X. Then $\varinjlim U_i \cong \bigcup_{i\in\mathbb{N}} U_i$ with the subspace topology and the canonical inclusions from U_i.*

- *In the category of abelian groups, the limit for every sequence*

$$A_0 \xrightarrow{f} A_1 \xrightarrow{f} A_2 \xrightarrow{\cdot} ..$$

exists and is given by the following set

$$\coprod A_i / \sim$$

where $a \in A_i \sim f(a) \in A_{i+1}$. The group structure is given by adding $a \in A_i$ and $b \in A_j$ as adding the images in some common A_k for $k \geq i, j$.

Proof. The first is clear. For the second assertion we simply note that a map □

Definition 1.8.7. For I a small category an I-indexed diagram in $\mathcal{C}$ is simply a functor $F : I \to \mathcal{C}$. A cone under such a diagram is given by an object $c \in \mathcal{C}$ together with morphisms $F(i) \to c$ for every i such that for every morphism $i \to j$ in I the resulting diagram $F(i) \to F(j) \to c$ commutes. A cone is called colimit cone if it satisfies the universal property: for every other cone d under F there exists a unique map $c \to d$ such that all the diagrams

$$\begin{tikzcd} F(i) \arrow[r] \arrow[dr] & c \arrow[d] \\ & d \end{tikzcd}$$

commute. In this case we also write $c = \mathrm{colim}_I F$.

The definitions so far are just special cases of this general definition (for different I's). We recommend working this out. Also of course in the general case colimits are, if they exist unique.

1.9 The Jordan Curve theorem

First recall that an embedding of topological spaces $f : X \to Y$ is an injective continuous map which induces a homeomorphism $X \to f(X)$. In this case we abusively write $Y \setminus X$ for $Y \setminus f(X)$. The classical Jordan Curve theorem states that an embedding $S^1 \to \mathbb{R}^2$ separates the plane $\mathbb{R}^2$ into two regions out of which one is bounded and the other is unbounded.

Picture....

In this section we want to give a proof of this theorem and a generalization thereof. But first we need to introduce some terminology.

Lemma 1.9.1. *Let $U_0 \subseteq U_1 \subseteq U_2 \subseteq ... \subseteq X$ be a sequence of open sets in X such that $\bigcup_{i \in \mathbb{N}} U_i = X$. Then we have*

$$H_n(X) \cong \varinjlim H_n(U_i)$$

Proof. Invoking the universal property we get a map

$$\phi : \varinjlim H_n(U_i) \to H_n(X)$$

We check that it is an isomorphism. If $C \subseteq X$ is compact, then $C \subseteq U_n$ for some n. Consider an element $[\alpha] \in H_n(X)$ with $\alpha \in C_n(X)$ a singular chain. Then since Δ^n is compact we have that $\alpha \in C_n(U_i)$ for some i.

Now let $[\alpha] \in H_n(X)$ with $\alpha \in C_n(U_i)$. But since Δ is compact we have $\alpha \in C_n(U_i)$ since its a sum of simplices. Hence ϕ is surjective. Conversely Let $[\alpha] \in H_k(U_i)$. If $\phi[\alpha] = 0$ then there is $\beta \in C_{n+1}(X)$ with $\partial\beta = \alpha$. But then $\beta \in C_n(U_i)$ for some i and thus $[\alpha] = 0$ in $\underrightarrow{\operatorname{colim}} H_n(U_i)$. □

Theorem 1.9.2 (Mayer-Vietoris Sequence)**.** *Let E be a generalized homology theory and $U, V \subseteq M$ an open covering of M. Then there are long exact sequences*

$$\to E_n(U \cap V) \to E_n(U) \oplus E_n(V) \to E_n(M) \to E_{n-1}(U \cap V) \to ...$$
$$\to \widetilde{E}_n(U \cap V) \to \widetilde{E}_n(U) \oplus \widetilde{E}_n(V) \to \widetilde{E}_n(M) \to \widetilde{E}_{n-1}(U \cap V) \to ...$$

with the maps given by....

Proof. Exercises. □

Remark 1.9.3. Along the same lines we obtain a reduced long exact sequence for a pair (X, A):

$$\to \widetilde{E}_n(A) \to \widetilde{E}_n(X) \to E_n(X, A) \to E_{n-1}(A) \to$$

Lemma 1.9.4 (Alexander, Dieudonné)**.** *Fix n and suppose Y is a compact space which has the following property (*)*

for every embedding $Y \to S^n$ we have $\widetilde{H}_*(S^n \setminus Y) = 0$

Then $I \times Y$ (with $I = [0, 1]$) also has the property (*).

Proof. Let $f : I \times Y \to S^n$ be an embedding. Note that the image is closed by virtue of compactness, thus the complement is open. We consider

$$U := S^n \setminus f([0, 1/2] \times Y) \qquad V := S^n \setminus f([1/2, 1] \times Y)$$

and have that

$$U \cap V = S^n \setminus f(I \times Y) \qquad U \cup V = S^n \setminus f(\{1/2\} \times Y)$$

Then by assumption we have that the reduced homology of $U \cup V$ vanishes. Thus by the Mayer-Vietoris sequence we deduced that

$$\widetilde{H}_*(U \cap V) \to \widetilde{H}_*(U) \oplus \widetilde{H}_*(V)$$

is injective.

Now assume that we have a non-trivial class $\alpha \in \widetilde{H}_i(U \cap V)$. We find that the image either in $\widetilde{H}_*(U)$ or $\widetilde{H}_*(V)$ is non-trivial. We replace the interval I by the respective half intervall I_1 and repeat the argument. By this we find a sequence of descending intervalls

$$I \supset I_1 \supset I_2 \supset I_3 \supset ...$$

of length 2^{-k} such that the image of alpha in the respective open set $\widetilde{H}_i(S^n \setminus f(I_k \times Y)$ is non-zero. As a result we conclude that also the image of α in

$$\operatorname{colim} \widetilde{H}_i(S^n \setminus f(I_k \times Y)) = \widetilde{H}_i(S^n \setminus f(\cap I_k \times Y)$$

is non-zero. But since $\cap I_k$ is a point this contradicts the assumption. □

Corollary 1.9.5. *If $f : D^r \to S^n$ is an embedding. Then $\tilde{H}_*(S^n \setminus D^r) = 0$, in particular $S^n \setminus D^r$ is connected.*

Proof. Clear for $r = 0$. Then by Lemma 1.9.1 if follows inductively for all $r > 0$ since $D^r \cong I \times D^{r-1}$. □

Theorem 1.9.6 (Generalized Jordan Curve theorem)**.** *If $S^r \to S^n$ is an embedding then*

$$\widetilde{H}_i(S^n \setminus S^r) \cong \widetilde{H}_i(S^{n-r-1}) \cong \begin{cases} \mathbb{Z} & \textit{for } i = n-r-1 \\ 0 & \textit{else} \end{cases}$$

Proof. We use induction on r. For $r = 0$ we have $S^n \setminus S^0 \cong \mathbb{R}^n \setminus 0 \simeq S^{n-1}$.

Assume the result is known for $r-1$, i.e. ... and that we have an embedding $f : S^r \to S^n$. Put

$$U := S^n \setminus f(D^r_+) \qquad V := S^n \setminus f(D^r_-)$$

We have $U \cap V \simeq S^n \setminus S^{r-1}$ and consider the reduced Mayer-Vietoris Sequence

$$... \to \widetilde{H}_n(S^n \setminus S^r) \to \widetilde{H}_n(U) \oplus \widetilde{H}_n(V) \to \widetilde{H}_n(S^n \setminus S^{r-1}) \to \widetilde{H}_{n-1}(S^n \setminus S^r) \to ...$$

Then the sums are zero by corollary 1.9.5, thus the morphisms are equivalences. □

Corollary 1.9.7 (Jordan Brouwer Separation Theorem)**.** *If $f : S^{n-1} \to S^n$ is an embedding then $S^n \setminus S^{n-1}$ consists of exactly two components, both of which are acyclic (i.e. have trivial reduced homology). Moreover S^{n-1} is the topological boundary of both components.*

Proof. By 1.9.6 (and Proposition 1.6.4) we know that $H_0(S^n \setminus S^{n-1}) = \mathbb{Z} \oplus \mathbb{Z}$ and zero in other degrees. This implies the first part.

Now let U and V be the connected components of $S^n \setminus S^{n-1}$ which are open subsets of S^n by compactness of S^{n-1}. Thus no point in U or V is contained in ∂U, i.e. $\partial U \subseteq f(S^{n-1})$.

Assume $x \in S^{n-1}$ is such that $f(x) \notin \partial U$. Find neighborhood $N \ni f(x)$ st $N \subseteq U^c$. Then there is ball $B \in S^{n-1}$ such that $f(B) \subseteq N \subseteq U^c$. Since $B^c \simeq D^{n-1}$ we find by Corollary 1.9.5 that $S^n \setminus f(B^c)$ is connected. But

$$S^n \setminus f(B^c) = (S^n \setminus f(S^{n-1})) \cup f(B) \subseteq (U \cup V) \cup N) = U \sqcup (V \cup N)$$

This is a contradiction. □

Corollary 1.9.8. *If $f : S^{n-1} \to \mathbb{R}^n$ for $n \geq 2$ is an embedding then $R^n \setminus f(S^{n-1})$ has exactly two components, one is bounded and acyclic and the other is unbounded and has the homology of S^{n-1}.*

Proof. For the number of components we consider the embedding $S^{n-1} \xrightarrow{f} \mathbb{R}^n \to S^n$. By 1.9.7 the complement has two components U and V. One of them contains $\infty \in S^n \setminus \mathbb{R}^n$, wlog $\infty \in U$. Then $V \subseteq \mathbb{R}^n$ is a connected component of $\mathbb{R}^n \setminus f(S^{n-1})$ and acyclic by 1.9.7. For the set U we use the sequence of the pair $(U, U \setminus \infty)$:

$$... \to \widetilde{H}_i(U \setminus \infty) \to \widetilde{H}_i(U) \to H_i(U, U \setminus \infty) \to \widetilde{H}_{i-1}(U \setminus \infty) \to ...$$

The homology of U is trivial by corollary 1.9.7 and thus

$$\widetilde{H}_i(U \setminus \infty) \cong H_{i+1}(U, U \setminus \infty) \simeq H_{i+1}(D^n, D^n \setminus 0) \cong \widetilde{H}_i(S^{n-1})$$

which shows that for $n \geq 2$ the set $U \setminus \infty$ is connected and the statement about the homology. □

Lemma 1.9.9. *Let $f : D^n \to S^n$ be an embedding. Then $f(\mathrm{int}D^n)$ is open in S^n and equals a connected component of $S^n \setminus f(S^{n-1})$.*

Proof. Let U and V be the open components of $S^n \setminus f(S^{n-1})$, i.e.

$$U \sqcup V = S^n \setminus f(S^{n-1}) = S^n \setminus f(D^n) \sqcup f(D^n) \setminus f(S^{n-1})$$

By Corollary 1.9.5 the set $S^n \setminus f(D^n)$ is connected, hence we have that

$$f(D^n) \setminus f(S^{n-1}) = f(\mathrm{int}D^n)$$

is either equal to U or V. □

Lemma 1.9.10. *If M^n is an n-manifold and $f : M^n \to \mathbb{R}^n$ is injective and continuous then f is open (note: f is not asssumed to be an embedding).*

Proof. Let $V \subseteq M^n$ open, $x \in V$. We have to show that $f(V)$ contains a neighborhood of $f(x)$.

Pick $U \subseteq V$ and homeo $g : \mathbb{R}^n \to U$ (chart of M^n) with $g(0) = x$. Consider the composition:

$$fg\mid_{D^n} : D^n \to \mathbb{R}^n \to U \to \mathbb{R}^n$$

This is an embedding by compactness of D^n. Thus using Lemma 1.9.8 we see that $f \circ g(\mathrm{int}D^n)$ is open in $\mathbb{R}^n$. Moreover it contains $f(x)$. Thus

$$f(V) \supset fg(\mathrm{int}D^n)$$

is a neighborhood of $f(x)$. □

Theorem 1.9.11 (Brouwer Invariance of Domain)**.** *If M^n and N^n are n-dimensional manifolds and $f : M^n \to N^n$ is injective and continuous. Then f is open.*

Proof. Let $x \in M^n$ and pick an open subset $U \subseteq N^n$ containing $f(x)$ such that $U \cong \mathbb{R}^n$ and consider

$$g : f^{-1}(U) \xrightarrow{f|_{f^{-1}(U)}} U \to \mathbb{R}^n$$

Then g is open by Lemma 1.9.10. Therefore let V be a neighbourhood of $x \in M^n$. Then $f(V \cap f^{-1}(U))$ is open in N^n. In particular it contains an open neighbourhood of $f(x)$. Thus f is open. □

Theorem 1.9.12 (Classical Schoenflies)**.** *Let $f : S^1 \to S^2$ be an embedding. Then the closure of each of the components of $S^2 \setminus f(S^1)$ is homeomorphic to the disk D^2.*

Proof. Without proof. □

Question: is this also true for embeddings $S^{n-1} \to S^n$?

Example 1.9.13. Counterexample: Alexander's horned sphere:

We will construct an embedding $g : D^3 \to \mathbb{R}^3$ such that the complement $\mathbb{R}^3 \setminus g(D^3)$ is not simply connected. Then for the boundary embedding $f : S^2 \to \mathbb{R}^3$ the space $\mathbb{R}^3 \setminus f(S^2)$ has the two connected components $f(\mathrm{int}(D^2))$ and $\mathbb{R}^3 \setminus g(D^3)$ which can not be homeomorphic to a disk.

Construct inductively subsets of $\mathbb{R}^3$

$$X_0 \supset X_1 \supset X_2 \supset X_3 \supset ...$$

and $B_i \subseteq X_i$ as follows:

- Start with a ball $B_0 = D^3 \subseteq \mathbb{R}^3$. Attach a handle $I \times D^2$ along $\partial I \times D^2$ to obtain X_0.
- To obtain X_1 from X_0 delete part of a splice such that two linked handles remain (Picture). Set B_1 as the union of the ball B_0 with the two 'horns'. Thus X_1 is obtained from B_1 by attaching two handles. Moreover choose a homeomorphism $h_1 : B_0 \to B_1$ which is the identity outside a small neigborhood of $B_1 \setminus B_0$.
- To obtain X_n from X_{n-1} remove a part of the 2^{n-1} short handles of X_{n-1}. Then B_n is the ball consisting of B_{n-1} and the 'Horns'. Now choose a homeomorphism $h_n : B_{n-1} \to B_n$.

Now we set $X_\infty := \bigcap X_i$. Moreover we consider the maps $g_n : B_0 \to R^3$ defined as the composition

$$B_0 \xrightarrow{g_1} B_1 \xrightarrow{g_2} ... \xrightarrow{g_n} B_n \subseteq \mathbb{R}^3$$

Now consider the map

$$g_\infty : B_0 \to \mathbb{R}^3 \qquad g_\infty(x) = \lim_{n\to\infty} g_n(x)$$

It is continuous since the convergence is uniform and its also not hard to see that f is injective. Thus it is an embedding with image X_∞.

It remains to compute $\pi_1(\mathbb{R}^3 \setminus X_\infty) = \pi_1\big(\bigcup_{n\in\mathbb{N}} \mathbb{R}^3 \setminus X_n\big)$. We only sketch this Set $G_n := \pi_1(\mathbb{R}^3 \setminus X_n)$. Then $G_0 = \mathbb{Z}$. To compute G_1 consider the space $Z := X_1 \setminus X_0$. We find that $\pi_1(Z) = \langle \alpha, \beta \rangle$ is the free group on two generators. We have the decomposition

$$\mathbb{R}^3 \setminus X_1 = (\mathbb{R}^3 \setminus X_0) \cup Z$$

with intersection and annulus A. Using Seifert-van-Kampen we find that

$$G_1 \cong \langle \alpha, \beta \rangle$$

with inclusion $G_0 \to G_1$ given by $1 \mapsto [\alpha, \beta]$. Using similar arguments we find that G_n is the free group on 2^n generators and the maps $G_n \to G_{n+1}$ sends each generator to a commutator of two new generators. These maps are injective. Thus we find that

$$\pi_1(\mathbb{R}^3 \setminus X_\infty) = \varinjlim G_n$$

is nontrivial. As a side note: abelianization is trivial.

Definition 1.9.14. An embedding of manifolds $f : M^r \to N^n$ is called locally flat (or tame), if for every point $x \in M^r$ there is a neighboorhood U^r such that $U \to M^r \to N$ can be extended to an embedding $U^r \times [0,1]^{n-r} \to N^n$.

Note that every smooth embedding is automatically locally flat.

Theorem 1.9.15 (Generalized Schoenflies Theorem)**.** *Let $f : S^{n-1} \to S^n$ be a locally flat embedding. Then for each component of $S^n \setminus f(S^{n-1})$ the closure is homeomorphic to a disk D^n.*

Proof. Without proof. □

1.10 CW complexes

Definition 1.10.1. Let (X, A) be a pair of topological spaces with A Hausdorff. A relative CW-structure on (X, A) is a filtration

$$A = X_{-1} \subseteq X_0 \subseteq X_1 \subseteq X_2 \subseteq .... \qquad (X_n \subseteq X)$$

such that

- (cell structure) for every $n \in \mathbb{N}$ there is a pushout of topological spaces

$$\begin{tikzcd} \bigsqcup_{i\in I_n} S^{n-1} \arrow[r, "q_i^n"] \arrow[d, "\bigsqcup_{i\in I_n} j_n"'] & X_{n-1} \arrow[d] \\ \bigsqcup_{i\in I_n} D^n \arrow[r, "Q_i^n"] & X_n \end{tikzcd}$$

  here $j_n : S^{n-1} \to D^n$ and the right vertical morphism $X_{n-1} \to X_n$ are the inclusions.

- (weak topology) X is the direct limit of the X_n's. In other words $X = \bigcup_{n\in\mathbb{N}} X_n$ and X has the weak limit topology for the filtration, i.e. $C \subseteq X$ is closed if $C \cap X_n$ is closed for all $n \in \mathbb{N}$.

Then (X, A) together with the filtration is called a relative CW-complex. If $A = \emptyset$ then X is a CW-complex.

A map of pairs $(X, A) \to (Y, B)$ between relative CW complexes (X, A) and (Y, B) is called cellular, if $f(X_n) \subseteq Y_n$ for all $n \geq -1$.

Remark 1.10.2.

- Note that the cell structure only asserts the existence of the pushouts. The pushout itself is not part of the structure of a relative CW-complex.

- We call X_n the n-skeleton of the (relative) CW-complex.

- For a cellular homeomorphism the inverse is automatically also cellular.

- The space $X_n \setminus X_{n-1}$ is homeomorphic to a disjoint union of open disks (indexed by the set I_n).

- We call a connected component $e \subseteq (X_n \setminus X_n) \subseteq X$ an open n-cell in X (note that it is in general not open in X). The closure of $\overline{e} \subseteq X_n \subseteq X$ is called the corresponding closed n-cell. The boundary of the cell is $\partial e = \overline{e} \setminus e \subseteq X_{n-1}$. The map

$$q_i^n : S^{n-1} \to \partial e \subseteq X_{n-1}$$

is called the attaching map of the cell e and the map

$$Q_i^n : D^n \to \overline{e} \subseteq X_{n+1}$$

is called the characteristic map of the cell.

Examples 1.10.3. 1. The sphere S^d has for every $x \in S^d$ a CW-structure given by the filtration

$$\{x\} \subseteq S^d$$

(i.e. $X_{-1} = \emptyset$, $X_1 = X_2 = ... = X_{d-1} = \{x\}$ and $X_d = X_{d+1} = ... = S^d$. There is precisely one 0-cell and one d-cell.

2. Another CW-structure on S^d is given by the filtration

$$S^0 \subseteq S^1 \subseteq S^2 \subseteq ... \subseteq S^d$$

with the standard inclusion $S^n \subseteq S^d$. Here we have precisely two n-cells for every $0 \leq n \leq d$. To see this, consider the pushout

$$\begin{tikzcd} S^{n-1} \sqcup S^{n-1} \arrow[r, "\mathrm{id}\sqcup\mathrm{id}"] \arrow[d] & S^{n-1} \arrow[d] \\ D^n \cup D^n \arrow[r] & S^n \end{tikzcd}$$

3. A CW structure on the 2-sphere S^2 is given by its dissection as a tetrahedron. Then we have four 0-cells, six 1-cells and four 2-cells. (Picture)

4. the real and complex projective spaces $\mathbb{R}P^d$ and $\mathbb{C}P^d$ admits cell structures given by the filtrations

$$\mathbb{R}P^0 \subseteq \mathbb{R}P^1 \subseteq \mathbb{R}P^2 \subseteq ... \subseteq \mathbb{R}P^d$$
$$\mathbb{C}P^0 \subseteq \mathbb{C}P^1 \subseteq \mathbb{C}P^2 \subseteq ... \subseteq \mathbb{C}P^d$$

In this CW structure $\mathbb{R}P^d$ has an n-cell for every $n \leq d$ and $\mathbb{C}P^d$ has a $2n$-cell for $n \leq d$. To see this use the pushout from the exercises

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d] & \mathbb{R}P^n \arrow[d] \\ D^{n+1} \arrow[r, "F"] & \mathbb{R}P^{n+1} \end{tikzcd}$$

5. We define $\mathbb{R}P^\infty := \underrightarrow{\mathrm{colim}}\mathbb{R}P^n$ and $\mathbb{C}P^\infty := \underrightarrow{\mathrm{colim}}\mathbb{C}P^\infty$, then cell structures...

6. Genus g-surface. Recall that the genus g surface is the quotient of the regular 4g-gon S_g by identifying the edges according to the word $a_1 b_1 a_1^{-1} b_1^{-1} \cdot ... \cdot a_g b_h a_g^{-1} b_g^{-1}$. Consider the filtration

$$\{pt\} \subseteq \bigvee_g S^1 \subseteq \Sigma_g$$

where the basepoint is the image of the vertices and $\bigvee_g S^1 = \partial S_g / \sim$ is the image of the boundary. This (clearly) defines a CW structure on Σ_g.

7. A CW-structure on $\mathbb{R}$ is given by the filtration
$$\mathbb{Z} \subseteq \mathbb{R}$$
For $\mathbb{R}^2$ by
$$\mathbb{Z}^2 \subseteq (\mathbb{Z} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Z}) \subseteq \mathbb{R}^2$$
and for $\mathbb{R}^n$ likewise.

8. $\mathbb{Q} \subseteq \mathbb{R}$ is not a CW complex, since otherwise it had to be locally path connected. Also the Hawaiian earring is not a CW complex.

Proposition 1.10.4. *Let X be a CW-complex.*

1. *The subset $X_n \subseteq X$ is closed.*
2. *as a set we can write X as a disjoint union of open cells, i.e. $X = \bigcup_{n \in \mathbb{N}, i \in I_n} e_i^n$ (note that this is not a decomposition as a topological space)*
3. *The space X is Hausdorff*
4. *a subset $C \subseteq X$ is closed if $C \cap \overline{e}$ is compact for each cell e.*
5. *a subset $C \subseteq X$ is compact if and only if it is closed and intersects only finitely many open cells.*
6. *the subsets $\overline{e}$ and ∂e are compact and intersect only finitely many cells (closure finite).*

Proof. Ad 1) By the weak topology axiom of a CW complex it suffices to check that X_n is closed in X_{n+m}. For this it suffice to check that $X_n \subseteq X_{n+1}$ is closed.

Choose a pushout as in the definition of a CW complex. We set $D := \bigsqcup_{i \in I_n} D^n$ and $\partial D := \bigsqcup_{i \in I_n} \partial D^n$. Then we have $X_n = X_{n-1} \cup_{\partial D} D$. Then the complement X_{n-1}^c is closed, since its preimage under
$$k_n : X_{n-1} \bigsqcup D \to X_n$$
is given by $\bigsqcup \mathrm{int} D_n$.
Ad 2) Clear
Ad 3) Easy (show inductively that each X_n is Hausdorff using the explicit decomposition)

Ad 4) If $C \subseteq X$ is compact, then $C \cap \overline{e}$ compact as a closed subset of a compact Hausdorff space. Conversely assume $C \cap \overline{e}$ is compact for each cell e. It is enough to show that $C \cap X_n$ is closed for each n. We will show that by induction: the case $n = -1$ is trivial. Thus for the inductive step consider the quotient map
$$k_n : X_{n-1} \bigsqcup D \to X_n$$
Then $C \cap X_n$ is closed in X_n if and only if
$$k_n^{-1}(C \cap X_n) = C \cap X_{n-1} \sqcup (Q_i^n)^{-1}(C \cap \overline{e})$$
is closed. This is true by inductive hypothesis for the left hand summand and by assumption of the statement for the right hand summand.

Ad 5) Assume $C \subseteq X$ is closed and intersects only finitely many open cells. Then consider the union K of closed cells $\overline{e}$ such that C intersects the interior e. Then K is compact since its a finite union of compact sets (the closed cells are images of compact cells under the characteristic maps). Thus C is a closed subset of a compact set, hence itself compact.

Let conversely $C \subseteq X$ be compact. Then as a compact subset of a Hausdorff space it is closed. Assume C hits infinitely many open cells $e_{i_k}^{n_k}$ for $k \in \mathbb{N}$. We can assume that $n_1 \leq n_2 \leq n_3 \leq ....$ Choose a sequence of pairwise different points $x_k \in e_{i_k}^{n_k} \cap C$ and let M be their union.

We distinguish two cases:

- we have $\lim_{k\to\infty} n_k = \infty$. Then we can assume (WLOG) that $n_1 < n_2 < n_3....$ Let e be any cell. Then $\overline{e} \subseteq X_n$ for some n. For every subset $N \subseteq M$ we have that $N \cap \overline{e}$ is finite and therefore closed. By the criterion (4) this implies that N is closed. Together this shows that every subset of M is closed, i.e. M is discrete. But since $M \subseteq C$ is also compact this implies that M is finite, which is a contradiction.

- if n_k is bounded, we can assume (WLOG) that $n := n_1 = n_2 = n_3 = ...$ Consider $N \subseteq M$ a subset. Then $N \subseteq X_n \setminus X_{n-1}$ and the intersection of N with a closed cell $\overline{e}$ contains at most one point and is therefore closed. Thus again N is discrete which is a contradiction. □

Lemma 1.10.5. *Given a sequence $\ldots \subseteq X_i \subseteq \ldots \subseteq X$ of closed subspaces of a Hausdorff space X, such that $X = \operatorname{colim} X_i$, and for each n, a map*

$$\coprod D^n \to X_n$$

that restricts to a homeomorphism $\coprod \mathring{D}^n \to X_n \setminus X_{n-1}$, then the diagram

$$\begin{array}{ccc} \coprod \partial D^n & \longrightarrow & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod D^n & \longrightarrow & X_n \end{array}$$

is already a pushout, and hence X a CW complex.

Proof. We do get a map $(\coprod D^n) \amalg_{\coprod \partial D^n} X_{n-1} \to X_n$, which by assumption is a bijection. To see that it is a homeomorphism, it is thus sufficient to check that it is closed. But this can be checked on the summands, and then follows since $X_{n-1} \to X_n$ is a closed inclusion, and the maps $D^n \to X_n$ are automatically closed since D^n is compact and X_n is Hausdorff. □

Lemma 1.10.6. *(i) Let $p : X \to Y$ be a covering where Y is a CW-complex. Then X becomes a CW-complex with the filtration $X_n := p^{-1}(Y_n)$.*

(ii) Let G be a (discrete) group that acts on X such that for every g and every open cell e with $g \cdot e \cap e \neq \emptyset$ we have $g \cdot x = x$ for each $x \in e$. Then X/G admits a CW-structure given by $(X/G)_n = X_n/G$.

Proof. Part (i) is done in the exercises. For (ii), G acts on $X_n \setminus X_{n-1}$, and thus on the set $\pi_0(X_n \setminus X_{n-1})$ of n-cells. We choose one representative for each orbit, and for each of those representatives a characteristic map $D^n \to X_n$. Now these extend uniquely to a well-defined equivariant map

$$\pi_0(X_n \setminus X_{n-1}) \times D^n \to X_n,$$

where the well-definedness uses the condition that cells that are sent to themselves are actually fixed by the action. We can now use Lemma 1.10.5 to obtain a pushout

$$\begin{tikzcd} \pi_0(X_n \setminus X_{n-1}) \times \partial D^n \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\ \pi_0(X_n \setminus X_{n-1}) \times D^n \arrow[r] & X_n, \end{tikzcd}$$

which, after passing to G-orbits, yields a pushout (since colimits commute)

$$\begin{tikzcd} \pi_0(X_n \setminus X_{n-1})/G \times \partial D^n \arrow[r] \arrow[d] & X_{n-1}/G \arrow[d] \\ \pi_0(X_n \setminus X_{n-1})/G \times D^n \arrow[r] & X_n/G. \end{tikzcd}$$

Finally, we have $X/G = \operatorname{colim} X_i/G$, again because colimits commute.

□

Example 1.10.7. The sphere $S^d \to \mathbb{R}P^d$ inherits a CW structure this way from the cell structure on $\mathbb{R}P^d$. This is the decomposition $S^0 \subseteq S^1 \subseteq S^2 \subseteq ... \subseteq S^d$. The real numbers inherit a CW structure from the covering $\mathbb{R} \to S^1$ which agrees with the structure given in Example 1.10.3.

Definition 1.10.8. A CW-complex X is called finite, if it has only finitely many cells. It is called of finite type, if it has only finitely many n-cells for each n. It is called finite dimensional if for some n we have $X_n = X$, otherwise its called infinite dimensional. The dimension is the smalles n such that $X_n = X$.

Clearly finite implies finite type and finite dimensional. But there are no other implications between the three notions.

Examples 1.10.9. S^d, $\mathbb{R}P^d$, $\mathbb{C}P^d$ are finite. The spaces $\mathbb{R}P^\infty$ and $\mathbb{C}P^\infty$ are finite type, but not finite or finite dimensional. The space $\mathbb{R}^n$ with the standard cell structure is finite dimensional, but not of finite type. Here is a more interesting (i.e. not contractible) example which is finite dimensional but not of finite type: Let $X := \bigvee_{k\in\mathbb{N}} S^d$ (the infinite wedge sum of circles) with filtration

$$\emptyset \subseteq \{x\} \subseteq \bigvee_{k\in\mathbb{N}} S^d$$

where $x \in S^d$ is the basepoint.

0-dimensional CW complexes are discrete spaces. 1-dimensional CW complexes are graphs (note that every graph is homotopy equivalent to a wedge of spheres). Genus g-surfaces are two dimensional.

Corollary 1.10.10. *A CW complex X is finite if and only if it is compact.*

Proof. This follows immediately from Proposition 1.10.4. □

Lemma 1.10.11. *For a relative CW-complex (X, A) the pair (X, A) is a neighborhood deformation retract, i.e. A is closed and admits a neighborhood U that deformation retracts onto A.*

Proof. We construct inductively for every n sets $U_n \subseteq X_n$ with the following properties:

- $U_n \subseteq X_n$ open.
- $U_n \cap X_{n-1} = U_{n-1}$
- $U_{n-1} \to U_n$ is a deformation retract.

We set $U_{-1} := A$. Construct inductively U_n as follows: choose a pushout

$$\begin{tikzcd} \coprod_{i\in I_n} S^{n-1} \arrow[r, "q_i^n"] \arrow[d, "\coprod_{i\in I_n} j_n"'] & X_{n-1} \arrow[d] \\ \coprod_{i\in I_n} D^n \arrow[r, "Q_i^n"] & X_n \end{tikzcd}$$

Set

$$U_n := \bigcup_{i\in I_n} \{r \cdot x \mid r \in (1/2, 1], x \in (q_i^n)^{-1} U_{n-1}\}$$

This is obviously a deformation retract.

Now set $U := \bigcup_{n\in\mathbb{N}} U_n = \underrightarrow{\operatorname{colim}} U_n$. Then U is open in X and $A \subseteq U$. Now the proof follows by the following lemma. □

Lemma 1.10.12. *Given subset $Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq ....$ of Y such that*

1. *$Y_n \to Y_{n+1}$ is a deformation retract*
2. *$Y = \underrightarrow{\operatorname{colim}} Y_n$.*

Then $Y_0 \to Y$ is a deformation retract.

Proof. We only sketch the proof. First choose retract $r_n : Y_{n+1} \to Y_n$. Then set $R_n : Y_{n+1} \to Y_0$ as the composition ...

Universal property: $R : Y \to Y_0$.

Homotopy: Choose $h_n : Y_n \to Y_n$ that deforms the identity onto r_n. Then set $H_t^n : Y_n \times [0, 1] \to Y_n$ by

- $H_t^0(x) := x$.
- $$H_t^n(y) := \begin{cases} y & \text{for } t \in [0, 2^{-n}] \\ h^n(x, 2^{-n} \cdot (t - 2^{-n}) & \text{for } t \in [2^{-n}, 2^{-n+1}] \\ H^{n-1}(r_n(x), t) & \text{else} \end{cases}$$

□

Lemma 1.10.13. *Let X and Y be CW-complex with either X or Y locally compact (as topological space). Then $X \times Y$ becomes a CW complex with*

$$(X \times Y)_n = \bigcup_{i+j=n} X_i \times Y_j$$

and the open cells of $X \times Y$ are given by $e_i^p \times e_j^q$ for $p + q = n$.

Proof. We first want to show that the respective pushouts are true. It suffices to show that the diagram

$$\begin{tikzcd}
\bigsqcup_{I_q^X \times I_q^Y} S^{p-1} \times D^q \cup D^p \times S^{q-1} \arrow[r] \arrow[d] & \bigcup_{i+j=n-1} X_i \times Y_j \arrow[d] \\
\bigsqcup_{I_q^X \times I_q^Y} D^p \times D^q \arrow[r] & \bigcup_{i+j=n} X_i \times Y_j
\end{tikzcd}$$

is a pushout (since we clearly have homeomorphisms $D^p \times D^q \cong D^{p+q}$ that send $S^{p-1} \times D^q \cup D^p \times D^{q-1}$ to the boundary sphere S^{p+q-1}). This follows from Lemma 1.10.5.

In the end, unifying all the cells we have to show that the map

$$f : \bigsqcup D^p \times D^q \to X \times Y$$

that runs over all cells is an identification map, i.e. the image has the quotient topology. We factor this map as

$$\bigsqcup D^p \times D^q \to X \times D^q \to X \times Y$$

The first map is the product of an identification map with the identity on D^p. The second is the product with an identification map with X. Now we use that the product of an identifying map with a locally compact space is again identifying (in this case X has to be locally compact, otherwise interchange X and Y). □

Example 1.10.14. The standard CW-structure on $\mathbb{R}^n$ (see example 1.10.3) is the product structure obtained from the CW structure on $\mathbb{R}$.

Consider the d-dimensional torus $T^d := S^1 \times ... \times S^1$. We obtain a CW structure from the CW structure on S^1 given by

$$\emptyset \subseteq \{*\} \subseteq S^1$$

In this CW structure on T^d there are precisely $\binom{d}{n}$ cells of dimension n. Compare this with the cell structure on the 2-torus obtained as identifying the square. Then the cell structure has one 0-cell, two 1-cells and one 2-cell.

1.11 Fundamental group of CW complexes

In this section we want to explain how to read off the fundamental group and the homology from a CW structure on a space X. But to do this we need an important result about fundamental groups.

Theorem 1.11.1 (Seifert-van-Kampen)**.** *Let X be the union of two open subsets $U, V \subseteq$ such that U, V and $U \cap V$ are connected*[3]*. Then for every point $x \in U \cap V$ the diagram*

$$\begin{tikzcd} \pi_1(U \cap V, x) \arrow[r] \arrow[d] & \pi_1(U, x) \arrow[d] \\ \pi_1(V, x) \arrow[r] & \pi_1(X, x) \end{tikzcd}$$

induced by the inclusions is a pushout of groups.

In order to understand what it means let us first try to understand the pushout of groups. The first instance is to understand the coproduct. This is given by the so-called free product. For any pair of (not-necessarily abelian) group G, H the free product $G * H$ is given as a set by all 'reduced words'

$$g_1 h_1 g_2 h_2 g_3 h_3 ... g_k h_k$$

with $g_i \in G$ and $h_i \in H$ where we assume that neither of the elements is the neutral element or similar expressions that start with an element of H and/or end with an element of G. This forms a group by concatenation of words and then reducing by multiplying adjacent elements from the same group and omitting neutral elements. The neutral element is the empty word. Moreover we have two inclusions $G \to G * H$ and $H \to G * H$ by sending elements to the word of lenght 1. Then these two maps exhibit $G * H$ as the coproduct of G and H in the category of groups.

Now as usual, we can bulit the pushout of groups from the coproduct. Explicitkty for a given diagram of the form

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d, "h"] & G \\ H & \end{tikzcd}$$

we can form the quotient of $G * H$ by the normal subgroup generated by words of the form

$$g(f)h(f^{-1}) \in G * H \ .$$

We denote this quotient by $G *_F H$ and call it the amalgamated product. We get a diagram

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d, "h"] & G \arrow[d] \\ H \arrow[r] & G *_F H \end{tikzcd}$$

where the maps are given by the compositions of the canonical maps to $G * H$ composed with the quotient maps. The diagram commutes since for $f \in F$ we have that in $G *_F H$ the equality

$$g(f) = g(f)g(f^{-1})h(f) = h(f) \ .$$

holds. Moreover one easily checks that this diagram is a pushout by verifying the universal property. Informally elements of $G *_F H$ are words in G and H where elements in F can free by considered as sitting in G or in H. Now we understand the statement of Seifert-van-Kampen, namely that the fundamental group of X is given by the amalgamated product of the fundamental groups of G and H.

[3]Recall that this means pathconnected according to our conventions.

Sketch of proof of Theorem 1.11.1. We supress the basepoint x in the notation. From the diagram

$$\begin{tikzcd} \pi_1(U\cap V) \arrow[r] \arrow[d] & \pi_1(U) \arrow[d] \\ \pi_1(V) \arrow[r] & \pi_1(X) \end{tikzcd}$$

we get a canonical map $\pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V) \to \pi_1(X)$ and we will show that is injective and surjective. For surjecitivy we have to show that any path $\gamma : (S^1, 1) \to (X, x)$ can be written as a product $\gamma = \gamma_1 * \gamma_2 * ... * \gamma_n$ of paths $\gamma_i : [0, 1] \to (X, x)$ where each of the γ_i's lies entirely in either U or V. To this end we first use compactness of $[0, 1]$ to write γ as a product $\gamma_1' * \gamma_2' * ... * \gamma_n'$ where all γ_i' lie alternatingly in either U or V but are not necessarily pointed. The basepoints $\gamma_i(0)$ will have to lie in the intersection $U \cap V$. Since this intrersection is connected we can choose paths $\alpha_i : S^1 \to U \cap V$ connecting the point $x \in U \cap V$ to $\gamma_i(0) = \gamma_{i-1}(1)$, i.e. $\alpha_i(0) = x$ and $\alpha_i(1) = \gamma_i(0)$ Then we consider

$$\gamma_i = \alpha_i * \gamma_i' * \alpha_{i+1}$$

We find that up to homotopy we have that

$$\gamma = \gamma_1 * \gamma_2 * ... * \gamma_n \ .$$

This shows surjectivity. Injectivity works similarly, but more complicated as one has to decompose homotopies (and the condition that U and V are connected has to enter). We skip this part. □

Example 1.11.2. $\pi_1(S^n) = 0$ or $n \geq 2$. This follows by decomposing into the upper and lower hemisphere with intersection S^1. Note that this method does not work for S^1 since the intersection S^0 is not connected.

Corollary 1.11.3. *If X and Y are well-pointed then*

$$\pi_1(X \vee Y) = \pi_1(X) * \pi_1(Y)$$

Proof. Chose contractible neighborhoods of the base points in $U_x \subseteq X$ and $U_y \subseteq Y$. Then we have an open decomposition

$$(X \vee U_y) \cup (U_x \vee Y) = X \vee Y$$

and the assumptions of SvK are satisfied. The intersection $U_x \vee U_y$ is contractible. Thus we get as fundamental group

$$\pi_1(X \vee Y) = \pi_1(X \vee U_y) *_{\pi_1(U_x \vee U_y)} \pi_1(U_x \vee Y) = \pi_1(X) * \pi_1(Y) \ .$$

□

Lemma 1.11.4. *Assume a space X is the union of a sequence of open sets*

$$U_0 \subseteq U_1 \subseteq U_2 \subseteq ...$$

and $x \in U_0$. In fact we can allow arbitrary filtered posets in place of $\mathbb{N}$ [4] Then $\pi_1(X, x) = \varinjlim \pi_1(U_i, x)$.

[4] A poset $(S, \leq)$ is called filtered if for every pair of elements $s, s' \in S$ there exists an element s'' with $s \leq s''$ and $s' \leq s''$.

Proof. Every path $S^1 \to X$ factors through a finite U_i by compactness. Simillary every homotopy $S^1 \times I \to X$ factors through a finite stage. $\square$

Corollary 1.11.5. *For an infinite wedge $X = \bigvee_{i \in I} X_i$ of well pointed spaces we have that*

$$\pi_1(X) = \bigstar_{i \in I} \pi_1(X_i)$$

where $\bigstar$ is the infinite free product or infinite coproduct of groups.

Proof. For finite I this follows by induction. For infinite I we write X as the union of thickened up versions of finite wedges and apply the last lemma to conclude that

$$\pi_1(X) = \mathrm{colim}_{S \subseteq I} \bigstar_{s \in S} \pi_1(X_s) \ .$$

which works since the set of all finite subsets of I, ordered by inclusion is a filtered poset. But this colimit is the infinite wedge (in general the infinite coproduct is the directed colimit of its finite subcoproducts). $\square$

Now we want to describe the fundamental group of a CW complex. As a warm up lets try to understand what happens if we attach a cell.

Example 1.11.6. Consider a pushout

$$\begin{tikzcd} S^{n-1} \arrow[r, "q"] \arrow[d] & X \arrow[d] \\ D^n \arrow[r, "Q"] & X' \end{tikzcd}$$

Choose an open cover of X given by $U = Q(\mathrm{int}D^n)$ and $V = X' \setminus Q(0)$. Then $U \cap V$. Then $U \simeq \mathrm{int}D^n \simeq pt$, $V \simeq X$ and $U \cap V \simeq \mathrm{int}D^n \setminus 0 \simeq S^{n-1}$. One should think of the diagram

$$\begin{tikzcd} U \cap V \arrow[r, "q"] \arrow[d] & V \arrow[d] \\ U \arrow[r, "Q"] & X' \end{tikzcd}$$

as a blown up version of the initial diagram and in fact the two diagrams are equivalent in the homotopy category. Seifert van Kampen implies that the effect on π_1 of the latter is a pushout. We conclude that the induced square

$$\begin{tikzcd} \pi_1 S^{n-1} \arrow[r, "q"] \arrow[d] & \pi_1 X \arrow[d] \\ \pi_1 D^n \arrow[r, "Q"] & \pi_1 X' \end{tikzcd}$$

is a pushout. This shows that

$$\pi_1 X' = \pi_1 X *_{\pi_1 S^{n-1}} e = \mathrm{coker}(\pi_1 S^{n-1} \to \pi_1 X)$$

Lemma 1.11.7. *1. Let X be a topological space and $q_i, \tilde{q}_i : S^n \to X$ for $i \in I$ be maps that are homotopic. Then the pushouts*

$$X \cup_{\coprod q_i} \coprod D^{n+1} \qquad \textit{and} \qquad X \cup_{\coprod \tilde{q}_i} \coprod D^{n+1}$$

are also homotopic.

2. *Let $f : X \to Y$ be a homotopy equivalence and $q_i : S^n \to X$ be maps. Then the pushouts*
$$X_{\coprod q_i} \coprod D^{n+1} \qquad \text{and} \qquad Y_{\coprod f\circ q_i} \coprod D^{n+1}$$
are homotopy equivalent as well.

Proof. Exercises. □

To describe the fundamental group of a CW complex we first assume that X is connected and pick a basepoint $x \in X_0$. We now want to describe a certain elements $r_i \in \pi_1(X, x)$ for $i \in I_2$ (the set of 2-cells). Therefore choose the following things: a pushout

$$\begin{tikzcd} \coprod_{i\in I_2} S^1 \arrow[r, "q_i^2"] \arrow[d, "\coprod_{i\in I_2} j_n"'] & X_1 \arrow[d] \\ \coprod_{i\in I_2} D^2 \arrow[r, "Q_i^2"] & X_2 \end{tikzcd}$$

and for every $i \in I_2$ a path γ_i from x to $q_i^2(1)$. Then set

$$r_i := [\gamma_i * q_i * \gamma_i^{-1}] \in \pi_1(X, x)$$

Theorem 1.11.8. *The map*
$$\pi_1(X_1, x) \to \pi_1(X, x)$$
is surjective and the kernel is (as a normal subgroup) generated by the elements g_i for $i \in I_2$. In other words
$$\pi_1(X, x) \cong \pi_1(X_1, x)/\langle r_i\rangle$$

Proof. We first claim that $\pi_1(X_2, x) \cong \pi_1(X_1, x)/\langle r_i\rangle$. To see this we first build another space $\tilde{X}_2$ as the pushout

$$\begin{tikzcd} \bigsqcup_{i\in I_2} S^1 \arrow[r, "\tilde{q}_i^2"] \arrow[d, "\bigsqcup_{i\in I_2} j_n"'] & X_1 \arrow[d] \\ \bigsqcup_{i\in I_2} D^2 \arrow[r, "Q_i^2"] & \tilde{X}_2 \end{tikzcd}$$

where $\tilde{q}_i^2 = \gamma_i * q_i^2 * \gamma_i^{-1}$. This new space is homotopy equivalent to X_2 by Lemma 1.11.7. Thus we have an isomorphism $\pi_1(\tilde{X}_2, x) \to \pi_1(X_2, x)$ which sends the element $\tilde{q}_i^2$ to $\tilde{q}_i^2$. Moreover the diagram

$$\begin{tikzcd} & \pi_1(X_1, x) \arrow[dl] \arrow[dr] & \\ \pi_1(\tilde{X}_2, x) \arrow[rr] & & \pi_1(X_2, x) \end{tikzcd}$$

commutes. This we can reduce the claim to $\tilde{X}_2$. Here we use Seifert van Kampen for the cover consisting of a neigborhood of X_1 and a neighbourhood of the complement to deduce that the diagram

$$\begin{tikzcd} \bigstar_{i\in I_2}\pi_1(S^1, 1) \arrow[r, "\tilde{q}_i^2"] \arrow[d, "\bigsqcup_{i\in I_2} j_n"'] & \pi_1(X_1, x) \arrow[d] \\ \bigstar_{i\in I_2}\pi_1(D^2, 1) \arrow[r, "Q_i^2"] & \tilde{\pi}_1(\tilde{X}_2, x) \end{tikzcd}$$

is a pushout of groups which shows the first part.

Now we use the same trick the show that the inclusions $X_n \to X_{n+1}$ induces a homotopy equivalences on π_1. To see we can (as before) assume that the attaching maps $\tilde{q}_i^n$ preserve basepoints. Then using Seifert-van Kampen we deduce...

Finally we claim that for every CW complex we have that $\pi_1(X) \cong \underrightarrow{\mathrm{colim}}\pi_1(X_n)$. This follows by applying Lemma 1.11.4 to a decomposition of X into $\bigcup U_n$ where U_n is an open neighborhood of the n-Skeleton X_n which deformation retracts onto X_n and is guaranteed to exists by Lemma 1.10.11 using that (X, X_n) is a relative CW pair (just ignore the lower cells). The claim now follows since $\underrightarrow{\mathrm{colim}}\pi_1(X_n) \cong \pi_1(X_2)$. □

Remark 1.11.9.

1. This implies in particular that the fundamental group only depends on the 2-skeleton of the CW complex.

2. Note that $\pi_1(X_1, x)$ is the fundamental group of a graph, which is easy to compute. In particular it is always a free group since every connected graph is homotopy equivalent to a wedge of spheres.

Definition 1.11.10. The datum of a number of generators g_i for $i \in I$ and a number of words r_k for $k \in K$ in the g_i (equivalently an element in the free group generated by the r_k) is called a presentation of a 2-complex. It describes (the homotopy type) of a 2-complex $X = X_{(g_i, r_i)}$ (called the presentation complex) as follows:

$$\{*\} \subseteq \bigvee_{g_i} S^1 \to X$$

where X is defined as the pushout

$$\begin{tikzcd} \coprod_{r\in K} S^1 \arrow[r] \arrow[d] & \bigvee_{g_i} S^1 \arrow[d] \\ \coprod_{r\in K} D^2 \arrow[r] & X \end{tikzcd}$$

The fundamental group of the presentation complex is by Theorem 1.11.8 given by $\pi_1(X, x) \cong \langle g_i \mid r_i \rangle$.

Corollary 1.11.11. *For every group G there is a connected, 2-dimensional CW complex X with $\pi_1(X, x) = G$. The complex can be choosen finite (i.e. compact) if and only if the group admits a finite presentation.*

Proof. Choose a presentation of the group G. That is an isomorphism $G \cong \langle g_i \mid r_i \rangle$.... Then build the presentation complex.

This complex is finite if and only if there is a finite number of generators and relations. Conversely it is clear that the fundamental group of a finite CW complex is finitely presented. □

Proposition 1.11.12. *Every connected 2-dimensional CW complex is homotopy equivalent to one of the form $X_{(g_i, r_i)}$.*

Proof. We use 1.11.7 and the fact that every graph is homotopy equivalent to a wedge of spheres. □

Warning 1.11.13. How can we see that two 2-complexes $X_{(g_i,r_i)}$ and $X_{(g'_i,r'_i)}$ are homotopy equivalent? A sufficient condition is that there is a bijection between the free groups

$$\phi : \langle g_i \rangle \longrightarrow \langle g'_i \rangle$$

that sends the relators to relators, i.e. $\phi(r_i) = r'_i$. Because then the 1-skeleta are equivalent and the attaching maps are homotopic (thus using Lemma 1.11.7). This condition is not necessary.

A necessary condition for $X_{(g_i,r_i)}$ and $X_{(g'_i,r'_i)}$ to be homotopy equivalent is certainly that the fundamental groups are equivalent. But this is not sufficient. Another problem is that it is in general algorithmically undecidable if two presentations yield isomorphic groups : word problem.

Examples 1.11.14.

1. For $X = X_g$ the genus g-surface we have $X_1 = \bigvee_{2g} S^1$. There is only one 2 cell and the attaching map can be described by the word $\alpha_1\alpha_2\alpha_1^{-1}$.... Thus the fundamental group is given by

$$\langle \alpha_1\alpha_2, ..., \alpha_{2g} \mid \alpha_1\alpha_2\alpha_1^{-1}... \rangle$$

2. The fundamental group of $\mathbb{R}P^n$ is isomorphic to the Fundamental group of $\mathbb{R}P^2$ for all $n \geq 2$ (even $n = \infty$ allowed) and$\mathbb{R}P^2 = X_{(g|2g)}$ thus $\pi_1(\mathbb{R}P^2) = \mathbb{Z}/2$.

We finish by a standard application of covering theory.

Proposition 1.11.15. *Every subgroup of a free group is also free.*

Proof. We have that a free group G is the fundamental group $\pi_1(\bigvee S^1)$ of a wedge of circles, one for each generator of G. Then a subgroup $U \subseteq G$ is the fundamental group of a covering space $E \to \bigvee S^1$. By Lemma 1.10.6 it follows that E is also a 1-dimensional CW complex. Thus its fundamental group is also free. □

We warn the reader that even finitely generated free groups G can contains free subgroups with countably many generators.

1.12 Cellular Homology

Let $B \subseteq A \subseteq X$ and (E_*, δ) a generalized homology theory. Then there is a canonical map

$$\delta : E_n(X, A) \to E_{n-1}(A, B)$$

defined as the composition

$$E_n(X, A) \xrightarrow{\delta} E_{n-1}(A) \xrightarrow{k_*} E_{n-1}(A, B)$$

where $k : (A, \emptyset) \to (A, B)$.

Theorem 1.12.1 (Triple long exact sequence)**.** *The sequence*

$$\xrightarrow{\delta} E_n(A, B) \xrightarrow{i_*} E_n(X, B) \xrightarrow{j_*} E_n(X, A) \xrightarrow{\delta_*} E_{n-1}(A, B) \to ...$$

is long exact.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
& E_n(A,B) \arrow[dr, "i_*"] \arrow[rr, bend left, "t_*"] & & E_{n-1}(B) \arrow[dr, "u_*"] \arrow[rr, bend left, "r_*"] & & E_{n-1}(X) \\
E_n(A) \arrow[ur, "s_*"] \arrow[dr, "a_*"] & & E_n(X,B) \arrow[ur, "q_*"] \arrow[dr, "j_*"] & & E_{n-1}(A) \arrow[ur, "a_*"] \arrow[dr, "s_*"] & \\
& E_n(X) \arrow[ur, "p_*"] \arrow[rr, bend right, "b_*"] & & E_n(X.A) \arrow[ur, "c_*"] \arrow[rr, bend right, "k_*"] & & E_{n-1}(A,B)
\end{tikzcd}$$

In which three of the four ‘braids’ are long exact. We claim that it follows that also the fourth is exact. First we note that the composite

$$E_n(A,B) \xrightarrow{i_*} E_n(X,B) \xrightarrow{j_*} E_n(X,A)$$

is zero since it factors through $E_n(A,A) = 0$ by the commutativity of the square

$$\begin{tikzcd}
(A,B) \arrow[r, "i"] \arrow[d] & (X,B) \arrow[d, "j"] \\
(A,A) \arrow[r] & (X,A) \ .
\end{tikzcd}$$

All other compositions are zero by commutativity of the diagrams. Now we want to show exactness in $E_n(X,B)$ by a diagram chase. The other spots work similar.

Let $x \in E_n(X,B)$ with $j_*(x) = 0$. Then $q_*(x) \in \ker(u_*)$ by the commutativity of the diagram. Thus there exists $y \in E_n(A,B)$ with

$$t_* y = q_* x.$$

We conclude that $i_* y - x \in \ker(q_*)$. Thus there exists $z \in E_n(X)$ with

$$p_* z = i_* y - x$$

and therefore $b_* z = j_*(i_* y - x) = 0$. As a result we find $w \in E_n(A)$ with

$$a_* w = z \ .$$

Now we find that

$$i_*(y - s_* a) = i_* y - p_* a_*(w) = i_*(y) - (i_* y - x) = x \ .$$

Thus x lies in the image of i_*. $\square$

Remark 1.12.2. Proving Theorem 1.12.1 directly for singular homology (and not from the axioms) is actually much easier than the diagram chase that we have just done: one simply takes the long exact sequence associated with the short exact sequence of chain complexes:

$$0 \to C_*(A,B) \to C_*(X,B) \to C_*(X,A) \to 0$$

In the following we let H_* be a homology theory with dimension axiom and with coefficient group $\mathbb{Z}$. Everything that follows has analogues for other coefficients.

Lemma 1.12.3. *Let (X, A) be a relative CW complex.*

1. *The group $H_k(X_n, X_{n-1})$ is zero for $k \neq n$ and is free abelian for $k = n$ with a basis corresponding to the n-cells of X.*
2. *The group $H_k(X_n, A) = 0$ for $k > n$.*
3. *The inclusion $X_n \to X$ induces an isomorphism $H_k(X_n, A) \to H_k(X, A)$ for $k < n$.*
4. *The inclusion $A \to X_n$ induces an isomorphism $H_k(X, A) \to H_k(X, A)$ for each $k > n + 1$.*

Proof. 1) $X_{n-1} \to X_n$ has a neighborhood that retracts onto X_{n-1} (by Lemma 1.10.11). Thus $H_n(X_n, X_{n-1}) \cong \widetilde{H}_n(X_n/X_{n_1})$ by Proposition 1.6.5. But X_n/X_{n-1} is a wedge of n-spheres, one for each n-cell.

2) Consider the pair sequence for (X_n, X_{n-1}):

$$H_{k+1}(X_n, X_{n-1}) \to H_k(X_{n-1}) \to H_k(X_n) \to H_k(X_n, X_{n-1})$$

By 1, for $k \neq n, n-1$ we have $H_k(X_n) \cong H_k(X_{n-1})$
For $k > n$ we have $H_k(X_n) = H_k(X_{n-1}) = ... = H_k(X_0) = 0$.

3) For $k < n$ we have

$$H_k(X_n) = H_k(X_{n+1}) = ... = H_k(X_{n+m})$$

proves 3 for finite dimensional. For arbitrary X Lemma 1.9.1 shows that

$$H_k(X) = \varinjlim{}_{n\in\mathbb{N}} H_k(X_n).$$

in ordinary homology. For an arbitrary homology theory this follows by noting that for a CW complex X we have that X is homotopy equivalent to the mapping telescope (see Hatcher in the proof of Lemma 2.3.4 or tom Dieck (Satz 11.3)). The later alwas has homology given by the colimit as we have seen in the erercises. The colimit is equal to $H_k(X_n)$ for any $n > k$.

4) Works similar to 3 using the long exact sequence. □

Definition 1.12.4. Let (X, A) be a relative CW-complex and H_* be a homology theory (with dimension axiom). Then we define a cellular chain complex $C_*^{cell}(X, A)$ as follows.

$$C_n^{cell}(X, A) := H_n(X_n, X_{n-1})$$

with boundary operator $\partial : H_n(X_n, X_{n-1}) \to H_{n-1}(X_{n-1}, X_{n-2})$ given by the boundary operator associated to the triple (X_n, X_{n-1}, X_{n-2}). The homology of this chain complex is called cellular homology and denoted

$$H_n^{cell}(X, Y) := H_n(C_*^{cell}(X, A))$$

Remark 1.12.5.

- First note the boundary operator squares to zero because it is by definition given by

$$H_{n+1}(X_{n+1}, X_n) \xrightarrow{\delta} H_n(X_n) \xrightarrow{k_*} H_n(X_n, X_{n-1}) \xrightarrow{\delta} H_{n-1}(X_{n-1}) \xrightarrow{k_*} H_{n-1}(X_{n-1}, X_{n-2})$$

and $\delta \circ k_*$ is zero.

- The cellular homology forms a functor from the category of CW complexes and cellular maps to graded abelian groups.
- By Lemma 1.12.3 the n-th group in the cellular chain complex is free abelian generated by the number of n-cells in the CW structure on X.

Theorem 1.12.6. *For each homology theory (H_*, δ) (with arbitrary coefficient group but satisfying the dimension axiom) and each relative CW complex (X, A) there is an isomorphism*

$$H_n^{cell}(X, A) \to H_n(X, A)$$

which is natural in cellular maps.

Proof. We look at the nth-step of the cellular chain complex

$$H_{n+1}(X_{n+1}, X_n) \xrightarrow{\delta_{n+1}} H_n(X_n, X_{n-1}) \xrightarrow{\delta_n} H_{n-1}(X_{n-1}, X_{n-2})$$

From the respective triple long exact sequnces we get that

$$\begin{aligned}\mathrm{Im}(\delta_{n+1}) &= \ker(H_n(X_n, X_{n-1}) \to H_n(X_{n+1}, X_{n-1}))\\ \ker(\delta_n) &= \mathrm{Im}(H_n(X_n, X_{n-2}) \to H_n(X_n, X_{n-1}))\end{aligned}$$

Thus we find that the n-th cellular hompology is isomorphic to

$$\frac{\mathrm{Im}(H_n(X_n, X_{n-2}) \to H_n(X_n, X_{n-1}))}{\ker(H_n(X_n, X_{n-1}) \to H_n(X_{n+1}, X_{n-1}))}$$

This is by the homomorphism theorem the same as the image of the composite map

$$H_n(X_n, X_{n-2}) \to H_n(X_{n+1}, X_{n-1})).$$

But this map fits into the square

$$\begin{tikzcd}
H_n(X_n, X_{n-2}) \arrow[r] \arrow[d] & H_n(X_n, X_{n-1}) \arrow[d] \\
H_n(X_{n+1}, X_{n-2}) \arrow[r] & H_n(X_{n+1}, X_{n-1})
\end{tikzcd}$$

as the clockwise composition. Thus the image also equals the image of the counterclockwise composition. But for this the first map is surjective (by the long exact sequence) and the second injective (also the long exact sequence), so that the image is equal to $H_n(X_{n+1}, X_{n-2}) = H_n(X, A)$. □

Corollary 1.12.7.

1. *For a finite type CW complex all homology groups are finitely generated.*
2. *For a finite dimensional CW complex X of dimension n the homology groups H_k are zero for $k > n$. Moreover $H_n(X)$ is a free abelian group.*

Proof. For a finite CW complex all the groups C_n^{cell} are finitely generated free. Hence the cellular homology is finitely generated. For $n > \dim X$ we have $C_n^{cell}(X) = 0$, hence $H_n^{cell}(X) = 0$. For $n = \dim X$ the complex looks like

$$0 \to C_n^{cell}(X) \to C_{n-1}^{cell}(X) \to ..$$

hence the cellular homology is a subgroup of a finitely generated free group, thus itself free. □

Remark 1.12.8. The corresponding statement for homology with values in a ring R is that $H_*(X;R)$ is a finitely presented R module and that $H_*(X;R)$ vanishes above the dimension of X. The top homology is moreover torsion free.

Corollary 1.12.9. *The complex projective space $\mathbb{C}P^d$ (where $d = \infty$ is allowed) has homology*

$$H_n(\mathbb{C}P^d) = \begin{cases} \mathbb{Z} & \text{for } n \text{ even, } 0 \le n \le 2d \\ 0 & \text{else} \end{cases}$$

Proof. The chain complex is given by

$$.. \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to \mathbb{Z} \to 0 \to \mathbb{Z}$$

All differentials are necessarily zero. □

Definition 1.12.10. Let X be a CW complex with choosen pushout squares

$$\begin{tikzcd} \bigsqcup_{i\in I_n} S^{n-1} \arrow[r, "q_i^n"] \arrow[d, "\bigsqcup_{i\in I_n} j_n"'] & X_{n-1} \arrow[d] \\ \bigsqcup_{i\in I_n} D^n \arrow[r, "Q_i^n"] & X_n \end{tikzcd}$$

Then the incidence numbers $\mathrm{inc}_{i,j}^n$ are defined as the degree of the maps:

$$f_{ij} : S^{n-1} \xrightarrow{q_i^n} X_{n-1} \to X_{n-1}/(X_{n-1} \setminus e_j^{n-1}) \xrightarrow{(Q_j^{n-1})^{-1}} D^{n-1}/S^{n-2} \xrightarrow{i} S^{n-1}$$

for $i \in I_n$ and $j \in I_{n-1}$. The incidence matrix is the matrix $\mathrm{INC}^n := (\mathrm{inc}_{i,j})_{i\in I_n, j\in I_{n-1}}$.

Remark 1.12.11. We choose the identification $i : D^{n-1}/S^{n-2} \cong S^{n-1}$ such that the generator

$$1 \in H_{n-1}(D^{n-1}/S^{n-2}) \cong H_{n-1}(D^{n-1}, S^{n-2}) \cong H_{n-2}(S^{n-2}) \cong \mathbb{Z}$$

maps to

$$1 \in H_{n-1}(S^{n-1}) \cong \mathbb{Z}$$

on the left obtained from the long exact sequence maps to the generator on the right.

One possible choice of such a map i is given by the standard stereographical projection...

Remember that for a CW complex (X, A) there is a canonical isomorphism

$$\phi : \oplus_{i\in I_n} \mathbb{Z} \to C_n^{cell}(X, A)$$

Proposition 1.12.12. *For a relative CW-complex (X, A) the following square commutes:*

$$\begin{tikzcd} \oplus_{i\in I_n}\mathbb{Z} \arrow[r, "\mathrm{INC}^n"] \arrow[d, "\Phi"] & \oplus_{j\in I_{n-1}}\mathbb{Z} \arrow[d, "\Phi"] \\ C_n^{cell}(X,A) \arrow[r, "\partial"] & C_{n-1}^{cell}(X,A) \end{tikzcd}$$

In other words: in the basis given by the cells, the boundary operator is represented by the matrix INC^n*.*

Proof. Immediate by unfolding the definitions in terms of long exact sequences using excision. □

1.13 Examples

Proposition 1.13.1. *Consider the genus g surface Σ_g. Its cellular chain complex is given by*

$$... \to 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z}^{2g} \xrightarrow{0} \mathbb{Z}$$

Thus we have $H_0(\Sigma_g) = \mathbb{Z}, H_1(\Sigma_g) \cong \mathbb{Z}^{2g}$ and $H_2(\Sigma_g) = \mathbb{Z}$.

Proof. The number of cells in the cell structure (see Example 1.10.3) gives the identification of the cellular groups. Then d_0 must be zero since Σ_g is connected. To compute d_1 we consider the maps

$$f_{i,j} : S^1 \to \bigvee_{2g} S^1 \xrightarrow{pr} S^1$$

for i the unique 2-cell and $j \in \{a_1, b_1, ..., a_g, b_g\}$. But $\deg(f_{ij}) = 0$. □

Proposition 1.13.2. *Let $X = X_{\langle g_i|r_k\rangle}$ with $i \in I$ and $k \in K$ be a presentation complex. Then the cellular chain complex has the form*

$$... \to 0 \to \oplus_K\mathbb{Z} \xrightarrow{\mathrm{INC}^2} \oplus_I\mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

where inc_{ki} is given by the sum of the exponents of the letter g_i in the word r_k.

Proof. As before we see that the differential ∂_1 has to be zero...

Now consider the maps

$$f_{ik} : S^1 \to \bigvee_K S^1 \xrightarrow{pr} S^1$$

These have the degree given by the number of letters of the respective letter in the word. □

Examples 1.13.3.

- The genus g surface has $2g$ generators $a_1, b_1, ..., a_g, b_g$ and the word is given by $a_1b_1a_1^{-1}b_1^{-1}....$
- Consider the presentation $\langle a \mid a^2\rangle$. Then the complex is

  $$\mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

  hence $H_2(X) = 0$, $H_1(X) = \mathbb{Z}/2$ and $H_0(X) = \mathbb{Z}$. Which space is this?

- Consider the presentation
$$\langle a, b, c, d, ..., y, z \mid hello \rangle$$
The cellular chain complex is, after a permutation of the generators starting with l, h, e, o given by
$$0 \to \mathbb{Z} \xrightarrow{(2,1,1,1,0,,...,0)} \mathbb{Z}^{26} \xrightarrow{0} \mathbb{Z}.$$
Thus we have that $H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}^{25}$ and $H_2(X) = 0$.

We need to establish a new technique for computing degrees of maps $f : S^n \to S^n$, called the local degree. Assume $f(x) = y$ and there is a neighborhood U of x that gets homeomorphically mapped to a neighborhood V of y.

Definition 1.13.4. The local degree of f at the point $x \in S^n$ is the unique integer $\deg_x f$ such that the diagram

$$\begin{tikzcd}
H_n(S^n) \arrow[r, "\cdot \deg_x f"] \arrow[d] & H_n(S^n) \arrow[d] \\
H_n(S^n, S^n \setminus \{x\}) & H_n(S^n, S^n \setminus \{y\}) \\
H_n(U, U \setminus \{x\}) \arrow[r, "f_*"] \arrow[u] & H_n(V, V \setminus \{y\}) \arrow[u]
\end{tikzcd}$$

commutes (note that the upper horizontal map is not induced by a map of spaces)

Remark 1.13.5. The local degree is only defined for points x which have a neighborhood of the given type. It (clearly?) does not depend on the choice of neighborhood U. It can only take the values ± 1 since f is a local homeomorphism.

Example 1.13.6. Consider the map f given by

$$S^k \xrightarrow{\phi} \mathbb{R}P^k \xrightarrow{q} \mathbb{R}P^k / \mathbb{R}P^{k-1} \cong (\mathbb{R}^k)^+ \cong S^k$$

where ϕ is the canonical covering and q is the quotient map. In coordinates the map is given by

$$(x_0, ..., , x_k) \mapsto [x_0, ..., x_k] \mapsto \begin{cases} \infty & \text{for } x_k = 0 \\ (x_0/x_k, ..., x_{k-1}/x_k)) & \text{for } x_k \neq 0 \end{cases}$$

The upper hemisphere
$$D^k_+ = \{(x_0, ..., x_k) \in S^k \mid x_k > 0\}$$
gets homeomorphically mapped to its image, which is $\mathbb{R}^k \subseteq (\mathbb{R}^k)^+$. Let $N = (0, 0, .., 1)$ be the northpole. Then the map
$$f|_{D^k_+} : D^k_+ \to \mathbb{R}^k$$
is homotopic to the standard inclusion $D^k_+ \to \mathbb{R}^k$. Thus the local degree is given by 1, i.e.

$$\deg_N f = 1$$

For the southpole we oberserve that we can write the map

$$f|_{D^k_-} : D^k_- \to \mathbb{R}^k$$

given by $(x_0, ..., x_k) \mapsto (x_0/x_k, ..., x_{k-1}/x_k))$ as the composition

$$g : D^k_- \to D^k_+ \to \mathbb{R}^k$$

with the antipode. Thus

$$\deg_S f = (-1)^{k-1} \ .$$

Theorem 1.13.7. *Let $f : S^n \to S^n$ and $y \in S^n$ such that the preimage $f^{-1}(y) = \{x_1, ..., x_n\}$ is finite and such that every x_i has a neighborhood U_i which gets homeomorphically mapped to a neighboorhood V of y. Then $\deg f = \sum_{i=1}^n \deg_{x_i} f$.*

Proof. Consider the diagram

$$\begin{tikzcd}
H_n(S^n) \arrow[r, "f_*"] \arrow[d] & H_n(S^n) \arrow[d, "\simeq"] \\
H_n(S^n, S^n \setminus f^{-1}(y)) \arrow[r, "f_*"] & H_n(S^n, S^n \setminus y) \\
\bigoplus H_n(U_i, U_i \setminus x_i) \arrow[r] \arrow[u, "\simeq"] & H_n(V, V \setminus y) \arrow[u, "\simeq"']
\end{tikzcd}$$

It follows that the upper map is the sum of the lower maps, which under the respective identifications (the left hand side is the diagonal map $\mathbb{Z} \to \oplus\mathbb{Z}$ under the identifications) are the local degrees. □

Example 1.13.8. The degree of the map $f : S^k \to S^k$ from example 1.13.6 is given by

$$\deg f = 1 + (-1)^{k-1},$$

i.e. it is 0 for k even and 2 for k odd.

Proposition 1.13.9. *The cellular chain complex of real projective space $\mathbb{R}P^n$ is given by*

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} ... \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

for n even and by

$$0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} ... \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

for n odd. Accordingly for $\mathbb{R}P^\infty$ we have ... Thus we have

$$H_k(\mathbb{R}P^n) = \begin{cases} \mathbb{Z} & \text{for } k = 0 \text{ and for } k = n \text{ odd} \\ \mathbb{Z}/2 & \text{for } k \text{ odd, } 0 < k < n \\ 0 & \text{else} \end{cases}$$

and more generall for the coefficients $\mathbb{Q}$ and $\mathbb{Z}/2$ the result is

$$H_k(\mathbb{R}P^n; \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{for } 0 \le k \le n \\ 0 & \text{else} \end{cases}$$

and

$$H_k(\mathbb{R}P^n; \mathbb{F}_2) = \begin{cases} \mathbb{F}_2 & \text{for } 0 \le k \le n \\ 0 & \text{else} \end{cases}$$

Proof. The groups in the cellular chain complex follow from the fact that $\mathbb{R}P^n$ has a cell structure with one cell in every degree (see Example 1.10.3). Attaching maps are $S^{k-1} \to \mathbb{R}P^{k-1}$. Thus the incidence number are by definition given by the degree of the maps

$$S^{k-1} \xrightarrow{\phi} \mathbb{R}P^{k-1} \xrightarrow{q} \mathbb{R}P^{k_1}/\mathbb{R}P^{k-2} \cong S^{k-1}$$

which where computed in Example 1.13.8. The degree of this map is given by 2 for k even and 0 for k odd. Thus by Proposition 1.12.12 (10.19) we are done. □

We now want to construct Moore spaces. A moore space $M(G, n)$ for $n \geq 2$ and G an abelian group is a simply connected CW complex with the property that $H_n(M(G, n)) = G$ and $\tilde{H}_k(M(G, n)) = 0$ for $k \neq n$. We will later see that Moore spaces are up to homotopy uniquely determined, but lets for now concentrate on constructing them.

Example 1.13.10. The n-sphere S^n for $n \geq 1$ is a moore space $M(\mathbb{Z}, n)$.

Example 1.13.11. Consider the case $G = \mathbb{Z}/k$. Choose a self map of $f : S^n \to S^n$ of degree k (this exist by the excercises) and form a CW complex X of the form. $X = D^{n+1} \cup_f S^n$. It has a single n and a single $n+1$-cell. Thus we find that the cellular chain complex is given by

$$0 \to \mathbb{Z} \xrightarrow{k} \mathbb{Z} \to 0 \to ... \to 0 \to \mathbb{Z}$$

and thus it is a Moore space $M(\mathbb{Z}/k, n)$.

Now we want to show the existence of Moore spaces for all n and G. To this end we need to existence of enough maps between wedges of spheres.

Lemma 1.13.12. *For every element $x \in \bigoplus_I \mathbb{Z}$ with I any indexing set and every n there is a map $f : S^n \to \bigvee_I S^n$ such that on H_n the induced map*

$$\mathbb{Z} = H_n(S^n) \xrightarrow{f_*} H_n\left(\bigvee_I S^n\right) \cong \bigoplus_I Z$$

sends 1 to x.

Proof. Every element $x \in \bigoplus_I \mathbb{Z}$ lies in a finite sum $\bigoplus_{I_0} \mathbb{Z} \subseteq \bigoplus_I \mathbb{Z}$ for $I_0 \subseteq I$ finite. The inclusion $\bigoplus_{I_0} \mathbb{Z} \subseteq \bigoplus_I \mathbb{Z}$ is the effect of the inclusion

$$\bigvee_{I_0} S^n \to \bigvee_I S^n$$

thus we can immediately reduced the statement to the case of I being finite, even $I = \{1, ..., m\}$, thus $x \in \mathbb{Z}^m$. But then there is the map $S^n \to S^n \vee ... \vee S^n$ which projects S^n to the wedge by shrinking n horizontal circles. This map induces on H_n the map

$$\mathbb{Z} \to \mathbb{Z}^m \qquad k \mapsto (k, ..., k)$$

Now if $x = (x_1, ..., x_m)$ then we choose pointed selfs maps $f_1, ..., f_m : S^n \to S^n$ of degree x_i and consider the composition

$$S^n \to S^n \vee ... \vee S^n \xrightarrow{f_1 \vee ... \vee f_m} S^n \vee ... \vee S^n \ .$$

It has the desired effect. □

Proposition 1.13.13. *A moore space $M(G, n)$ exists for every $n \geq 2$ and every G.*

Proof. Choose a surjection $\oplus_I \mathbb{Z} \to G$ with kernel K. This is free abelian, i.e. $K = \oplus_J \mathbb{Z}$. Thus we get a short exact sequence

$$0 \to \bigoplus_J \mathbb{Z} \xrightarrow{g} \bigoplus_I \mathbb{Z} \to G \to 0$$

Now the map $\bigoplus_J \mathbb{Z} \to \bigoplus_I \mathbb{Z}$ is given by a family of maps $g_j : \mathbb{Z} \to \bigoplus_I \mathbb{Z}$. According to the last lemma we can for every element $b \in J$ choose a map $f_j : S^n \to \bigvee_I S^n$ such that on H_n this has the effect of the map g_j. Grouping together these maps thus gives us a map

$$f : \bigvee_J S^n \to \bigvee_I S^n$$

that has on homology the effect g. Then we form the CW complex X obtained as the pushout

$$\begin{tikzcd} \coprod_J S^n \arrow[r, "f"] \arrow[d] & \bigvee_I S^n \arrow[d] \\ \coprod_J D^{n+1} \arrow[r] & X \end{tikzcd}$$

It is simply connected and the cellular chain complex has the form

$$0 \to \oplus_J \mathbb{Z} \xrightarrow{g} \oplus_I \mathbb{Z} \to 0 \to ... \to 0 \to \mathbb{Z}$$

which shows that it has the correct homology. □

Remark 1.13.14. One can show that two Moore spaces $M(G, n)$ and $\widetilde{M}(G, n)$ are homotopy equivalent. Moreover there is also s space $M(G, 1)$ with $\widetilde{H}_*(M(G, 1)) = G$ for $* = 1$ and 0 else. But this space is not unique. The construction is exactly the same: choose a generator and relations descriptions

$$0 \to \oplus_J \mathbb{Z} \xrightarrow{g} \oplus_I \mathbb{Z} \to G \to 0$$

and a map $\coprod_J S^1 \to \bigvee_I S^1$ realizing the map g on homology. Then the 2-dimensional CW complex with this attaching map does the job. Note that the fundamental group is not determined by this procedure. One could (and some sources do) call this also a Moore space. Then only simply connected Moore spaces are unique.

Corollary 1.13.15. *For every sequence of abelian groups $G_1, G_2, G_3, ...$ there exists a connected CW complex X with $H_i(X) = G_i$ for $i \geq 1$. If $G_1 = 0$ then we can arrange for X to be simply connected.*

Proof. We choose spaces $M(G_i, i)$ for each i (see the last remark for the case $i = 1$). Then we set

$$X := \bigvee_{i=1}^{\infty} M(G_i, i)$$

which has the desired property. □

1.14 Euler Characteristic

Recall that the rank of an abelian group A is the dimension of its rationalization:

$$\mathrm{rank}(A) := \dim_{\mathbb{Q}}(A \otimes \mathbb{Q})$$

If A is finitely generated, then we have $A \cong \mathbb{Z}^n \oplus T$ where T is the torsion subgroup. Then $\mathrm{rank} A = n$, since $A_{\mathbb{Q}} = A \otimes \mathbb{Q} \cong \mathbb{Q}^n$. The most important property for us is that for a short exact sequence

$$0 \to A \to B \to C \to 0$$

we have $\mathrm{rank}(B) = \mathrm{rank}A + \mathrm{rank}C$. This easily follows from the fact that

$$0 \to A_{\mathbb{Q}} \to B_{\mathbb{Q}} \to C_{\mathbb{Q}} \to 0$$

is also exact and a count of dimensions.

Definition 1.14.1. A space X is called homologically finite over $\mathbb{Q}$, if all homology groups have finite rank and if only finitely many ranks are nonzero.

For a homologically finite space X the Euler characteristic of X is defined to be

$$\chi(X) := \sum_{i=0}^{\infty} (-1)^n \ \mathrm{rank} H_i(X) \in \mathbb{Z}$$

Remark 1.14.2. Finite CW complexes are homologically finite by Corollary 1.12.7, therefore many authors define the Euler characteristic only for finite CW complexes. Note however that the Euler characteristic does not depend on the CW structure. Compact topological manifolds are also homologically finite as we will see in the next section. Also the space $\mathbb{R}P^\infty$ is homologically finite (as the next example shows).

Examples 1.14.3.

- For n even we have

$$\chi(S^n) = \mathrm{rank} H_0(S^n) + \mathrm{rank} H_n(S^n) = 2$$

and for n odd:

$$\chi(S^n) = \mathrm{rank} H_0(S^n) - \mathrm{rank} H_n(S^n) = 0$$

- Complex projective space

$$\chi(\mathbb{C}P^n) = \mathrm{rank} H_0(\mathbb{C}P^n) + \mathrm{rank} H_2(\mathbb{C}P^n) + ... + \mathrm{rank} H_n(\mathbb{C}P^n) = n+1$$

- $\chi(\mathbb{R}P^n) = 1$ for n even and $\chi(\mathbb{R}P^n) = 0$ for n odd.

- For the genus g surface we have

$$\chi(\Sigma_g) = \mathrm{rank} H_0(\Sigma_g) - \mathrm{rank} H_1(\Sigma_g) + \mathrm{rank} H_2(\Sigma_g) = 2 - 2g$$

Note that two genus g surfaces are homoeomorphic iff they are homotopy equivalent iff there Euler characteristics agree.

Theorem 1.14.4. *(Euler-Poincaré) If X is a finite CW complex with a_i the number of i-cells. Then*

$$\chi(X) = \sum_{i=0}^{\infty}(-1)^n a_i$$

Proof. In order to prove this claim we consider the cellular chain complex C_*^{cell} and apply the following claim: for a general finite length chain complex if finitely generated abelian groups

$$0 \to C_n \xrightarrow{\partial_{n-1}} C_{n-1} \to ... \xrightarrow{\partial_0} C_0 \to 0$$

we have that $\sum_{i=0}^{n}(-1)^i \text{rank} C_i = \sum_{i=0}^{n}(-1)^i \text{rank} H_i(C)$. In order to prove this we first note that for every map $\partial_i : C_i \to C_{i-1}$ we have a short exact sequence

$$0 \to \ker(\partial_i) \to C_{i+1} \to \text{Im}(\partial_i) \to 0$$

and thus

$$\text{rank}\ker(\partial_i) + \text{rank}\text{Im}(\partial_i) = \text{rank}C_i.$$

We also have a short exact sequence

$$0 \to \ker(\partial_{i-1}) \to \text{Im}(\partial_i) \to H_i(C) \to 0$$

and thus

$$\text{rank}H_i(C) = \text{rank}\text{Im}(\partial_i) - \text{rank}\ker(\partial_{i-1}) \ .$$

Now we conclude that

$$\begin{aligned}
\sum_{i=0}^{n} \text{rank}(-1)^i C_i &= \sum_{i=0}^{n}(-1)^i \left(\text{rank}\ker(\partial_i) + \text{rank}\text{Im}(\partial_i)\right) \\
&= \sum_{i=0}^{n}(-1)^i \left(\text{rank}\text{Im}(\partial_i) - \text{rank}\ker(\partial_{i-1})\right) \\
&= \sum_{i=0}^{n}(-1)^i \text{rank}H_i(C) \ .
\end{aligned}$$

□

Examples 1.14.5.
- The spheres have one cell in dimension 0 and one cell in dimension n. Thus we directly get $\chi(S^n) = 2$ for n even and $\chi(S^n)$ for n odd.
- Complex projective space $\chi(\mathbb{C}P^n) = ...$
- $\chi(\mathbb{R}P^n) = 1$ for n even and $\chi(\mathbb{R}P^n) = 0$ for n odd.
- For the genus g surface we have
- For a presentation complex $\langle g_1, ..., g_i \mid r_1, .., r_k \rangle$ we have the Euler characteristic

$$\chi(X) = 1 + i - k = 1 + \text{number of generators} - \text{number of relations}$$

Corollary 1.14.6. *(Euler) For every CW structure on the 2-sphere with F 2-cells, E 1-cells and V 0-cells we have $F - E + V = 2$.*

Example 1.14.7. Pictures of Tetrahedron, octahedron, hexahedron

A regular polyhedron is a compact, convex subset of $\mathbb{R}^3$ bounded by regular n-gons such that: two n-gons can meet in at most one edge and in every vertex exactly m different n-gons meet.

polyhedron	V	E	F	m	n
tetrahedron	4	6	4	3	3
octehedron	6	12	8	4	3
hexahedron	8	12	6	3	4
dodecahedron	20	30	12	3	5
icosahedron	12	30	20	5	3

Proposition 1.14.8. *The platonic solids are the only regular polyhedra (up to the obvious notion of equivalence).*

Proof. We will show that the values appearing in the table are the only possibilities.

From Euler's formula above we get $V - E + F = 2$. Also we have $mV = 2E = nF$. It follows that

$$\frac{1}{m} + \frac{1}{n} = \frac{V}{2E} + \frac{F}{2E} = \frac{2+E}{2E} = \frac{1}{E} + \frac{1}{2} > \frac{1}{2}$$

Since $m, n \geq 3$ this can only work if m and n take the values given in the tabular. But n and m determine all the other values (by the above equations) $\square$

Proposition 1.14.9. *The Euler characteristic has the following properties*

1. *It is homotopy invariant, i.e. for $X \simeq Y$ we have $\chi(X) = \chi(Y)$.*
2. *For a product $X \times Y$ of spaces which admit finite CW structures we have $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$.*
3. *Consider an n-fold covering $X \to Y$ and assume that Y admits a finite CW structure. Then we have $\chi(X) = n \cdot \chi(Y)$.*

Proof. The first is clear, for the second let a_i and a'_i be the number of i cells of X and Y. Then $X \times Y$ has a cell structure with $\sum_{i+j=k} a_i a'_j$-cells of dimension k. Thus we get that

$$\chi(X \times Y) = \sum_k (-1)^k \sum_{i+j=k} a_i a'_j = \sum \sum_{i+j} (-1)^i a_i (-1)^j a'_j = (\sum_i (-1)^i a_i) \cdot (\sum_j (-1)^j a'_j) \ .$$

Similarly, if Y has a_i cells of dimension i then X admits a CW structure with na_i cells of dimension i which immediately implies the result. $\square$

Example 1.14.10.
- The d-dimensional torus $T^d := S^1 \times ... \times S^1$ has Euler characteristic $\chi(T^d) = 0$.
- The Lens space $L_{p,q} = S^3/(\mathbb{Z}/p)$ has Euler characteristic

$$\chi(L_{p,q}) = 0$$

Warning 1.14.11. The second part of the proposition 1.14.9 is true for all spaces such that $\chi(X)$ and $\chi(Y)$ are defined (as we will see later). The third can fail as the example $S^\infty \to \mathbb{R}P^\infty$ shows. Here $S^\infty = \underrightarrow{\operatorname{colim}} S^n$. Hence we have $\widetilde{H}_k(S^\infty) = 0$. Thus $\chi(S^\infty) = 1$ (in fact S^∞ is contractible). But $S^\infty \to \mathbb{R}P^\infty$ is a 2-fold covering. Thus

$$\chi(S^\infty) = 1 \neq 2 = 2 \cdot \chi(\mathbb{R}P^\infty)$$

Proposition 1.14.12. *Let*

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_2 \arrow[r] & X \end{tikzcd}$$

be a pushout such that the Euler characteristic of X_0, X_1 and X_2 is defined and such that $X_0 \to X_2$ is a subspace inclusion which admits a neighboorhood that retracts onto X_0. Then the Euler characteristic of X is defined (i.e. X is homologically finite over $\mathbb{Q}$) and we have

$$\chi(X) = \chi(X_1) + \chi(X_2) - \chi(X_0)$$

Proof. We prove consider the Mayer-Vietoris seqeunce

$$... \to H_i(X_0) \to H_i(X_1) \oplus H_i(X_2) \to H_i(X) \to H_{i-1}(X_0) \to ...$$

This is an exact sequence, thus the alternating sum of the ranks vanishes (e.g. using that this can be considered as a chain complex with trivial homology and the Euler characteristic is the same as the alternating sums of the ranks of the homologies). This alternating sum is given by

$$\begin{aligned} &\sum_i \big((-1)^{3i}\mathrm{rank} H_i(X_0) + (-1)^{3i+1}(\mathrm{rank} H_i(X_0) + \mathrm{rank} H_i(X_1)) + (-1)^{3i+2}\mathrm{rank} H_i(X)\big) \\ &\quad = \sum_i \big((-1)^{i}\mathrm{rank} H_i(X_0) + (-1)^{i+1}(\mathrm{rank} H_i(X_0) + \mathrm{rank} H_i(X_1)) + (-1)^{i}\mathrm{rank} H_i(X)\big) \\ &\quad = \chi(X_0) - \chi(X_1) - \chi(X_2) + \chi(X). \end{aligned}$$

□

1.15 Homotopy invariance of singular homology

In this section and the next, we finally want to prove the EIlemberg-Steenrod Axioms. First we deal with the Homotopy invariance, therefore remember its statemnent:

If $f \simeq g : (X, A) \to (Y, B)$ then $f_* = g_* : H_n(X, B) \to H_n(Y, B)$

We will have to develop some theory first.

Definition 1.15.1. Let $f, g : A_* \to B_*$ be maps of chain complexes. Then a *chain homotopy* is a sequence of homomorphisms $D_n : A_n \to B_{n-1}$ (we will most often only write D) such that $\partial \circ D_n + D_{n-1} \circ \partial = f - g$.

Picture.

We say f and g are chain homotopic and write $f \simeq g$. A chain map $f : A \to B$ is called a chain equivalence, if there is a chain map $g : B \to A$ such that $gf \simeq \mathrm{id}$ and $fg \simeq \mathrm{id}$.

Proposition 1.15.2. *1. If $f \simeq g$ then $f_* = g_* : H_*(A) \to H_*(B)$.*

2. If $f : A \to B$ is a chain equivalence then $f_* : H_*A \to H_*B$ is an isomorphism.

3. Chain homotopy is an equivalence relation

4. *If D is a chain homotopy between $f, g : A \to B$. Let moreover $h : B \to C$ be a chain map. Then $hD : A_* \to C_{*+1}$ is a chain homotopy between hf and hg.*

Proof. ad 1: If $a \in A_n$ with $\partial a = 0$ then we have

$$f_*[a] - g_*[a] = [(f-g)(a)] = [\partial D_n(a) + D_{n-1}\partial a] = [\partial D_n(a)] = 0.$$

Hence $f_*[a] = g_*[a]$.
ad 2: follows from 1 since this implies that g_* is inverse to f_*.
ad 3: if D is a chain homotopy between f and g and D' is a chain homotopy between g and h then $D'' := D + D'$ is a chain homotopy between f and h since

$$f - h = f - g + g - h = \partial D_n + D_{n-1}\partial + \partial D'_n + D'_{n-1}\partial = \partial D''_n + D''_{n-1}\partial$$

This shows transitivity. Symmetry is clear (insert a sign) and reflexivity as well. ad 4: we have

$$\partial h D + h D \partial = h(\partial D + D\partial) = h(f-g) = hf - hg$$

□

Now we make the first towards homotopy invariance of singular homology.

Theorem 1.15.3. *If X is a contractible topological space, then $H_n(X, G) = 0$ for all $n \neq 0$.*

Proof. Consider a contracting homotopy $H : X \times [0,1] \to X$ with $H_0 = \mathrm{id}$ and $H(x,1) = x_0$ for some point $x_0 \in X$. We define for each singular simplex $\sigma : \Delta^{n-1} \to X$ a simplex $D\sigma : \Delta^n \to X$ given by

$$D(\sigma)\left(\sum_{i=0}^{n} \lambda_i e_i\right) = H\left(\sigma(\sum_{i=1}^{n} \frac{\lambda_i}{1-\lambda_0} e_{i-1}), \lambda_0\right)$$

and extend by $D(\sigma)(e_0) = x_0$. This can be pictures as follows:
We get an induced map

$$D : C_{n-1}(X) \to C_n(X)$$

We compute that for $n > 1$ we have

$$\begin{aligned}
\partial(D(\sigma)) &= \sigma^{(0)} + \sum_{i=1}^{n} (-1)^i (D\sigma)^{(i)} \\
&= \sigma + \sum_{i=1}^{n} (-1)^i D(\sigma^{(i-1)}) \\
&= \sigma - D(\partial\sigma)
\end{aligned}$$

and for $n = 1$ we get that

$$\partial(D\sigma) = \sigma - \sigma_0$$

where σ_0 is the point x_0 considered as a 0-simplex. Thus we find that

$$\partial D\sigma + D\partial\sigma = \mathrm{id} - \epsilon$$

where $\epsilon : C_*(X) \to C_*(X)$ is the chain map that is zero in positive degrees and sends every 0-simplex to σ_0. The induced map is clealy zero in positive degrees. □

Now in order to prove homotopy invariance of singular homology we need another tool, the so called cross product, which is a bilinear map

$$H_p(X) \times H_q(Y) \to H_{p+q}(X \times Y)$$

Its construction requires some choice which happen to be irrelevant in the end. The rough idea is to consider for a p-simplex σ in X and a q-simplex τ in Y the product $\Delta^p \times \Delta^q \to X \times Y$. If we subdivide this into $p+q$ simplices then we can consider their sum as a singular $p+q$-chain on $X \times Y$.

Proposition 1.15.4. *There exist bilinear maps*

$$\times : C_pX \times C_qY \to C_{p+q}(X \times Y)$$

such that

1. *For $x \in X$ and $\sigma : \Delta^p \to Y$ the element $x \times \sigma$ is described as the p simplex $\Delta^p \to X \times Y$ with $w \mapsto (x, \sigma(w))$. Likewise for $\tau \times y$ with $y \in Y$ and $\tau : \Delta^q \to X$.*
2. *The map $\times$ is natural in X and Y, i.e. for $f : X \to X'$ and $g : Y \to Y'$ we have*
$$C_*(f \times g)(a \times b) = C_*f(a) \times C_*g(b)$$
3. $$\partial(a \times b) = \partial a \times b + (-1)^{\deg a} a \times \partial b$$

Proof. Requirement (1) uniquely determines maps

$$C_0X \times C_qY \to C_q(X \times Y) \qquad C_pX \times C_0Y \to C_{p+q}(X \times Y)$$

for each pair X, Y by linearity. Now we want to inductively construct the maps

$$\times : C_pX \times C_qY \to C_{p+q}(X \times Y)$$

under the assumption that

$$C_{p-1}X \times C_qY \to C_{p+q-1}(X \times Y) \qquad \text{and} \qquad C_pX \times C_{q-1}Y \to C_{p+q-1}(X \times Y)$$

have already been constructed. By freeness of the singular chain complex it will suffices to construct the value $\sigma \times \mu$ for $\sigma : \Delta^p \to X \in C_p(X)$ and $\mu : \Delta^q \to Y \in C_q(Y)$. Since $\times$ is supposed to be natural we can assume that $X = \Delta^p$ and $\sigma = \mathrm{id}_{\Delta^p}$ as well as $Y = \Delta^q$ and $\mu = \mathrm{id}_{\Delta^q}$, so that we need to define the value

$$\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q} \in C_{p+q}(\Delta^p \times \Delta^q)\ .$$

By (2) it has to satisfy the relation

$$\partial(\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q}) = \partial(\mathrm{id}_{\Delta^p}) \times \mathrm{id}_{\Delta^q} + (-1)^p \mathrm{id}_{\Delta^p} \times (\partial \mathrm{id}_{\Delta^q})$$

where the right hand term is already defined by induction. The right hand term clearly is a cycle in $C_{p+q}(\Delta^p \times \Delta^q)$ as one easily checks. But since $\Delta^p \times \Delta^q$ is contractible we can employ Theorem 1.15.3 to deduce that this cycle is a boundary, i.e. we can make a choice of $(\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q})$ which satisfies the relation. This then uniquely determines a natural map

$$\times : C_pX \times C_qY \to C_{p+q}(X \times Y)$$

which also satisfies relation (3) by naturality. $\square$

Proposition 1.15.5. *The cross product induces a bilinear map*

$$H_p(X,A) \times H_q(Y,B) \to H_{p+q}(X \times Y, X \times B \cup A \times Y)$$

defined by $[[a]] \times [[b]] \mapsto [[a \times b]]$ *where* $a \in C_p(X)$ *and* $b \in C_q(Y)$*. In particular it induces bilinear maps*

$$H_p(X) \times H_q(Y) \to H_{p+q}(X \times Y) \ .$$

Proof. Assume $a \in C_p(X)$ has the property that $\partial a \in C_{p-1}(A)$. This means it represents a cycle in $C_p(X,A)$. Similar assume that $b \in C_q(Y)$ with $\partial b \in C_{q-1}(B)$. Then

$$\partial(a \times b) = \partial a \times b \pm a \times \partial b \in C_{p+q-1}(A \times Y \cup X \times B).$$

Thus $a \times b$ represents a cycle in $C_*(X \times Y, A \times Y \cup X \times B)$. It remains to check that the class $[[a \times b]]$ is independent of the choice of representatives a and b (for the classes $[[a]] \in H_p(X,A)$ and $[[b]] \in H_q(Y,B)$. This is clear under adding chains in A and B (by naturality of $\times$). Hence consider for $a, x \in C_*(X)$ and $b, y \in C_*(Y)$ the term

$$(a + \partial x) \times (b + \partial y) = a \times b + \partial x \times b + a \times \partial y + \partial x \times \partial y =: \xi$$

We also compute

$$\begin{aligned} &a \times b + (-1)^{\deg a}\partial(a \times y) + \partial(x \times b) + \partial(x \times \partial y) \\ &= \xi + (-1)^{\deg a}\partial a \times y + (-1)^{\deg x} x \times \partial b \end{aligned}$$

If a and b are relative cycles, then the last summands lie in $C_*(X \times B \cup A \times Y)$. Thus we have that $[[\xi]] = [[a \times b]]$ which verifies the claim. □

Remark 1.15.6. The map $\times : C_p(X) \times C_q(Y) \to C_{p+q}(X \times Y)$ depends on choices. One can show that these choices don't matter too much, in the sense that the conditions of Prop. 1.15.4 uniquely characterize $\times$ up to some notion of chain homotopy of bilinear maps.

This implies in particular that the maps

$$H_p(X,A) \times H_q(Y,B) \to H_{p+q}(X \times Y, X \times B \cup A \times Y)$$

are independent of all the choices.

The most important case for us is the case $X = I = \Delta^1$. Then we have a map

$$D : C_q(X) \to C_{q+1}(I \times X)$$

given by $D(c) = I \times c$ where $I \in C_1(I)$ is the identity simplex (the id_{Δ^1} from the proof of 1.15.5). We claim that D is a chain homotopy between the maps $C_*(i_0)$ and $C_*(i_1)$ where i_0, i_1 are the two inclusions $X \to I \times X$ given by $i_k(x) = (k, x)$. To see this note that

$$\partial(I \times c) = 1 \times c - 0 \times c - I \times \partial c.$$

Hence

$$\partial D(c) + D\partial(c) = 1 \times c - 0 \times c = i_1(c) - i_0(c)$$

using for the last equality property (1) of Proposition 1.15.4. Now for $A \subseteq X$ the chain homotopy D also descends to a chain homotopy

$$D : C_q(X,A) \to C_{q+1}(I \times X, I \times A)$$

between the corresponding inclusions by naturality of $\times$.

Theorem 1.15.7. *If f and g are homotopic maps $(X, A) \to (Y, B)$ then $C_*(f)$ and $C_*(g)$ are chain homotopic maps $C_*(X, A; G) \to C_*(Y, B; G)$. In particular we have that $f_* = g_* : H_*(X) \to H_*(Y)$.*

Proof. We will prove that for $G = \mathbb{Z}$. For other groups G it follows by tensoring with G on the level of singular chains (which preserves chain homotopies).

Let $H : (X \times I, A \times I) \to (Y, B)$ be a homotopy between f and g. We consider the map

$$C_*(X, A) \xrightarrow{D} C_{*+1}(X \times I, A \times I) \xrightarrow{C_{*+1}H} C_{*+1}(Y, B)$$

This then is a chain homotopy between $C_*(H \circ i_0) = C_*(f)$ and $C_*(H \circ i_1) = C_*(g)$ by Proposition 15.2 (4). $\square$

Corollary 1.15.8. *Singular homology (with arbitrary coefficients) satisfies the Homotopy Axiom (from the ES Axioms 1.6.1).*

1.16 Excision for singular homology

In this section we want to prove the excision axiom. Recall its statement

> Given (X, A) and $Z \subseteq A$ with $\overline{Z} \subseteq \mathrm{int}(A)$, then the morphisms
>
> $$H_n(X \setminus Z, A \setminus Z) \to H_n(X, A)$$
>
> induced by the inclusion is an isomorphism.

This will be Theorem 1.16.8 below. The problem is with large simplices not contained in either A or $X \setminus Z$ (picture). Note that A and $X \setminus Z$ form a cover of X, in the sense that their interiors $\mathrm{int}(A)$ and $X \setminus \overline{Z}$ cover X. The idea is to 'subdivide' all simplices to make them smaller until they lie completely within one of the two. To this end we want to construct a subdivision operator $\Upsilon : C_*(X) \to C_*(X)$ and a chain homotopy T from $\Upsilon \simeq \mathrm{id}_{C_*(X)}$.

Let $L_*(\Delta^q) \subseteq C_*(\Delta^q)$ be the subcomplex spanned by affine simplices, i.e. simplices of the form

$$\Delta^p \to \Delta^q \qquad \sigma(\sum_{i=0}^{n} \lambda_i e_i) = \sum_{i=0}^{p} \lambda_i v_i$$

We will denote this affine simplex, by $[v_0, ..., v_n]$ (as in Definition 1.2.3). The groups $L_*(\Delta^q)$ form a subcomplex since each face of an affine simplex is again affine. Let $v \in \Delta^q$ and $\sigma = [v_0, ..., v_n] : \Delta^p \to \Delta^q$ be affine. The *cone on σ from v* is then defined as

$$v\sigma := [v, v_0, ..., v_n] : \Delta^{p+1} \to \Delta^q$$

Picture...
By linear extension we get a group homomorphism

$$v(-) : L_p(\Delta^q) \to L_{p+1}(\Delta^q)$$

Lemma 1.16.1. *We have*

$$\partial(vc) = \begin{cases} c - v(\partial c) & \text{if } \deg(c) > 0 \\ c - \epsilon(c)v & \text{if } \deg(c) = 0 \end{cases}$$

Here $\epsilon : C_0(\Delta^q) \to \mathbb{Z}, \sum n_\sigma \sigma \mapsto \sum n_\sigma$ *is the augmentation (as studied in Section 1.3).*

Proof. We compute

$$\begin{aligned} \partial[v, v_0, ..., v_n] &= [v_0, ..., v_n] - \sum_{i=0}^{n} (-1)^i [v, v_0, ..., \hat{v}_i, ..., v_n] \\ &= [v_0, ..., v_n] - v(\partial[v_0, .., v_p]) \end{aligned}$$

By linear extension this shows the case $\deg(c) > 0$. In degree 0 we compute

$$\partial v\sigma = \sigma - v = \sigma - \epsilon(\sigma)v$$

which shows by linear extension the case $\deg(c) = 0$. □

We now define the 'barycentric subdivision' operator $\Upsilon : L_p(\Delta^q) \to L_p(\Delta^q)$ on simplices inductively by

$$\Upsilon(\sigma) = \begin{cases} \sigma & \text{for } p = 0 \\ \underline{\sigma}\Upsilon(\partial\sigma) & \text{for } p > 0 \end{cases}$$

where $\underline{\sigma} := \frac{1}{p+1} \sum_{i=0}^{p} v_i$ is the 'barycenter' of the simplex $\sigma = [v_0, ..., v_n]$. An operator $T : L_p(\Delta^q) \to L_{p+1}(\Delta^q)$ is defined inductively on simplices by the formula

$$T(\sigma) = \begin{cases} 0 & \text{for } p = 0 \\ \underline{\sigma}(\Upsilon\sigma - \sigma - T(\partial\sigma)) & \text{for } p > 0 \end{cases}$$

(Maybe discuss some cases for $p = 0, 1, 2$)

Lemma 1.16.2. $\Upsilon : L_p(\Delta^q) \to L_p(\Delta^q)$ *is a chain map and* T *is a chain homotopy from* Υ *to* 1.

Proof. To show that Υ is a chain map we have to show that $\Upsilon(\partial\sigma)) = \partial\Upsilon(\sigma)$ for each $\sigma : \Delta^p \to \Delta^q$. We will prove that by induction on p. For $p = 0$ we have

$$\Upsilon(\partial\sigma) = \Upsilon(0) = 0 = \partial\sigma = \partial\Upsilon(\sigma)$$

For $p = 1$ we have

$$\Upsilon(\partial\sigma) = \partial\sigma$$

and using Lemma 1.16.1

$$\partial\Upsilon(\sigma) = \partial(\underline{\sigma}(\partial\sigma)) = \partial(\sigma) - \epsilon(\partial\sigma)\sigma = \partial\sigma$$

For $p > 1$ again using Lemma 1.16.1 (second equality) and inductive hypothesis (third equality) we get

$$\partial\Upsilon(\sigma) = \partial(\underline{\sigma}\Upsilon(\partial\sigma)) = \Upsilon(\partial\sigma) - \underline{\sigma}\partial\Upsilon(\partial\sigma) = \Upsilon(\partial\sigma) - \underline{\sigma}\Upsilon(\partial\partial\sigma) = \Upsilon(\partial\sigma)$$

Together we have shown that Υ is a chain map. Now we want to show that T is a chain homotopy, i.e. that

$$\partial T + T\partial = \Upsilon - \mathrm{id}$$

For $p = 0$ we have

$$\partial T\sigma + T\partial\sigma = 0 = \sigma - \sigma = \Upsilon(\sigma) - \sigma.$$

For $p = 1$ we have

$$\partial T\sigma + T\partial\sigma = \partial(\underline{\sigma}(\Upsilon\sigma - \sigma - T(\partial\sigma)) = \Upsilon\sigma - \sigma - \underline{\sigma}(\partial(\Upsilon\sigma - \sigma)) = \Upsilon\sigma - \sigma$$

For $p > 1$ we have

$$\partial T\sigma = ... = \Upsilon\sigma - \sigma - T\partial\sigma - \underline{\sigma}(\partial\Upsilon\sigma - \partial\sigma - \partial T\partial\sigma)$$

But $\partial T\partial\sigma = (\Upsilon - \mathrm{id} - T\partial)(\partial\sigma) = \Upsilon\partial\sigma - \partial\sigma$. Thus

$$\partial T\sigma + T\partial\sigma = \Upsilon\sigma - \sigma - T\partial\sigma + T\partial\sigma = \Upsilon\sigma - \sigma$$

□

Proposition 1.16.3. *There are unique maps $\Upsilon : C_*(X) \to C_*(X)$ and $T : C_*(X) \to C_{*+1}(X)$ for every X satisfying the following properties:*

1. *The operators Υ, T are natural in X, i.e. for $f : X \to Y$ we have*

$$\Upsilon C_*(f) = C_*(f)\Upsilon \qquad TC_*(f) = C_*(f)T$$

2. *Υ and T extend the operators defined above on $L_*(\Delta^q)$.*

3. *Υ is a chain map and T a chain homotopy $\Upsilon \simeq 1$.*

Proof. The first two properties already characterize $\Upsilon(\sigma) = C_*(\sigma)(\Upsilon^{\mathrm{old}}(\Delta^p))$, where $\Delta^p \in L_p(\Delta^p)$ is the identity p-simplex, $\sigma : \Delta^p \to X$ is any singular simplex, and Υ^{old} refers to the geometrically defined subdivision operator on $L_*(\Delta^p)$. So Υ is uniquely determined, and analogously for T.

This formula also defines an operator Υ, we just have to check that it coincides with Υ^{old} on $L_*(\Delta^q)$. But this follows from the fact that for affine $\sigma : \Delta^p \to \Delta^q$, the map $C_*(\sigma) : C_*(\Delta^p) \to C_*(\Delta^q)$ preserves the subcomplexes L_*, and preserves barycenters and cones, so commutes with Υ^{old}. In particular, $C_*(\sigma)(\Upsilon^{\mathrm{old}}(\Delta^p)) = \Upsilon^{\mathrm{old}}(\sigma)$. Analogously, T coincides with T^{old} for affine simplices.

Finally, the formulas $\partial\Upsilon = \Upsilon\partial$ and $\partial T + T\partial = \Upsilon - \mathrm{id}$ follow from the previously proven variants on $L_*(\Delta^p)$ and naturality. □

Corollary 1.16.4. *The map $\Upsilon^k : C_*(X) \to C_*(X)$ is chain homotopic to the identity. The chain homotopy T_k can be chosen natural in X.*

Proof. This follows since the composition of chain homotopies is a chain homotopy as well. Concretely, we set $T_1 = T$, and then inductively $T_k = \Upsilon T_{k-1} + T$. Then

$$\partial T_k + T_k\partial = \Upsilon(\partial T_{k-1} + T_{k-1}\partial) + \partial T + T\partial = \Upsilon(\Upsilon^{k-1} - \mathrm{id}) + \Upsilon - \mathrm{id} = \Upsilon^k - \mathrm{id}$$

by induction. □

Lemma 1.16.5. *1. If $\sigma = [v_0, ..., v_p] : \Delta^p \to \Delta^q$ is an affine simplex, then any simplex in the chain $\Upsilon(\sigma)$ has diameter of at most $\frac{p}{p+1}\mathrm{diam}(\sigma)$.*

2. For an affine simplex σ, each simplex in the chain $\Upsilon^k(\sigma) \in L_p(\Delta^q)$ has a diameter of at most $\left(\frac{p}{p+1}\right)^k \mathrm{diam}(\sigma)$.

3. Let X be a space, $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of X and $\sigma : \Delta^p \to X$ be a simplex. Then there exists a $k \in \mathbb{N}$ such that $\Upsilon^k(\sigma) \in C_p(X)$ is $\mathcal{U}$-small, i.e. that every simplex of $\Upsilon^k(\sigma)$ has image contained entirely in one of the U_i's.

Proof. Every simplex in $\Upsilon\sigma = \underline{\sigma}\Upsilon(\partial\sigma)$ is a simplex of some $\underline{\sigma}\Upsilon(\sigma^{(i)})$. Inductively we see that every simplex in $\Upsilon\sigma$ has the form $[\underline{\sigma_0}, \underline{\sigma_1}, \underline{\sigma_2}, ..., \underline{\sigma_p}]$ where $\sigma_0 = \sigma : \Delta^p \to \Delta^q$ and $\sigma_{i+1} : \Delta^{p-i-1} \to \Delta^q$ is some face of $\sigma_i : \Delta^{p-i} \to \Delta^q$. By reordering the vertices v_k we can arrange that

$$\sigma_i = [v_0, ..., v_i]$$

and thus $\underline{\sigma_i} = \frac{1}{i+1}\sum_{k=0}^{i} v_k$. The diameter of the simplex $[\underline{\sigma_0}, \underline{\sigma_1}, \underline{\sigma_2}, ..., \underline{\sigma_p}]$ is then bounded by the distance of two vertices. Thus in order to prove the statement we have to show the inequality

$$\|\underline{\sigma_i} - \underline{\sigma_j}\| = \left\| \frac{1}{i+1}\sum_{k=0}^{i} v_k - \frac{1}{j+1}\sum_{k=0}^{j} v_k \right\| \leq \frac{p}{p+1}\,\mathrm{diam}(\sigma)$$

We may assume $i < j$ because of symmetry. Now we compute

$$\begin{aligned}
\|\underline{\sigma_i} - \underline{\sigma_j}\| &= \left\| \frac{1}{i+1}\sum_{k=0}^{i} v_k - \frac{1}{j+1}\sum_{k=0}^{j} v_k \right\| \\
&= \left\| \left(\frac{1}{i+1} - \frac{1}{j+1}\right)\sum_{k=0}^{i} v_k - \frac{1}{j+1}\sum_{k=i+1}^{j} v_k \right\| \\
&= \left\| \frac{j-i}{(i+1)(j+1)}\sum_{k=0}^{i} v_k - \frac{1}{j+1}\sum_{k=i+1}^{j} v_k \right\| \\
&= \frac{j-i}{j+1} \cdot \left\| \frac{1}{i+1}\sum_{k=0}^{j} v_k - \frac{1}{j-i}\sum_{k=i+1}^{j} v_k \right\|
\end{aligned}$$

The terms $\frac{1}{i+1}\sum_{k=0}^{j} v_k$ and $\frac{1}{j-i}\sum_{k=i+1}^{j} v_k$ are the barycenters of $[v_0, \ldots, v_i]$ and $[v_{i+1}, \ldots, v_j]$ respectively. As such, they are both contained in the convex σ, and thus their distance is bounded by $\mathrm{diam}(\sigma)$. We conclude

$$\|\underline{\sigma_i} - \underline{\sigma_j}\| \leq \frac{j-i}{j+1}\,\mathrm{diam}(\sigma) \leq \frac{p}{p+1}\,\mathrm{diam}(\sigma),$$

where the last inequality holds since $i \geq 0$ and $j \leq p$. This proves the first claim.

Ad 2: The second assertion follows from the first immediately by iteration.

Ad 3: By naturality of Υ^k we have $\Upsilon^k(\sigma) = C_*(\sigma)\Upsilon^k(\Delta^p)$. Since $C_*(\sigma)$ is defined by postcomposition, i.e. by composing a simplex $\Delta^p \to \Delta^p$ with the map $\sigma : \Delta^p \to X$, we can pull back the open covering $\mathcal{U} = \{U_i\}_{i\in I}$ to an open covering $\sigma^{-1}\mathcal{U} = \{\sigma^{-1}U_i\}_{i\in I}$ and find that the condition that each simplex of $\Upsilon^k(\sigma)$ is contained in one of the $U_i's$ is equivalent to the condition that each simplex of $\Upsilon^k(\Delta^p)$ is contained in one open set of $\sigma^{-1}\mathcal{U}$. By the Lebesgue Lemma (or directly by compactness of Δ^p) there exist for such a covering of the simplex an $\epsilon > 0$ such that every subset of Δ^p of diameter less or equal to ϵ is entirely contained in an open of $\sigma^{-1}\mathcal{U}$. Now we choose k such that every simplex in $\Upsilon^k(\Delta^p)$ has diameter less then ϵ which can be done by (2). Then we are done. □

Definition 1.16.6. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be a covering of X such that the interiors still cover X. Let $C_*(X)^{\mathcal{U}} \subseteq C_*(X)$ be the subcomplex spanned by the $\mathcal{U}$-small singular simplices. For $A \subseteq X$ let $C_*^{\mathcal{U}}(X, A) := C_*(X)^{\mathcal{U}}/C_*(A)^{\mathcal{U}\cap\mathcal{A}}$. Then we define

$$H_*^{\mathcal{U}}(X, A) := H_*(C_*^{\mathcal{U}}(X, A))$$

Theorem 1.16.7. *In the situation of 1.16.6 the morphism induced by the inclusion $H_*^{\mathcal{U}}(X, A) \to H_*(X, A)$ is an isomorphism.*

Proof. We first prove the statement for $A = \emptyset$. Let $c \in C_p(X)^{\mathcal{U}}$ with $\partial c = 0$. Suppose $c = \partial e$ for $e \in C_{p+1}(X)$. This means c represents a class in the kernel of $H_*^{\mathcal{U}}(X, A) \to H_*(X, A)$. We must show that this class is zero, hence that $c = \partial e'$ for $e' \in C_{p+1}(X)^{\mathcal{U}}$. By Lemma 1.16.5(3) there is k such that $\Upsilon^k(e) \in C_{p+1}(X)^{\mathcal{U}}$ and

$$\Upsilon^k(e) - e = T_k(\partial e) + \partial T_k(e) = T_k(c) + \partial T_k(e)$$

Therefore

$$\partial\Upsilon^k(e) - \partial e = \partial T_k(c)$$

and thus

$$c = \partial e = \partial(\Upsilon^k(e) - T_k(c))$$

We set $e' := \Upsilon^k(e) - T_k(c)$ which is in $C_{p+1}(X)^{\mathcal{U}}$ by naturality of T_k.

To see that the map is surjective let $c \in C_p(X)$ be a cycle, i.e. $\partial c = 0$. Then we try to find $c' \in C_p(X)^{\mathcal{U}}$ with $c - c' = \partial e$ for some $e \in C_{p+1}(X)$. Find k such that $\Upsilon^k(c) \in C_p(X)^{\mathcal{U}}$. Note that we c can be a linear combination of finitely many simplices. Thus we find a seperate k for every of these simplicies and then take the maximum of all these k's. Then we have that

$$\Upsilon^k(c) - c = T_k(\partial c) + \partial T_k(c) = \partial T_k(c)$$

This proves surjectivitiy.

Now to prove the relative case for $A \subseteq X$ consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & C_*(A)^{\mathcal{U}\cap\mathcal{A}} \arrow[r] \arrow[d] & C_*(X)^{\mathcal{U}} \arrow[r] \arrow[d] & C_*(X, A)^{\mathcal{U}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C_*(A) \arrow[r] & C_*(X) \arrow[r] & C_*(X, A) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. This induces a morphism of long exact sequences

according to Theorem 1.5.8. By the first part we know that the morphisms ... are isomorphisms. Thus by the 5-Lemma (Lemma 1.5.14) the relative morphism is an isomorphism as well. □

Theorem 1.16.8 (Excision)**.** *Given (X, A) and $Z \subseteq A$ with $\overline{Z} \subseteq \operatorname{int}(A)$ then the morphisms*

$$H_n(X \setminus Z, A \setminus Z) \to H_n(X, A)$$

induced by the inclusion is an isomorphism.

Proof. Consider $\mathcal{U} = \{A, X \setminus Z\}$. This is a covering of X in the sense that the interiors $\operatorname{int}(A)$ and $X \setminus \overline{Z}$ still cover X. Thus by Theorem 1.16.7 we find that the morphism

$$H_*^{\mathcal{U}}(X, A) \to H_*(X, A)$$

is an isomorphism. Note that

$$C_*^{\mathcal{U}}(X) = C_*(A) + C_*(X \setminus U), \quad C_*^{\mathcal{U} \cap A}(A) = C_*(A), \quad C_*(A \setminus U) = C_*(A) \cap C_*(X \setminus U)$$

It follows that

$$\frac{C_*(X \setminus U)}{C_*(A \setminus U)} \cong \frac{C_*^{\mathcal{U}}(X)}{C_*(A)}$$

Since this isomorphism is induced by the inclusion we find that the following diagram commutes:

$$\begin{tikzcd} H_*(X \setminus U, A \setminus U) \arrow[r, "\sim"] \arrow[dr] & H_*^{\mathcal{U}}(X, A) \arrow[d, "\sim"] \\ & H_*(X, A) \end{tikzcd}$$

So the diagonal map is an isomorphism, too, which we wanted to prove. □

We have now given a full proof of the Eilenberg-Steenrod axioms for singular homology, and that they uniquely determine the homology groups (in the appropriate sense) for CW complexes. In other words, we have given a full proof of Theorem 1.6.2.

1.17 Manifolds and orientations

Recall that by Definition 1.7.4 a topological manifold of dimension n is a second countable Hausdorff space M such that every point in M has a neighborhood homeomorphic to $\mathbb{R}^n$.

Remark 1.17.1 (Without proof)**.** All topological manifolds M of dimension $n \neq 4$ and all smooth manifolds are CW complexes of dimension n. Thus Corollary 1.12.7 together with Corollary 1.10.10 implies that all compact manifolds of dimension $\neq 4$ have finitely generated homology groups in degree $\leq n$ and vanishing homology groups above. Since we do not want to use these hard results and since we do not want to exclude dimension 4 we give a more direct argument.

Proposition 1.17.2. *Let M be a compact topological manifold. Then the homology groups $H_k(M)$ are finitely generated. More generally for any manifold M with a compact subset $K \subseteq M$ the image of the morphism $H_k(K) \to H_k(M)$ is finitely generated.*

Proof. We use nduction on k. Thus assume that for each manifold M and each compact subset K we already know that the image of

$$H_{k-1}(K) \to H_{k-1}(M)$$

is finitely generated. Now we want to prove that the image

$$H_k(K) \to H_k(M)$$

is finitely generated for each $K \subseteq M$ compact. In fact, we will prove a slighlty stronger statement:

> Each compact subset $K \subseteq M$ admits an open neighborhood $U \supseteq K$ such that the image
> $$H_k(U) \to H_k(M)$$
> is finitely generated.

This implies the claim since subgroups of finitely generated abelian groups are also finitely generated.

To see the last statement we note that (since every manifold is locally compact) each point $m \in M$ admits a compact neighborhood which satisfies the claim (since we can chose a contracitble open neighborhood first). Since every compact subset $K \subseteq M$ is then a finite union of such neighborhoods it suffices to prove that if two compact subsets $K_1, K_2 \subseteq M$ satisfy the claim, then so does their union. We choose open subsets $U_1 \supset K_1$ and $U_2 \supset K_2$ such that these have finitely generated image. Then we can also find smaller open subsets

$$K_1 \subseteq U_1' \subseteq \overline{U_1'} \subseteq U_2$$

and

$$K_2 \subseteq U_2' \subseteq \overline{U_2'} \subseteq U_2$$

with the closures $\overline{U_1'}$ and $\overline{U_2'}$ being compact and consider the following part of the Mayer-Vietoris sequences

$$\begin{tikzcd}
 & H_k(U_1' \cup U_2') \arrow[r] \arrow[d] & H_{k-1}(U_1' \cap U_2') \arrow[d] \\
H_kU_1 \oplus H_k(U_2) \arrow[r] \arrow[d] & H_k(U_1 \cup U_2) \arrow[r] \arrow[d] & H_{k-1}(U_1 \cap U_2) \\
H_kM \oplus H_kM \arrow[r] & H_kM &
\end{tikzcd}$$

The images of the outer vertical maps are finitely generated, the left one by choice of U_1 and U_2 and the right one by induction hypothesis (since $U_1 \cap U_2$ is itself a manifolds and the map factors through the map $H_{k-1}(\overline{U_1'} \cap \overline{U_2'}) \to H_{k-1}(U_1 \cap U_2)$). The middle horizontal sequence is exact, thus the claim follows from the next Lemma. □

Lemma 1.17.3. *Assume that we have a commutative diagram of abelian groups*

$$\begin{tikzcd}
 & A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & E \\
F \arrow[r] & G &
\end{tikzcd}$$

such that the vertical maps have finitely generated images and the middle horizontal sequence is exact (not necessarily short exact). Then also the image of $A \to G$ is finitely generated.

Proof. We denote the maps in the diagram by φ_{AB} and similarly for the other maps, indicating source and target.

By assumption the image of φ_{BE} is finitely generated, thus the image of φ_{AE}, which is a subset is also finitely generated. Choose generators $e_1, ..., e_n$ with preimages $a_1, .., a_n \in A$. Also choose generators $f_1, ..., f_k$ for the image of φ_{CF}. Then we consider the subgroup

$$G' := \langle \varphi_{AG}(a_1), .., \varphi_{AG}(a_n), \varphi_{FG}(f_1), ..., \varphi_{FG}(f_k) \rangle \subseteq G \ .$$

We claim that the image of φ_{AG} is contained in G' which will finish the proof, since then the image is also finitely generated. Consider an element $a \in A$. We have that

$$\varphi_{AE}(a) = \sum_{i=1}^{n} \mu_i e_i$$

with $\mu_i \in \mathbb{Z}$. Then consider

$$d := \varphi_{AD}(a) - \sum_{i=1}^{n} \mu_i \varphi_{AD}(a_i) \ .$$

We have that $\varphi_{DE}(d) = 0$ by construction, thus we find an element $c \in C$ with $\varphi_{CD}(c) = d$. But then we have that

$$\varphi_{DG}(d) = \varphi_{CG}(c) \in \langle \varphi_{FG}(f_1), ..., \varphi_{FG}(f_k) \rangle \subseteq G'$$

It follows that

$$\varphi_{AF}(a) = \varphi_{DG}(d) + \sum_{i=1}^{n} \mu_i \varphi_{AG}(a_i) \in G'.$$

which finishes the proof. □

Remark 1.17.4. The same proof shows that $H_k(M, R)$ is finitely generated over R for any noetherian ring R, in particular for $R = \mathbb{Q}$ and $R = \mathbb{Z}/n$.

Definition 1.17.5. A chart of a topological manifold M is an open subset $U \subseteq M$ together with a homeomorphism $\phi : U \to V \subseteq \mathbb{R}^n$.

An atlas $\mathcal{A}$ of a topological manifold is a family of charts $\{\phi_i : U_i \to V_i\}_{i \in I}$ such that $\bigcup_{i \in I} U_i$ cover M (by definition every topological manifold admits an atlas).

The transition functions of an atlas $\mathcal{A}$ are the homeomorphisms

$$\varphi_{ij} : \phi_i(V_j) \to \phi_j(V_i) \qquad x \mapsto \varphi_j \circ \varphi_i^{-1}(x)$$

Note that we have the so-called cocycle condition $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$. In particular $\varphi_{ji} = \varphi_{ij}^{-1}$.

Example 1.17.6.
1. $M = \mathbb{R}^n$. Atlas with one chart $\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$. No (interesting) transition functions.

2. The sphere S^n. We have an atlas $\mathcal{A}$ consisting of two charts as follows:

$$U_0 := S^n \setminus N := \{(x_0, ..., x_n) \in S^n \mid x_0 \neq 1\}$$

and $\varphi_0 : U_0 \to \mathbb{R}^n$ the stereographic projection (to a equatorial hyperplane) is given by

$$\varphi_0(x_0, ..., x_n) = \left(\frac{x_1}{1 - x_0}, ..., \frac{x_n}{1 - x_0}\right)$$

Similarly we have

$$U_1 := S^n \setminus S := \{(x_0, ..., x_n) \in S^n \mid x_0 \neq -1\}$$

and

$$\varphi_1(x_0, ..., x_n) = \left(\frac{x_1}{1 + x_0}, ..., \frac{x_n}{1 + x_0}\right)$$

The transition function

$$\varphi_{01} : \mathbb{R}^n \setminus 0 \to \mathbb{R}^n \setminus 0$$

is given by

$$(y_1, ..., y_n) \mapsto -\frac{1}{|y|^2}(y_1, ..., y_n)$$

3. Another possibility is an atlas for S^n consisting of $(2n + 2)$ charts as follows:

$$U_i^+ = \{(x_0, ..., x_n) \in S^n \mid x_i > 0\}$$

$$U_i^- = \{(x_0, ..., x_n) \in S^n \mid x_i < 0\}$$

with $\varphi^\pm : U_i^\pm \to D^n$ given by $(x_0, ..., x_n) \mapsto (x_0, ..., \hat{x}_i, ..., x_n)$. Picture

4. Real projective space $\mathbb{R}P^n = \mathbb{R}^{n+1}/\sim = \{[x_0 : ... : x_n]\}$. Set

$$U_i := \{[x_0 : ... : x_n] \mid x_i \neq 0\}$$

and

$$\varphi_i : U_i \to \mathbb{R}^n \qquad [x_0 : ... : x_n] \mapsto \left(\frac{x_0}{x_i}, ..., \widehat{\frac{x_i}{x_i}}, ..., \frac{x_n}{x_i}\right)$$

The transition functions φ_{ij} for $i < j$ are given by

$$\varphi_{ij} : \mathbb{R}^n \setminus \{y_j \neq 0\} \to \mathbb{R}^n \setminus \{y_{i+1} \neq 0\}$$

$$(y_1, ..., y_n) \mapsto (\frac{y_1}{y_j}, ..., \frac{y_i}{y_j}, \frac{1}{y_j}, ..., \widehat{\frac{y_j}{y_j}}, ..., \frac{y_n}{y_j})$$

5. Similarly for $\mathbb{C}P^n$ we have $n + 1$-charts.

Definition 1.17.7. The local degree in $x \in U$ for a homeomorphisms $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^n$ is defined as the number $\deg_x f$ such that $f_* : H_n(U, U \setminus \{x\}) \to H_n(V, V \setminus \{y\})$ is given by multiplication with $\deg_x f$ (in the standard identification with $\mathbb{Z}$). We call f orientation preserving, if the local degree of f in x for each point $x \in U$ is one. Otherwise its called orientation reversing.

Remark 1.17.8.
- Note that this is an extension of our old local degree. Indeed if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism (in particular it induces a map $f^+ : S^n = (\mathbb{R}^n)^+ \to (\mathbb{R}^n)^+ = S^n$) then the local degree of f in $x \in \mathbb{R}^n$ agrees with the local degree defined in Definition 1.13.4. In this case it also agrees with the (global) degree of the map f^+ by Theorem 1.13.7. Conversely if we start with a map $f : S^n \to S^n$ and $U \subseteq S^n$ that gets homeomorphically mapped to $V \subseteq S^n$. Then we can identify U and V with subsets of $\mathbb{R}^n$ and reduce the old local degree to the new one (we will see soon how all these degrees nicely fit together)
- If U is path connected then the local degree is independent of the point x. This means that we have to check orientability only in one point. To see this pick a nicely embedded path γ connecting x and x' and consider the diagram:

$$\begin{tikzcd}
H_n(U, U \setminus \{x\}) \arrow[r] & H_n(V, V \setminus \{f(x)\}) \\
H_n(U, U \setminus \gamma I) \arrow[r] \arrow[u, "\simeq"] \arrow[d, "\simeq"'] & H_n(V, V \setminus f\gamma(I)) \arrow[u] \arrow[d] \\
H_n(U, U \setminus \{x'\}) \arrow[r] & H_n(V, V \setminus \{f(x)\})
\end{tikzcd}$$

Example 1.17.9. The map

$$\mathbb{R}^n \to \mathbb{R}^n \qquad (y_1, ..., y_n) \mapsto (-y_1, ..., y_n)$$

is orientation reversing (by Lemma 1.7.10). .

Lemma 1.17.10. *Let A be an invertible $n \times n$-matrix. Then the local degree of the mapping $A : \mathbb{R}^n \to \mathbb{R}^n$ in every point $x \in \mathbb{R}^n$ (which is the same as the degree of the map $A^+ : S^n \to S^n$) is given by the sign of the determinant of A.*

Proof. We check this in the point 0. Clearly for a product of matrices AB we have $\deg_0(AB) = \deg_0(A) \cdot \deg_0(B)$. Thus we can reduce the Lemma the the case of elementary matrices. There are three forms of elementary matrices: First consider E_i^λ (λ on diagonal). Such a matrix is homotopic (through invertible matrices) to the matrix E_i^λ with $\lambda = \pm 1$. In this case the statement follows from Lemma 1.7.10. The matrix E_{ij}^λ (lambda offdiagonal) is homotopic (as an invertible map $\mathbb{R}^n \to \mathbb{R}^n$) to the identity. The final case of the interchanging of two rows is homotopic to a reflection as well, thus also follows from Lemma 1.7.10. □

Theorem 1.17.11. *Let $f : U \subseteq \mathbb{R}^n \to V \subseteq \mathbb{R}^n$ be a smooth diffeomorphism. Then the local degree of f in $x \in U$ is the sign of the determinant of the derivative $df_x : \mathbb{R}^n \to \mathbb{R}^n$.*

Proof. By a translation we can assume that $0 \in U$ and $f(0) = 0$. Using the last Lemma we can compose the map f with a linear map such that we can assume wlog that $df_x = \mathrm{id}$. Thus we have to show that under these conditions the map f has local degree 1 in the point 0. Using Taylor's theorem we write f as

$$f(x_1, ..., x_n) = (x_1, ..., x_n) + g(x_1, ..., x_n)$$

where for some $\epsilon > 0$ we have $|g(x)| < 1/2|f(x)|$ for $0 < |x| < 2\epsilon$.

Then define

$$F(x,t) := \begin{cases} f(x) & 2\epsilon < |x| \\ f(x) - t(2 - |x|/\epsilon)g(x) & \epsilon \leq |x| \leq 2\epsilon \\ f(x) - tg(x) & \text{else} \end{cases}$$

The map $F(x,1) : U \to V$ is the identity near 0, hence has local degree 1 in 0. The map $F(x,0)$ agrees with the map f around 0, hence has the same degree as f in zero. Moreover one can check (and this requires some thinking) that F is indeed a homotopy of pairs. Thus we find that $\deg_0 f = 1$. □

Examples 1.17.12. Consider the map

$$\varphi_{ij} : \mathbb{R}^n \setminus \{y_j = 0\} \to \mathbb{R}^n \setminus \{y_{i+1} = 0\}$$
$$(y_1, ..., y_n) \mapsto \left(\frac{y_1}{y_j}, ..., \frac{y_i}{y_j}, \frac{1}{y_j}, \frac{y_{i+1}}{y_j}, ..., \widehat{\frac{y_j}{y_j}}, ..., \frac{y_n}{y_j}\right)$$

for $i \leq j$. We consider the case $i = 0, j = 1$, i.e. $f := \varphi_{01}$ is given by

$$(y_1, ..., y_n) \mapsto \left(\frac{1}{y_1}, \frac{y_2}{y_1}, ..., , \frac{y_n}{y_1}\right)$$

Thus we find that the Jacobian is

$$df_y = \begin{pmatrix} -\frac{1}{y_1^2} & * & & & \\ & \frac{1}{y_1} & * & & \\ & & \frac{1}{y_1} & * & ... \\ & & & ... & ... \end{pmatrix}$$

and therefore the determinant

$$\det df_y = -\frac{1}{y_1^{n+1}}$$

Hence the morphism f is not orientation preserving, since the local degree in the point $(1, 0, ..., 0)$ is -1. More generally we find by interchanging the rows that

$$\det d\varphi_{ij} = (-1)^{i+j} \frac{1}{y_j^{n+1}}$$

Definition 1.17.13. An atlas $\mathcal{A}$ of a manifold M is called oriented if the transition functions are oriented . Two oriented atlases $\mathcal{A}$ and $\mathcal{A}'$ are called (oriented) equivalent if their union $\mathcal{A} \cup \mathcal{A}'$ is again an oriented atlas of M. An orientation on a manifold is an equivalence class of oriented atlases. A manifold is called orientable if it admits an orientation.

Remark 1.17.14. There are manifolds that do not admit orientations as we will see soon. If a connected manifold is oriented then there are exactly two orientations.

Examples 1.17.15. The standard atlases of $\mathbb{R}^n$, Σ_g and $\mathbb{C}P^n$ are oriented (as we see from Examples 1.17.6 and 1.17.9). These are the standard orientations on these manifolds.

The standard atlas of $\mathbb{R}P^n$ is not oriented as the last example 1.17.12 shows (since the transition functions are not oriented). Lets consider another atlas on $\mathbb{R}P^n$ with the same open sets U_i but the maps $\tilde{\varphi}_i := (-1)^i \varphi_i$. We find that the transition functions $\tilde{\varphi}_{ij}$ are given by

$$\tilde{\varphi_{ij}} = (-1)^{i+j} \varphi_{ij}$$

Thus we get

$$\det d\tilde{\varphi_{ij}} = (-1)^{n\cdot(i+j)}(-1)^{i+j}\frac{1}{y_1^{n+1}} = (-1)^{(n+1)\cdot(i+j)}\frac{1}{y_1^{n+1}}$$

For n odd this is always positive, hence $\mathbb{R}P^n$ is orientable for n odd.

The next result is the main result about oriented manifolds.

Theorem 1.17.16. *For an n-dimensional, connected manifold M we have $H_k(M, G) = 0$ for $k > n$ and*

$$H_n(M, G) \cong \begin{cases} G & \text{if } M \text{ is compact and orientable} \\ G[2] & \text{if } M \text{ is compact and not orientable} \\ 0 & \text{of } M \text{ is not compact} \end{cases}$$

where $G[2] = \{g \in G \mid 2g = 0\}$.

In particular a compact, connected manifold is orientable if and only if $H_n(M, \mathbb{Z}) = \mathbb{Z}$.

Example 1.17.17.

1. For the genus g-surface we have seen in Proposition 1.13.1 that $H_2(\Sigma_g) \cong \mathbb{Z}$. Thus it is orientable.
2. For projective space $\mathbb{R}P^n$ we have seen that $H_n(\mathbb{R}P^n) = \mathbb{Z}$ for n odd and $H_n(\mathbb{R}P^n) = 0$ for n even. Thus $\mathbb{R}P^n$ is orientable precisely for n odd (we had already seen that it was orientable in Example 1.17.15).
3. The surface Σ_g^- for $g \geq 1$ is obtained as the quotient of the regular 2g-gon Q_g divided out by the relation $a_1^2 a_2^2 a_3^2 ... a_g^2$. It is not hard to check that this defines a topological manifold of dimension 2. For example we have that $\Sigma_1^- = \mathbb{R}P^2$ and Σ_2^- is the Klein bottle.

   Abstractly Σ_g^- is the presentation complex corresponding to the presentation $\langle a_1, ..., a_g \mid a_1^2 ... a_g^2 \rangle$. We obtain the cellular chain complexes using Proposition 1.13.2 as

   $$... \to 0 \to \mathbb{Z} \xrightarrow{(2,2,...,2)} \mathbb{Z}^g \xrightarrow{0} \mathbb{Z}$$

   and therefore $H_0(\Sigma_g^-) = \mathbb{Z}, H_1(\Sigma_g^-) = \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2$ and $H_2(\Sigma_g^-) = 0$.

   In particular the Σ_g^- are all non orientable. We can also conclude that all the Σ_g and Σ_g^- are pairwise non-homotopy equivalent (exercise).
4. For complex projective space we can immediately see that $H_{2n}(\mathbb{C}P^n)$ has the be $\mathbb{Z}$ from the fact that it is oriented without computing.
5. The Lens space $L_{p,q}$ has a CW structure with one cell in dimensions 0,1,2 and 3 as was shown in the exercises. Thus its cellular chain complex $C_*^{cell}(L_{p,q})$ has the form

   $$... \to 0 \to \mathbb{Z} \xrightarrow{\partial_3} \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z}$$

Since $L_{p,q}$ is connected we have $\partial_1 = 0$. Since $H_1(L_{p,q}) = \mathbb{Z}/p$ we have $\partial_2 = p$. The third differential ∂_3 is given by multiplication by some m. Now we have $0 = \partial_2\partial_3 = pm$. Thus we have $m = 0$. We get

$$... \to 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

We conclude that $H_3(L_{p,q}) = \mathbb{Z}$, $H_2(L_{p,q}) = 0$, $H_1(L_{p,q}) = \mathbb{Z}/p$ and $H_0(L_{p,q}) = \mathbb{Z}$. In particular $L_{p,q}$ is orientable for all values of p and q.

Corollary 1.17.18 (Invariance of dimension II)**.** *For a compact n-dimensional manifold M we have $H_n(M, \mathbb{Z}/2) \cong \mathbb{Z}/2$. Thus if two compact manifolds M and N are homotopy equivalent, we have that* $\dim M = \dim N$.

Definition 1.17.19. For an oriented manifold the canonical generator of $H_n(M;\mathbb{Z}) \cong \mathbb{Z}$ is called a fundamental class of M and denoted $[M] \in H_n(M;\mathbb{Z})$.

Remark 1.17.20.

1. Later we will get much stronger results from the existence of a fundamental class (called Poincare duality). For example we will see that for an oriented manifold of dimension n we have $\mathrm{rank} H_k(M) = \mathrm{rank} H_{n-k}(M)$.

2. If we have a compact oriented n-manifold M and a map $M \to X$ then we get an induced class in $[M] \in H_n(X)$ as the image of the fundamental class of $[M] \in H_n(X)$. In this way manifolds describe classes in singular homology. In fact a lot of classes (but not all) can be described this way.

1.18 Classification of surfaces

Theorem 1.18.1. *Each compact, connected 2-manifold is homeomorphic to one of the surfaces*

- Σ_g *for* $g = 0, 1, 2, ...$
- Σ_g^- *for* $g = 1, 2, 3....$

We already know that the surfaces Σ_g and Σ_g^- are non-homeomorphic. The easiest way to see this is to use the following argument: the Σ_g are orientable (Example 1.17.17) and the Σ_g^- are non-orientable. Hence they can not be homotopy equivalent (note that these are compact manifolds). To distinguish the Σ_g among each other or the Σ_g^- among each other it suffices to consider the Euler characteristic, which is

$$\chi(\Sigma_g) = 2 - 2g \qquad \text{and} \qquad \chi(\Sigma_g^-) = 2 - g$$

Corollary 1.18.2. *For connected, compact surfaces F and F' the following are equivalent:*

- *F and F' are homeomorphic*
- *F and F' are homotopy equivalently*
- *F and F' have the same orientation behaviour (i.e. both orientable or both non-orientable) and $\chi(F) = \chi(F')$.*

Definition 1.18.3. Let M and N be n-dimensional manifolds which are oriented or non-orientable. We choose embeddings $i_M : \mathbb{R}^n \to M$ and $i_N : \mathbb{R}^n \to N$ such that i_M is orientation preserving when this makes sense and i_N is orientation reversing when this makes sense. The we define the connected sum as

$$M\#N := (M \setminus \mathrm{int}D^n) \cup_{S^{n-1}} (N \setminus \mathrm{int}D^n)$$

where $S^{n-1} \subseteq D^n \subseteq \mathbb{R}^n$ are the sphere and the closed disk inside of $\mathbb{R}^n$ the glueing is understood using the embeddings i_M and i_N.

Proposition 1.18.4. *Let M and N be oriented or non-orientable manifolds of dimension n. Then $M\#N$ has Euler characteristic $\chi(M\#N) = \chi(M) + \chi(N) - 2$ for n even or $\chi(M) + \chi(N)$ for n odd. The connected sum is orientable, precisely of M and N are both orientable. For two compact, connected manifolds the connected sum is also compact and connected.*

Proof. This was exercise 11.4. □

Corollary 1.18.5. *Every surface is homeomorphic to one of the manifolds:*

1. S^2 *or* $T\#T\#...\#T \cong \Sigma_g$
2. $\mathbb{R}P^2\#\mathbb{R}P^2\#...\#\mathbb{R}P^2 \cong \Sigma_g^-$.

Proof. It suffices to show that Σ_g is the connected sum of tori and Σ_g^- is the connected sum of $\mathbb{R}P^2$'s. To this end we use the last Proposition and deduce that $T\#T...\#T$ with $g+1$ summands is orientable and has Euler characteristic $2-2g$ which can be shown inductively. Then we use the classification of 1.18.2 to get the result. The same argument applies for the connected sum of $\mathbb{R}P^2$'s. □

There is one last open point, namely that the connected sum was defined using embeddings. But the notation already suggests that it should not depend on these choices.

Theorem 1.18.6. *For two n-dimensional manifolds M and N, the connected sum is well defined. That means that different choices of embeddings lead to homeomorphic connected sums.*

Proof. This statement is true for general topological n-dimensional manifolds and follows from Kirby's annulus theorem. We can only prove this for $n = 1$ and for $n = 2$ in the compact case. For $n = 1$ it follows from the classification that $S^1\#S^1$ is homeomorphic to S^1 (and thus cannot depend on any choice). The same shows that $S^1\#\mathbb{R} \cong \mathbb{R}$ and $\mathbb{R}\#\mathbb{R} \cong \mathbb{R} \sqcup \mathbb{R}$. For 2-manifolds the classification shows that

$$\Sigma_g\#\Sigma'_g \cong \Sigma_{g+g'}$$

and

$$\Sigma_g\#\Sigma^-_{g'} \cong ?$$

□

Remark 1.18.7. The connected sum depends on choices, one can show that $\mathbb{C}P^2\#\mathbb{C}P^2$ and $\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$ are not homotopy equivalent using the so-called signature.

1.19 Uniqueness of homology

M and N are oriented manifolds of dimension n.

Definition 1.19.1. If M and N are compact oriented with fundamental classes $[M]$ and $[N]$ then the degree of a map $f : M \to N$ is defined as the interger $\deg f$ such that

$$f_*[M] = \deg f \cdot [N]$$

in $H_n(N) \cong \mathbb{Z}$.

Definition 1.19.2. If M and N are not necessarily compact (e.g. open subsets of compact manifolds) and $f : M \to N$ is a homeomorphism then the local degree of f in a point $m \in M$ is defined as $+1$ if f sends the local orientation in $m \in M$ to the local orientation in $f(m)$. In this case we say that f is orientation preserving in m. Otherwise we set $\deg_m f = -1$ and say f is orientation reversing in m.

Remark 1.19.3. These definitions include all the old degrees we have defines. For example for a sphere S^n we take the canonical orientation and then the degree of Definition **??** is precisely the degree of Definition 1.19.1. In fact, if we took the opposite orientation the degrees would agree since then the fundamental class obtains a minus sign which affects the equation twice.

For an open set $U \subseteq \mathbb{R}^n$ or $U \subseteq S^n$ we have a canonical induced orientation and then local degree of Definition 1.13.4 clearly reduces to our local degree.

Now we have the exact same tools as in Theorem 1.13.7 to compute the degree.

Theorem 1.19.4. *Let $f : M \to N$ and $y \in N$ such that the preimage $f^{-1}(y) = \{x_1, ..., x_n\}$ is finite and such that every x_i has a neighborhood U_i which gets homeomorphically mapped to a neighboorhood V of y. Then* $\deg f = \sum_{i=1}^{n} \deg_{x_i} f$.

Proof. Exactly the same as in Theorem 1.13.7. □

Theorem 1.19.5. *If $f : M \to N$ is a diffeomorphism between smooth manifolds then the local degree in $m \in M$ is the sign of the determinant of the Jacobian of f computed in positively oriented charts.*

Proof. Without loss of generality we can reduce to a coordinate neighboorhood. Then it precisely reduces to Theorem 1.17.11. □

We now want to return to the study of homology theories, which satisify the Eilenberg-Steenrod axioms as in Definition 1.6.1. The proof of the last two theorem holds for all generalized homology theories. Since the local degree of smooth diffeomorphisms is computed in terms of a Jacobian, it does not depend at all on the specific choice of homology theory H_*.

Corollary 1.19.6. *The local degree if a smooth diffeomorphism $f : M \to N$ does not depend on the choice of homology theory H_**

Theorem 1.19.7 (Sard)**.** *Every smooth map $M \to N$ admits a value $n \in N$ (in fact those values are dense) such that every point in the preimage has a point that gets diffeomorphically maps to a neighboorhood of n. If M is compact the preimage of n is finite.*

Note that such points $n \in N$ are regular. This gives us a way to compute the degree of smooth maps out of compact manifolds, since we can first find such an n, then apply Theorem 1.19.5 and 1.19.4 to reduce it to a sum of signs of Jacobians.

Theorem 1.19.8. *Every continuous map $f : M \to N$ between smooth manifolds M and N is homotopic to a smooth map.*

Corollary 1.19.9. *For every map $f : M \to N$ the degree is independent of the homology theory.*

Proof. Replace f first by a homotpic smooth map (which does not change the degree). Then ... □

Theorem 1.19.10. *If H and H' are both homology theories in the sense of the axioms . Then for every relative CW complex $A \subseteq X$ the groups $H_*(X, A)$ and $H'_*(X, A)$ are canonically isomorphic.*

Proof. By Theorem 1.12.6 the homology groups are isomorphic to the respective cellular homology groups $H_*^{cell}(X, A)$ and $(H')_*^{cell}(X, A)$. But by Proposition 1.12.12 the cellular chain complex (and hence the cellular homology) is canonically isomorphic to a complex which only depends on the number of cells and the incidence numbers of (X, A). Now the incidence numbers are degrees of maps $S^n \to S^n$ which by the last corollary are the same for H and H'. □

Proposition 1.19.11. *The only group that can act freely on S^{2n} is the group $\mathbb{Z}/2$.*

1.20 Proof of Theorem 1.17.16

Question: How many orientations exist on M? How can we know that a manifold is not orientable?

Remark 1.20.1. A local orientation of a top. manifold M in $m \in M$ is a choice of a generator $\mathfrak{o}_m$ of $H_n(M, M \setminus \{m\})$. Every chart $\varphi : U \to \mathbb{R}^n$ around m induces such a local orientation $\mathfrak{o}_m(\varphi)$ since it induces an isomorphisms

$$H_n(M, M \setminus \{m\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{\varphi(m)\}) \cong \mathbb{Z}$$

Then $\mathfrak{o}_M(\varphi)$ is the element corresponding to $1 \in \mathbb{Z}$ under this isomorphism. Conversely every local orientation $\mathfrak{o}_m$ singles out a class of charts around m which we call oriented, namely those φ with $\mathfrak{o}_m(\varphi) = \mathfrak{o}_m$.

We want to assemble together to an oriented atlas. Therefore we need a local orientation in every point. But these local orientations have to depend continuously on the point, they can not be allowed to jump. We want to make this precise now.

Definition 1.20.2. Let M be a topological manifold of dimension n and G an abelian group. For $x \in M$ set $\Theta_x \otimes G := H_n(M, M \setminus \{x\}; G) \cong H_n(M, M \setminus \{x\}; \mathbb{Z}) \otimes G$. Set

$$\Theta_M \otimes G := \bigcup_{x \in M} \Theta_x \otimes G$$

and let $p : \Theta_M \otimes G \to M$ be the map that takes $\Theta_x \otimes G$ to $\{x\}$. For $G = \mathbb{Z}$ we just write Θ. We call Θ the orientation bundle (with values in G).

Give $\Theta_M \otimes G$ the following topology: Let $U \subseteq M$ be open and $\alpha \in H_n(M, M \setminus U)$. Assembling together the images under

$$H_n(M, M \setminus U) \to H_n(M, M \setminus \{x\})$$

defines a mapping $s_\alpha : U \to \Theta_M \otimes G$. We give $\Theta_M \otimes G$ the finest topology such that all s_α become continuous.

Proposition 1.20.3. *The images $U_\alpha := s_\alpha(U)$ for all $U \in M$ and $\alpha \in H_n(M, M \setminus U; G)$ form a basis of the topology. Moreover $p : \Theta \otimes G \to M$ is a covering map and fibrewise addition is continuous.*

Proof. Let $U \subseteq M$ be a convex ball in some coordinate neighborhood. Then by excision the morphism

$$H_n(M, M \setminus U) \to H_n(M, M \setminus \{x\})$$

is an isomorphism for every $x \in U$. Thus we have an isomorphism

$$\begin{tikzcd} U \times H_n(M, M \setminus U; G) \arrow[rr, "\varphi"] \arrow[dr] & & p^{-1}(U) \arrow[dl] \\ & U & \end{tikzcd}$$

We equip $H_n(M, M \setminus U; G)$ with the discrete topology. Then this morphism is by definition of the topology continuous. It is also bijective and it is straightforward (but a bit annoying) to check that it is a homeomorphism.

Since every point $x \in M$ is contained in some convex coordinate ball this shows that the morphism $p : \Theta_M \to M$ is locally a product with a discrete sets (namely the set $\mathbb{Z}$) and therefore a covering map. The statement about the basis of the topology can easily be derived from that. The fibrewise addition corresponds to the addition on the product factor and is therefore also continuous. □

Definition 1.20.4. The orientation cover of M is the subspace $\hat{M} \subseteq \Theta$ consisting of all elements $H_n(M, M \setminus x)$ that are generators.

Corollary 1.20.5. *The orientation cover $\hat{M} \to M$ is a 2-sheeted covering of M (with not necessarily connected total space). Hence for connected M it is classified by a group homomorphism $w_1 : \pi_1(M) \to \mathbb{Z}/2$ which we call the orientation character.*

Proof. Use the same proof as the the last Proposition to see that locally it is reduced to the generators. For the covering statement note that in general 2 sheeted coverings are described by a index 2-subgroup of the fundamental group which in turn is the same as a group homomorphism to $\mathbb{Z}/2$ (since all index 2 subgroups are normal). □

Remark 1.20.6. Note that the orientation character can equivalently be described as a group homomorphism $w_1 : H_1(M) \to \mathbb{Z}/2$ since the group $\mathbb{Z}/2$ is abelian and hence by the universal property of the abelianization. Also note that there is unique non-trivial covering transformations $\tau : \hat{M} \to \hat{M}$.

Definition 1.20.7. A section of a covering $p : N \to M$ is a morphism $s : M \to N$ such that $ps = \mathrm{id}$. For a closed subspace $A \subseteq M$ we denote by

$$\Gamma(A, \Theta_M \otimes G) = \{s : A \to \Theta_M \otimes G \mid ps(a) = a\}$$

the groups of sections of the orientation bundle (with pointwise addition) and by

$$\Gamma_c(A, \Theta_M \otimes G) \subseteq \Gamma(A, \Theta_M \otimes G)$$

the subgroup of sections with compact support, i.e. those s for which the set $\overline{\{x \in A \mid s(x) \neq 0\}}$ is compact.

Lemma 1.20.8. *For a connected manifold M the following are equivalent :*

1. *The orientation character w_1 is trivial.*
2. *The orientation cover $\hat{M}$ admits a section.*
3. *We have a homeomorphism $\hat{M} \to M \sqcup M$ that commutes with the maps to M.*
4. *We have a homeomorphism $\Theta_M \cong M \times \mathbb{Z}$ that commutes with the morphisms to M and the group structures.*
5. *For every abelian group G we have a homeomorphism $\Theta_M \otimes G \cong M \times G$ that commutes with the morphisms to M and the group structures.*

Proof. The first three assertions are equivalent for every 2-fold covering by standard covering theory. To see the equivalence of (3) and (4) note that if we have a homeomorphism

$$\phi : \hat{M} \sqcup M \to M \cong M \times \{-1, 1\}$$

we can extend it to a homeomorphism

$$\Phi : \Theta_M \to M \times \mathbb{Z}$$

by setting $\Phi(x) = \Phi(ny) = n\phi(y)$ for a generator $y \in H_n(M, M \setminus \{m\})$. This is easily seen to be well-defined and commutes with the group structures. Conversely assume we have such a Φ. Then its restriction to the subset $\hat{M} \subseteq \Theta_M$ has image $M \times \{-1, 1\}$ (since these are the generators). Thus it induces the desired equivalence.

To go from (4) to (5) we tensor the equivalence fibrewise with G. Conversely (4) is clearly a special case of (5). $\square$

Theorem 1.20.9. *There is a 1-1 correspondence between orientations on M and sections of the orientation cover.*

Proof. We describe two explicit maps

$$\{\text{Oriented atlases of } M\} \longleftrightarrow \{\text{sections of } \hat{M} \to M\}$$

Given an orientation $[\mathcal{A}]$ send it to the section $s_{\mathcal{A}}$ defined as follows: for $m \in M$ choose a chart $\varphi : U \to \mathbb{R}^n$ from $\mathcal{A}$ and set $s_{\mathcal{A}}(m) := \mathfrak{o}_M(\varphi) \in H_n(M, M \setminus \{m\})$. This is well-defined since for another chart φ' the transition function is orientation preserving and thus leads to the same generator $\mathfrak{o}_M(\varphi')$. Now we have to show that $s_{\mathcal{A}}$ is continuous. For this it sufficed to proof this locally. But for a chart $\varphi : U \to \mathbb{R}^n$ choose an open $U' \subseteq U$ such that $\overline{U'} \subseteq U$ and such that $\varphi(U')$ is a ball. Then we can describe the restriction $s_{\mathcal{A}} \mid_{U'}$ as the section $s_\alpha : U' \to \hat{M} \subseteq \Theta_M$ corresponding to the element $\alpha \in H_n(M, M \setminus U')$ which maps to 1 under the isomorphism

$$H_n(M, M \setminus U') \cong H_n(U, U \setminus U') \cong H_n(U, U \setminus U') \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B) \cong \mathbb{Z}$$

Thus by definition of the topology on Θ_M the restriction $s_{\mathcal{A}} \mid_{U'}$ is continuous and therefore a section of $\hat{M} \to M$.

The converse construction works as follows: for a given section $s : M \to \hat{M}$ we define an atlas $\mathcal{A}_s$ a follows: $\mathcal{A}_s$ contains all charts

$$\varphi : U \to \mathbb{R}^n$$

such that for every one $m \in U$ we have that $\mathfrak{o}_m(\varphi) = s(m)$. First note that this already implies that we have $\mathfrak{o}_m(\varphi) = s(m)$ for all $m \in U$ since U is connected and since both sides are sections of a covering. We have to check that $\mathcal{A}_s$ defines an atlas. Thus for every $m \in M$ there has to be a chart in $\mathcal{A}_s$ around m. Pick an arbitrary chart φ around m of the manifold M (which is not necessarily in M). Make a new chart φ' by postcomposing with an orientation reversing homeomorphism $\mathbb{R}^n \to \mathbb{R}^n$. Then we have $\varphi'(m) \neq \varphi(m)$. But since the group $H_n(M, M \setminus \{m\})$ has only two generators one of the two elements has to agree with $s(m)$ and hence the corresponding chart is in $\mathcal{A}_s$. To see that the atlas $\mathcal{A}_s$ is oriented we use a similar straightforward argument.

Now it remains to show that the two constructions are inverse to each other. First to see that $s_{\mathcal{A}_s} = s$ we reduce this to a local chart, where it follows immediately from the local descriptions given in the first part of the proof. For the converse, i.e. that $\mathcal{A}_{s_{\mathcal{A}}}$ is equivalent to $\mathcal{A}$ one has to check that the transition functions are orientable, which is immediate unwinding the definitions. □

Corollary 1.20.10. *If a manifold M is orientable and connected, then it has precisely two orientations. Moreover if two orientations agree in one point (meaning that the induced local orientations in that point agree) then they agree globally. In other words: an orientation is uniquely determined by a single chart resp. a single local orientation.*

Example 1.20.11. We have already stated that the genus g-surface is orientable (we will see an independent proof soon). Recall that the genus g-surface was defined as a quotient $P_g/\sim$ of the regular 4g-gon. This gives as a canonical chart

$$\mathrm{int} P_g \to \Sigma_g$$

which we want to be oriented. This uniquely fixes the orientation.

Corollary 1.20.12. *If a manifold M is simply connected or if $H_1(M, \mathbb{Z})$ contains no subgroups of index 2 then it is orientable.*

Proof. By Lemma 2.9.8 and Theorem 2.9.9 M is orientable precisely if $w_1 : H_1(M) \to \mathbb{Z}/2$ is trivial, which is automatically the case if there is no subgroup of index 2. □

Example 1.20.13. We can now directly see (without having to write down an oriented atlas) that the following manifolds are all orientable: S^n for $n \geq 2$, $\mathbb{C}P^n$, $L(p,q)$ for p not divisible by 2. Recall that we have $\pi_1(L_{p,q}) = H_1(L_{p,q}) = \mathbb{Z}/p$.

Again all of these spaces have preferred orientations which are described using (one) preferred chart. Also products of these manifolds are orientable, e.g. $S^2 \times S^3$.

The case of $\mathbb{R}P^n$ (at least for n odd) still remains open (since $H_1(\mathbb{R}P^n) \cong \mathbb{Z}/2$).

Corollary 1.20.14. *The total space of the orientation cover $\hat{M}$ is a manifold that is canonically oriented. The involution τ is orientation reversing.*

Proof. Exercise. □

For $A \subseteq M$ closed define a homomorphism

$$J_A : H_n(M, M \setminus \{A\}; G) \to \Gamma_c(A, \Theta_M \otimes G) \tag{1.2}$$

as before (it sends $\alpha \in H_n(M, M \setminus \{A\}; G)$ to the section $s_\alpha : A \to \Theta_M \times G$). We must show that this section is continuous and has compact support. Continuity can be checked locally i.e. for opens $U \subseteq A$ with $U = V \cap A$. Then $\alpha \mid_U \in H_n(M, M \setminus U)$ can be extended to $H_n(M, M \setminus V)$ for U and V sufficiently small. Thus we have that s_α is the restriction of the section induced form the extension. The fact that the section has compact support follows from the fact that every representative of α in $C_n(M, M \setminus A)$ factors through some compact subset $B \subseteq M$ since Δ_n is compact. Then the section s_α has also support in B.

Theorem 1.20.15. *Let M be a topological manifold of dimension n. Then the homology groups $H_k(M; G)$ are zero for $k > n$ and the canonical morphism 2.1 is a group isomorphism*

$$H_n(M; G) \cong \Gamma_c(M, \Theta_M \otimes G)$$

Proof. We give the proof for $G = \mathbb{Z}$ and the other cases works exactly similar. For a topological manifold M with a closed subset A we consider the following statement:

> $S(M, A)$: $H_k(M, M \setminus A) = 0$ for all $k > \dim M$ and $J_A : H_n(M, M \setminus \{A\}) \to \Gamma_c(A, \Theta_M)$ is an isomorphism.

The Theorem is $S(M, M)$.

1. $S(M, A)$ holds for A compact and convex in some coordinate chart of M....
2. If $S(M, A_1), S(M, A_2)$ and $S(M, A_1 \cap A_2)$ hold, then also $S(M, A_1 \cup A_2)$
3. All compact, then

Now the Theorem follows by the following Lemma. □

Lemma 1.20.16. *(Bootstrap Lemma) Let $S(M; A)$ be a statement about pairs (A, M) where M is an n-manifold and A a closed subset. If (1)-(5) hold then $S(M, A)$ holds for all A and M.*

Proof. From (1)-(2) we conclude that $S(M, A)$ holds for all compact subsets A since these can be covered by a finite union of closed balls in coordinate neighborhoods.

□

Proof of Theorem 1.17.16. For the orientable case we have an isomorphism $\Theta_M \cong M \times \mathbb{Z}$ by Lemma 2.9.8 and Theorem 2.9.9. Thus $H_n(M, \mathbb{Z}) \cong \Gamma(M, M \times \mathbb{Z}) \cong \mathbb{Z}$. For the unoriented case note that if there is a section $s : M \to \Theta$ which is not zero (in one point r equivalently everywhere) then it can be divided by n such that it becomes a section of $\hat{M}$. But then M would be orientable. Thus there can not be a section different from zero. □

1.21 Simplicial complexes and triangulations

In this section we want to provide the preliminaries for the classification of 1 and 2-dimensional manifolds. The central tool is the notion of simplicial complex, and triangulations, which is closely related to CW complexes.

Definition 1.21.1. We call points $v_0, ..., v_k$ in $\mathbb{R}^n$ affinely independent if their convex hull is a k-simplex, i.e. if $\sum \lambda_i v_i = 0$ with $\sum \lambda_i = 0$ imples that all $\lambda_i = 0$. If this is not the case then one of the points lies in the affine space spanned by the other points (explain). For affinely independent points we denote the affine simplex spanned by the point $v_0, ..., v_k$ by $(v_0, ..., v_k) \subseteq \mathbb{R}^n$. Note that this is just the image of the map $[v_0, ..v_k]$ of Definition 1.2.3. Any subsimplex $(v_{i_0}, ..., v_{i_s}) \subseteq (v_0, ..., v_k)$ is called a face.

Definition 1.21.2. An euclidean (or geometric) simplicial complex K is a collection of affine simplices (in some $\mathbb{R}^n$) such that

1. $\sigma \in K$ implies that every face of σ is also in K.

2. $\sigma, \tau \in K$ implies that $\sigma \cap \tau$ is a face of both σ and τ (or is empty).

For a simplicial complex K the associated polyhedron is the topological space $|K| = \bigcup_{\sigma \in K} \sigma$ (with the subspace topology).

Draw some examples (as in Lees 93). Also note that a simplex in $\mathbb{R}^n$ the faces both define simplicial complexes. The tetrahedron defines a simplicial complex whose polyhedron is homeomorphic to the 2-sphere.

Definition 1.21.3. An (abstract) simplicial complex K consist of a set V and a set $S \subseteq P(V)$ of finite, nonempty subsets of E such that

1. Each singleton $\{v\}$ is in S

2. $\sigma \subseteq \tau$ and $\tau \in S$ implies σ in S.

The elements of V are called the vertices of K and the elements of S the simplices. A subset in S with $k+1$-elements is said to have dimension k. We say that K is finite/countable if S is finite/countable. It is called locally finite if each vertex is only contained in finitely many simplices. A subcomplex of K consists of a set of simplices such that for every $s \in L$ also all faces are contained in L.

Example 1.21.4.
- Every euclidean simplicial complex K defines an abstract simplicial complex with the set V given by the set of vertices of K. The set S then consists of all subset $\{v_0, ..., v_k\}$ such that $(v_0, ..., v_k)$ is in K.
- Picture from above
- Explicitly for $\Delta^n \subseteq \mathbb{R}^{n+1}$ we get $V = \{0, 1, ..., n\}$ and $S := P(V)$. For $\partial\Delta^n \subseteq \mathbb{R}^{n+1}$ we get $V = \{0, 1, ..., n\}$ and $S := \{s \subseteq V \mid s \neq V\}$.
- The 1-dimensional complex K_n is defined as $V := \{1, 2, ..., n\}$. The 1-simplices are $\{1, 2\}, \{2, 3\}, \{3, 4\}, ...\{n, 1\}$. This corresponds to the circle with n-points (picture). The complex K_∞ has $V = \mathbb{Z}$ and 1-simplices $\{n, n+1\}$. This corresponds to $\mathbb{R}$ with the marking....
- Let M be a (finite) metric space. For every $\epsilon > 0$ we define a simplcial complex with $V = M$ and $s \subseteq M$ a simplex if the diameter of s is less or equal to ϵ. This is called the Vietoris–Rips complex and plays an important role in big data analysis and image processing.

We now want to define a topological space, whose 'blueprint' is K.

Definition 1.21.5. The geometric realization $|K|$ of a simplicial complex K is defined (as a set) as the set of functions $\alpha : V \to [0, 1]$ such that

1. $\{v \in V \mid \alpha(v) > 0\}$ is a simplex of K
2. $\sum_{v\in V} \alpha(v) = 1$

We now define two different topologies on the set $|K|$. First we define a metric by

$$d(\alpha, \beta) := \sqrt{\sum_{v\in V} \big(\alpha(v) - \beta(v)\big)^2}$$

The set $|K|$ with the metric topology is denoted by $|K|_d$. This topology is well-suited for locally finite simplcial complexes, but in general it turns out that we have to consider another topology which can be described as follows: for $s \in S$ let $|s| \subseteq |K|$ be the subset

$$|s| := \{\alpha \in |K| \mid \alpha \text{ is supported in } s \subseteq V\}$$

If we consider this as a subset of $\mathbb{R}^S$ then it is given by the standard simplex

$$\Delta^{|S|-1} \cong \left\{ \sum_{v\in s} \lambda_v \delta_s \mid \sum_{v\in S} \lambda_v = 1, \lambda_v \in [0, 1] \right\} \subseteq \mathbb{R}^S$$

For different simplices $s_1, s_2 \in S$ there are three subspaces topologies on $|s_1|_d \cap |s_2|_d = |s_1 \cap s_2|_d$ which agree. We now declare a topology on the set $|K|$ be letting $A \subseteq |K|$ be closed precisely if the intersection $A \cap |s|$ is closed as a subset of $|s|_d$ for each simplex $s \in S$. We call this the complex topology and denote the set $|K|$ with this topology abusively also by $|K|$. This is the topology we want to consider in the following.

Lemma 1.21.6. *1. The canonical morphism $\bigsqcup_{s\in S} |s|_d \to |K|$ (which is on each simplex the inclusion) becomes an identification.*

2. *A map $|K| \to Y$ is continuous precisely if the restriction to each closed simplex is continuous. In particular $|K| \to |K|_d$ is continuous.*

Proof. The first is clear by definition of the topology and the second follows from the first. □

For a subcomplex $L \subseteq K$ we get a continuous inclusion map $|L| \to |K|$. Now define a filtration

$$|K_0| \subseteq |K_1| \subseteq |K_2| \subseteq .... \subseteq |K|$$

of $|K|$ by letting K_i be the subcomplex consisting of all $\sigma \in K$ with $\dim \sigma \leq i$.

Proposition 1.21.7. *The above filtration defines a CW structure on $|K|$.*

Proof. Choose an ordering on the vertices. If $s = \{v_0, ..., v_n\}$ is a simplex of K with $v_0 < v_1 < ... < v_n$ then we have an identification $\Delta^n \cong |s|$ and define the characteristic map for an n-cell to be

$$Q_i^n : \Delta^n \to |s|_d \to |K|.$$

Then the diagram

$$\begin{tikzcd} \bigsqcup_{s\in K, \dim s = n} \partial\Delta^n \arrow[r] \arrow[d] & |K|_{n-1} \arrow[d] \\ \bigsqcup_{s\in K, \dim s = n} \Delta^n \arrow[r] & |K|_n \end{tikzcd}$$

is a pushout (of sets) as one easily checks. The fact that the topology is the right one follows from the last Lemma (1) (recall that it suffices to check that the map from the closed cells to the total space is an identification). □

We now get all the nice properties from CW complexes. In particular from Corollary 1.10.10 we get:

Corollary 1.21.8. *The space $|K|$ is compact, if and only if it $|K|$ has only finitely many cells. Moreover each compact subset $C \subseteq |K|$ hits only finitely many simplices.*

For each simplicial complex K and each set of points $x_v \in \mathbb{R}^n$ for $v \in V$ there is a canonical continuous map $f : |K| \to \mathbb{R}^n$ defined by

$$f(\alpha) = \sum_{v\in V} \alpha(v)x_v$$

If this is an embedding then the image is an euclidean simplex. If conversely K was an euclidean simplex from the start then the canonical such map is an embedding. This opens the question, whether each abstract simplicial complex can be realized as an euclidean simplex. Recall that the dimension of a simplicial complex is defined as the dimension of the highest simplex.

Proposition 1.21.9. *Let K be countable, locally finite and of dimension n. Then K can be realized as an euclidean simplex in $\mathbb{R}^{2n+1}$.*

Proof. Choose a set of points $x_i \in \mathbb{R}^{2n+1}$ with the following properties:

1. Each subset of $2n + 2$ points is affinely independent

2. For each compact $C \subseteq \mathbb{R}^{2n+1}$ there is j such that C is disjoint to the convex hull of $\{x_i \mid i \geq j\}$

Such points exist: choose a sequence of halfplanes $H_1 \supset H_2 \supset H_3 \supset ....$ with $\bigcap H_i = \emptyset$. Assume $p_i \in H_i$ for $i < k$ is choosen such that (1) holds. Then there are only finitely many hyperplanes which contain 2n+1 points from $\{p_1, ..., p_{k-1}\}$. Choose p_k to not lie in these hyperplanes. This way we get inductively such a sequence.

Now choose a numbering of the vertices of V and consider the map

$$f : |K| \to \mathbb{R}^{2n+1}$$

induced as above, i.e. $f(v_i) = x_i$. Because of (1) each simplex $|s| \subseteq |K|$ is mapped linearly and injective to $\mathbb{R}^{2n+1}$. The same argument also shows that f is injective on $|s| \cup |t|$. Hence f is injective. Now we want to show that f is closed, hence an embedding. Let $|A| \subseteq |K|$ be closed. Then $f(A)$ is closed if $f(A) \cap C$ is closed for all compact C. Because of (2) and because K is locally finite it follows that $f^{-1}(C)$ is contained in a finite subcomplex. Hence $f^{-1}(C)$ is compact as a closed subset of a compact space (by Proposition 1.21.9). Thus $f(A) \cap C = f(A \cap f^{-1}(C))$ is compact, hence closed. □

Definition 1.21.10. A triangulation of a space X consists of an abstract simplicial complex K together with a homeomorphism $|K| \to X$.

Remark 1.21.11.

1. By Proposition 1.21.7 every triangulation of a space X also induces a CW structure on X. Conversely we can ask whether a CW structure induces a triangulation. It can in fact be shown that every CW structure can be 'refined' to a triangulation. We will not show this here but as we have noticed already, a triangulation needs much more 'cells' than a CW structure. This is why CW structures play a more important role in practice than triangulations.

2. It is an important question whether manifolds admit a traingulation or not. We will show that this is possible for 1 and 2-dimensional manifolds which will eventually lead to the classification. For 3-dimensional manifolds this is a deep theorem of Moise (proved in the 70's). There are topological 4-manifolds for which it is known that they do not admit a triangulation (constructed by Freedman). Smooth manifolds always admit a triangulation, which can be shown using transversality methods.

For what follows let us introduce some more terminology. For each simplex s of K let us denote the open simplex $\langle s \rangle \subseteq |K|$ defined as

$$\langle s \rangle := \{\alpha \in |K| \mid \alpha(v) \neq 0 \Leftrightarrow v \in s\}$$

Note that this is not open in $|K|$ in general. It corresponds to open cells in CW complexes under the CW structure. Obviously we have $\langle s \rangle \subseteq |s|$ and we set $\partial := |s| \setminus \langle s \rangle$. For each $v \in V$ we set

$$\mathrm{St}(v) := \{\alpha \in |K| \mid \alpha(v) \neq 0\}$$

This is open in $|K|$ and $|K|_d$ since $\alpha \mapsto \alpha(v)$ is continuous in both topologies. It is easy to see that $\mathrm{St}(v) = \bigcup \langle s \rangle$ where the union is over all simplices s that contain v. The link of v is

$$\mathrm{Lk}(v) = \overline{\mathrm{St}(v)} \setminus \mathrm{St}(v)$$

Lemma 1.21.12. *If for a simplicial complex K the space $|K|$ is an n-dimensional manifold, then K is countable, locally finite and n-dimensional.*

Proof. If $|K|$ is a n-dimensional manifold, then every point $x \in |K|$ has countable neighborhood basis. Choose a vertex v and choose such a neighboorhood basis $U_0 \supset U_1 \supset U_2 \supset ...$ for the associated point $v \in |K|$. Assume v is contained in infinitely many simplices $s_1, s_2, ...$ For each i we find that $\langle s_i \rangle \cap U \neq \emptyset$. Then we find a sequence $a_i \in U_i$ which converges to v. But the set $\{x, a_1, a_2, ...\}$ is discrete since it hits every open simplex in precisely in one point.

Let t be a simplex of K and v be a vertex of t. Since K is locally finite we find a simplex s of maximal dimension with t as a face. Then $\langle s \rangle$ is open, hence by the invariance of dimension (Theorem 1.7.6) we have $\dim t \leq \dim s = n$.

The set
$$U(v) = \{\alpha \in |K| \mid \alpha(v') < \alpha(v) \text{ for each simplex } v' \neq v\}$$
is an open neighborhood of v in $|K|$. Since these sets are disjoint for different v we find that there can only by countable many v's. □

Theorem 1.21.13. *For a simplicial complex K the space $|K|$ is a 1-manifold if and only if K is 1-dimensional, countable and each vertex lies in exactly two edges (i.e. 1-simplices).*

Proof. If K is 1-dimensional, countable and has the property that each vertex lies in two edges, then clearly every point of $|K|$ has a neighborhood homeomorphic to $\mathbb{R}$ and is Hausdorff. Countability implies second countability. Thus we show the converse.

The above Lemma 1.21.12 implies that K is countable. Now Let v be a vertex of K. Consider the star $\mathrm{St}(v) \subseteq |K|$. Because $|K|$ is a 1-manifold, there is an open neighborhood $U \subseteq V$ of v homeomorphic to $\mathbb{R}$. Then $U \setminus \{v\}$ has exactly two components. But $U \setminus \{v\}$ can be written as a disjoint union of $\langle e \rangle \cap U$ where e runs through all 1-simplices of K containing v. Since all these sets are non-empty this show that there have to be exactly two edges containing v . □

Definition 1.21.14. A simplicial complex is called a surface complex, if it has the following properties:

1. K is countable, locally finite and 2-dimensional
2. Each 0-simplex is face of at least one 1-simplex
3. Each 1-simplex is face of exactly two 2-simplices
4. If a is a 0-simplex and $x_1, ..., x_k$ are the 2-simplices which have a as a face. Then the faces of $x_1, ..., x_k$ which do not have a as a face form a connected graph.

The last condition is equivalent to

4'. The 2-simplices $x_1, ..., x_k$ can be indexed such that x_i and x_{i+1} have a 1-simplex in common for $i = 1, ..., k - 1$.

Consider the tetrahedron. This satisifies all the conditions.... check.

Two tetrahedra connected at a vertex do not satisfy condition (4).

Theorem 1.21.15. *The space $|K|$ is a surface (i.e. a 2-dimensional topological manifold) if and only if K is a surface complex.*

Proof. Assume $|K|$ is a surface. We have to check conditions (1)-(4). Condition (1) follows from Lemma 1.21.12. Isolated 0-simplices would be in contradiction to the invariance of dimension (Theorem 1.7.6) (since a neighborhood would be zero dimensional). Thus condition (2) is clear. An isolated 1-simplex can not occur by the same argument. Thus a simplex is the face of at least one 2-simplex. But it can also not be the face of exactly one 1-simplex, because then locally the spaces would be homeomorphic to a Half plane $\{x \in \mathbb{R}^2 \mid x_1 \leq 0\}$ where the 1-simplex gets mapped to the x_2-axis. But this is not a manifold since the boundary points do not have a neighborhood homeomorphic to $\mathbb{R}^2$. Thus every 1-simplex is the face of at least two 2-simplices. Now assume a 1-simplex t is the face of more than two 2-simplices, which we denote by $x_1, ..., x_k$ (note that its finitely many since $|K|$ is locally finite). Let x be a point in the interior of $|t|$. Then each neighborhood of t intersects all the 2-simplices $x_1, ..., x_k$. We also have that $U := \langle s_1 \rangle \cup \langle t \rangle \cup \langle s_2 \rangle$ is abstractly homeomorphic to $\mathbb{R}^2$. By the invariance of domain Theorem 1.9.11 we find that U has to be open in $|K|$. But this would imply that $\langle s_3 \rangle \cap U \neq \emptyset$.

Let a be a vertex of K. Condition (4) can be reformulated as follows: for $\overline{\mathrm{St}(a)} = |x_1| \cup |x_2| \cup ... \cup |x_k|$ is $\overline{\mathrm{St}(a)} \setminus \mathrm{St}(a)$ the geometric realization of the relevant complex. We have to show that it is path connected. But it is homotopy equivalent to $\overline{\mathrm{St}(a)} \setminus \{a\}$. For a surface and a connected open subset U, the set $U \setminus \{x\}$ is also path connected for every $x \in U$.

Now let conversely $|K|$ be a surface complex. Then $|K|$ has for every point which is not a vertex a neighborhood which is homeomorphic to $\mathbb{R}^n$. For a vertex a we find by (4') that $St(a)$ is homeomorphic to an open k-gon. □

Theorem 1.21.16. *Every 1-dimensional manifold admits a triangulation by a simplicial complex K.*

Proof. We construct a sequence $G_n \subseteq M$ of compact subspaces which satisfies the following:

1. $\bigcup_{n=1}^{\infty} G_n = M$.
2. Each G_n is a finite graph, i.e. is triangulated by a finite, 1-dimensional cellular complex.
3. For each n, G_n is a subcomplex of G_{n+1}.
4. For each n there exists $m > n$ such that $G_n \subseteq \mathrm{Int} G_m$.

□

Theorem 1.21.17 (Radó)**.** *Every 2-dimensional manifold admits a triangulation by a 2-dimensional simplicial complex K.*

1.22 Classification of surfaces

The goal in this section is the give a full classification of surfaces, but lets start with a little warm-up:

Theorem 1.22.1. *A connected 1-manifold is homeomorphic to S^1 if it is compact and homeomorphic to $\mathbb{R}$ if it is not.*

Proof. By the triangulation theorem (Theorem 1) 1.21.16 and 1.21.13 each 1-dimensional manifold M is homeomorphic to $|K|$ for K a 1-dimensional simplicial complex in which each vertex lies in exactly two edges. We will show that K is isomorphic to one of the simplicial complexes K_n or K_∞ from Example 1.21.4 (3). This implies the claim since the realizations of these are homeomorphic to S^1 or $\mathbb{R}$ as we argued there.

Let v_0 be an arbitrary vertex in K. It is contained in two edges, so we can label the other vertices of these edges arbitrarily as v_{-1} and v_1. Now we let v_2 be the vertex of the edge containing v_1 other than $\{v_0, v_1\}$. Iterating this preocess we get a sequence of vertices $(, ...v_{-2}, v_{-1}, v_0, v_1, v_2, ...)$. By connnectedness of M all the vertices of K must be contained in this sequence, since otherwise the realization of this subcomplex would define a disjoint connected component of $|K|$. Now we distinguish two cases:

Case 1: If $v_n \neq v_k$ for any $n \neq k$ then clearly the sequence given an isomorphism from K_∞ to K.

Case 2: If $v_n = v_{n+m}$ for some $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$. We assume m is minimal (note that $m \leq 3$). Inductively we see that we have $v_k = v_{k+m}$ for all k. But then its clear that this complex is isomorphic to K_n.

□

The first step towards the classification of 2-manifolds is the following statement. Morally this says that we can cut the surface open along curves to be able to unfold it onto the plane.

Proposition 1.22.2. *Let F be a compact, connected surface. Then F can be written as the quotient of a regular $2n$-gon modulo linear identification of the 1-dimensional faces.*

Proof. Since F admits a triangulation by Radó's theorem we haven $F \cong |K|$ for a finite simplicial complex K. We first claim that we can number the 2-simplices as $D_1, D_2, ..., D_s$ such that for $i > 1$ the triangle D_i has an edge in common with one of the triangles $D_1, ..., D_{i-1}$. Assume this is not the case, then choose a maximal system $D_1, ..., D_k$ with this property (such systems exist). Then we find that every other 2-simplex has no face in common with the given ones, thus $|K|$ is disconnected. Contradiction.

From the list $D_1, ..., D_s$ we construct inductively a regular polygon. For $i > 1$ let T_i be an edge of D_i that is among the edges of $T_1, ..., T_{i-1}$. The surface F is a quotient of $\bigsqcup_{j=1}^s |D_j|$. For each edges T that is an edge of D_i and D_j the two subspaces of $|D_i|$ and $|D_j$ are identified. We will use that the composition of identification maps is again an identification, i.e. we can successively identify the edges. We want to show inductively the following: if we have carried out in $\bigsqcup_{j=1}^i |D_j|$ the identifications belonging to $T_2, T_3, ..., T_i$ then the resulting space $E(i)$ is homeomorphic to a regular $(i + 2)$-gon. The $i + 2$-edges of $E(i)$ are then the edges which have not yet been identified.

For $i = 1$ this is clear, since we just have a triangle. For the transition from i to $i + 1$ we have to identify a triangle along a common edge with the $i + 2$-gon $E(i)$ which clearly results in a $i + 3$-gon (use the next Lemma) . (Bild)

Finally we obtain an $s+2$-gon. Now note that the number s has to be even, since according to Theorem 1.21.15 the complex K has to be a surface complex, which means in particular that each edge has to be contained in precisely two of the triangles. But the s-triangles have $3s$ edges. Thus s has to be even. □

Lemma 1.22.3. *Let E_1 and E_2 be spaces which are homeomorphic to D^2. Let $A_1 \subseteq \partial E_1$, $A_2 \subseteq \partial E_2$ be subsets homeomorphic to $[0,1]$ and let $h : A_1 \to A_2$ be a homeomorphism. Then the glued space $E = E_1 \cup_h E_2$ is homeomorphic to D^2 as well.*

Proof. We use half circles $D^+ := \{(x,y) \in D^2 \mid y \geq 0\}$ and $D^- := \{(x,y) \in D^2 \mid x \leq 0\}$. Let $\varphi : [-1,1] \to A_1$ be a homeomorphisms. There is a homeomorphism

$$\Phi^+ : D^+ \to E_1$$

which extends φ. Now chose $\psi : [-1,1] \to A_2$ such that $h = \psi^{-1}\varphi$ and extend ψ to $\Phi^- : D^- \to E_2$. Then Φ^+ and Φ^- can be glued together to give a homeomorphism $D^2 \to E$. □

Lets now try to understand the combinatorics of the glueing the 2n-gon P. This is very similar to presentation complexes, but lets be explicit. Lets walk along the boundary of P starting at some point. We number the edges successively by $a_1, a_2, ...$ If we come to an edge which has to be identified with a_1 we label it with a_1^{-1} or a_1 depending on whether the direction has to be changed or not. Since every edge has to occur precisely twice, the word will consist of the letters $a_1, ..., a_n$ and each letter will occur exactly twice in the word.

Definition 1.22.4. We call a word w in letters $a_1, ..., a_n$ a surface word, if it has length $2n$ and each letter occurs precisely twice. For such a word w we get a surface F_w as the quotient of the polygon which is homeomorphic to the presentation complex of $\langle a_1, ..., a_n \mid w \rangle$. We call two surface words w and w' equivalent if the associated surfaces are homeomorphic.

Corollary 1.22.5. *Every surface is homeomorphic to a presentation complex described by a surface word.*

Example 1.22.6. Recall that the following surfaces are described by words as follows:

1. The sphere: aa^{-1}
2. The torus: $aba^{-1}b^{-1}$
3. The projective plane: aa
4. The Klein bottle: $abab^{-1}$
5. The genus g-surface: $a_1b_1a_1^{-1}b_1^{-1} \ldots a_gb_ga_g^{-1}b_g^{-1}$
6. The non-orientable genus g surface: $a_1a_1b_1b_1 ... a_gb_g$

Note that Σ_2^- has not the same surface word as the Klein bottle!

Now the task is to find modifications of the word that do not change the equvalence class, i.e. the homeomorphism type of the presentation complex. First it is clear that by our description as a quotient of a polygon that the surface does not depend where we start labeling the edges. This means that we have

$$b_1, ..., b_n \sim b_2, ..., b_n, b_1$$

We also can replace a symbol c in the word by c^{-1} (where we take the convention that $c^{-1^{-1}} = c$). Finally we can change the direction in which we run through the word, i.e.

$$b_1, ..., b_n \sim b_n^{-1}, ..., b_1^{-1}$$

Lemma 1.22.7. *Eliminating neighboured letters* $...cc^{-1}...$ *in a word leads to an equivalent word as long as the word has at least four letters.*

When identifying the edges of the polygon according to a word certain vertices get identified. As a example note that from the word $aba^{-1}b^{-1}$ we get one point in the quotient. In $aa^{-1}bb^{-1}cc^{-1}$ (the tetrahedron) we get 4 points.

Lemma 1.22.8. *Each surface word is equivalent to a word which identifies all vertices.*

Proof. Assume all neighboured letters cc^{-1} have been eliminnated. Then there are two neighboored points P and Q which do not get identified in the Quotient. □

We now call a word reduced if the reductions of the last two lemmas have been executed. We call a letter a in the word w of first type if it occurs twice as $...a...a...$ (or $...a^{-1}...a^{-1}...$) and of second type if it occures as $...a...a^{-1}...$ (or $...a^{-1}...a...$).

Lemma 1.22.9. *A word can be modified to an equivalent word such that letters of the first type are neighboured. Reduced words stay reduced after that operation. In particular* $K \cong \Sigma_2^-$ *(recall that the corresponding words are* $abab^{-1}$ *and* $xxbb$*).*

If there are no letters of the second type we can thus bring a word into the form $a_1a_1...a_na_n$.

Lemma 1.22.10. *Assume the operations before have been carries out. Assume a word has at least four letters. If there are letters of the second type then then there are at least two such letters and they are seperated, i.e. the word has the form*

$$...c...d...c^{-1}...d^{-1}...$$

1

Lemma 1.22.11. *Pairs of the second type can be made neighboured, i.e. of the form* $...aba^{-1}b^{-1}....$

If the word has no letters of the first kind it thus can be brought in the form $a_1b_1a_1^{-1}b_1^{-1}...a_nb_na_n^{-1}b_n^{-1}$.

Lemma 1.22.12. *The word* $...xxaba^{-1}b^{-1}...$ *is equivalent to the word* $...zzyyuu....$ *(These are* $\mathbb{R}P^2\#T^2$ *and* Σ_3^-*).*

In case that the word really has letters of the first and second kind we can always by rotations and changing directing bring ot to the form of the last lemma. This way we can eliminate all letters of the second kind. Altogether we have proven the following theorem:

Proposition 1.22.13 (Classification of surface words)**.** *Any surface word is equivalent to a word of the form:*

- aa^{-1}
- $a_1b_1a_1^{-1}b_1^{-1} \ldots a_gb_ga_g^{-1}b_g^{-1}$
- $a_1a_1a_2a_2 \ldots a_gb_g$

Chapter 2

Topology II, Summer term 2025

Organizational matters:

- Oral exams at the end of the semester. Allowance criteria:
  1. Participation in the Exercise classes
  2. Present a solution
- Literature:
  - Bredon: Topology and Geometry
  - Lück: Algebraische Topologie
  - Hatcher: Algebraic Topology
  - tom Dieck: Algebraic Topology
  - Weibel: Homological Algebraic
  - Boot, Tu: Differential forms in algebraic topology
  - Dold: Lecture Notes in Math
- This is the follow-up lecture of Topology I. We will assume everything that has been covered in Topology I. If we refer to specific results we will cite it as Theorem 1.2.4. This means Theorem 2.4 in Topology I. As a consequence we start the enumeration this term with 2 in front of all numbers.

2.1 Preface Cohomology

Last term: Homology groups: X topological space, functorial assigmment of groups $H_n(X, R)$ for R an abelian group (in practice $R = \mathbb{Z}, R = \mathbb{F}_p$ or $R = \mathbb{Q}$). Used to distinguish spaces, e.g. S^n and S^m can not be homotopy equivalently

$\rightsquigarrow$ invariance of dimension, invariance of domain, classification of surfaces etc.

Problem: Homology can not distinguish all spaces.

Example 2.1.1. Are $S^2 \vee S^4$ and $\mathbb{C}P^2$ homotopy equivalent? Compute homology:

$$\begin{aligned} H_*(S^2 \vee S^4, R) &= \begin{cases} R & \text{for } * = 0,2,4 \\ 0 & \text{else} \end{cases} \\ &= H_*(\mathbb{C}P^2, R) \end{aligned}$$

Example 2.1.2. Lens spaces $L_{p,q}$ have homology

$$H_*(L_{p,q}) = \begin{cases} \mathbb{Z} & \text{for } * = 0,3 \\ \mathbb{Z}/p & \text{for } * = 1 \\ 0 & \text{else} \end{cases}$$

Independent of q. When are $L_{p,q}$ and $L_{p,q'}$ homotopy equivalent?

Need other, better invariant. But $H_*(-)$ is 'easy' to compute, has good formal properties: Eilenberg-Steenrod axioms, Mayer-Viertoris-Sequence, Fundamental class for oriented manifolds,... We need an invariant that is also computable, but more refined
$\rightsquigarrow$ cohomology $H^*(X)$.

Plan for the lecture :

1. First define abelian groups $H^*(X, R)$ by 'dualizing' the definition of the groups $H_*(X, R)$. This leads to formal properties similar to Homology (Eilenberg-Steenrod-axioms, Mayer-Vietoris Sequence...)

2. Provide tools to compute $H^*(X)$ from $H_*(X)$ and vice versa. In particular we will see that the groups $H^*(X)$ can not distinguish more spaces than $H_*(X)$ (Universal coefficient theorems).

3. Equip the groups $H^*(X)$ with a graded ring structure, i.e. multiplication maps

$$H^p(X) \times H^q(X) \to H^{p+q}(X)$$

As a ring $H^*(X)$ is a much more powerful invariant than $H_*(X)$. For example we will see that

$$H^*(S^2 \vee S^4) = \frac{\mathbb{Z}[x,y]}{(x^2, y^2, xy)} \qquad |x| = 2, |y| = 4$$

but

$$H^*(\mathbb{C}P^2) = \frac{\mathbb{Z}[t]}{(t^3)} \qquad |t| = 2$$

Since these rings are not isomorphic, the spaces $\mathbb{C}P^2$ and $S^2 \vee S^4$ can not be homotopy equivalent.

4. Discuss the Künneth theorem, which allows to compute $H^*(X \times Y)$ in terms of $H^*(X)$ and $H^*(Y)$. We will also see a similar result for $H_*(X \times Y)$.

5. Last term we saw (or at least stated) that for a compact, oriented n-dimensional manifold M we have an isomorphisms $H_n(M) \cong \mathbb{Z}$. This is a special case of Poincare duality which states that
$$H_k(M) \cong H^{n-k}(M)$$
for all k. In order to prove this result we will introduce locally finite homology and cohomology with compact support.

6. Poincare Duality leads to the 'intersection form'
$$\lambda : H_n(M) \times H_n(M) \to \mathbb{Z}$$
for an oriented $2n$-dimensional manifold M and the 'linking form'
$$lk : TH_n(M) \times TH_n(M) \to \mathbb{Q}/\mathbb{Z}$$
for an oriented $(2n+1)$-dimensional manifold M (where TH is the torsion submodule). For the Lens space $L_{p,q}$ we will find that the bilinear form
$$lk : \mathbb{Z}/p \times \mathbb{Z}/p \to \mathbb{Q}/\mathbb{Z}$$
is given by $lk(1,1) = q/p$. We conclude that $L_{p,q}$ is homotopy equivalent to $L_{p,q'}$ precisely if
$$q \cdot q' = \pm n^2 \mod p$$

7. We will prove the de Rham theorem, which allows to express $H^*(X, \mathbb{R})$ in terms of differential forms for smooth manifolds M. This can often be used to describe classes in $H^*(M, \mathbb{R})$ very explicitly.

8. Depending on the time we might cover basic homotopy theory.

2.2 Singular Cohomology

Recall that for a space X the singular chain complex
$$C_*(X) = \quad ... \to C_2(X) \xrightarrow{\partial} C_1(X) \to C_0(X)$$
Defines a functor
$$C_* : \mathrm{Top} \to \mathrm{Ch}$$
We want to 'dualize' this complex.

Definition 2.2.1. The n-th singular cochain group of a pair (X, A) with values in an abelian group R is the abelian group
$$C^n(X, A; R) := \mathrm{Hom}_{\mathrm{Ab}}(C_n(X, A), R)$$
the differential is the homomorphism $d^n : C^n(X, A; R) \to C^{n+1}(X, A; R)$ given by $\mathrm{Hom}_{\mathrm{Ab}}(\partial, R)$, i.e. explicitily
$$f \in \mathrm{Hom}_{\mathrm{Ab}}(C_n(X, A), R) \mapsto f \circ \partial_n \in \mathrm{Hom}_{\mathrm{Ab}}(C_{n+1}(X, A), R)$$
We write $C^n(X, A)$ for $C^n(X, A; \mathbb{Z})$ and $C^n(X; R)$ for $C^n(X, A; R)$.

Note that we have $\partial \circ \partial = 0$ since the corresponding relation holds for ∂_*. Also note that the group $C^n(X)$ can be explicitly described using the fact that $C_n(X)$ is a free abelian group: a cochain $c \in C^n(X)$ can be described as an assignment of an integer $c(\sigma)$ to every simplex $\sigma : \Delta^n \to X$ without any further constraints. The group structure is pointwise addition.

Definition 2.2.2. A cochain complexes is a graded abelian group C^* together with a sequence of morphisms $d : C^i \to C^{i+1}$ such that $d^2 = 0$. A morphism of cochain complexes is a sequence of morphisms $f_i : C^i \to D^i$ such that $df = fd$ i.e. the ladder diagram commutes. The category of cochain complexes is denoted CoCh.

Proposition 2.2.3. *For every space X the cochain groups form a cochain complex $C^*(X)$. For every morphism of topological spaces $f : X \to Y$ there is an induced morphism of cochain complexes $C^*(f) : C^*(Y) \to C^*(X)$ defined as postcomposition with $C_*(f)$. We have*

$$C^*(f \circ g) = C^*(g) \circ C^*(g) \qquad \textit{and} \qquad \mathbb{C}^*(\mathrm{id}) = \mathrm{id}$$

Remark 2.2.4.

- The last proposition can be restated in saying that the assigment $X \mapsto C^*(X)$ is a *contravariant functor* from Top to CoCh.

  To define contravariant functors formally we define for every category C the opposite category C^{op} as follows: objects of C^{op} are objects of C. Morphisms sets are given by $\mathrm{Hom}_{C^{op}}(a, b) = \mathrm{Hom}_C(b, a)$. Identites are the same as in C and composition is the composition 'reversed'.

  A contravariant functor from C to D is then by definition a functor $C^{op} \to D$.

- The construction which sends a chain complex C_* to its dual C^* with $C^n = \mathrm{Hom}_{\mathrm{Ab}}(C_n, \mathbb{Z})$ is a contravariant functor from $\mathrm{Ch} \to \mathrm{CoCh}$. Then the functor functor $\mathrm{Top}^{op} \to \mathrm{CoCh}$ sending X to $C^n(X)$ can be written as the composition of the functor $X \mapsto C_n(X)$ and this functor.

Definition 2.2.5. For every cochain complex C^* the cohomology groups are defined as

$$H^n(C^*) := \frac{\ker(d : C^n \to C^{n+1})}{\mathrm{im}\ (d : C^{n-1} \to C^n)}$$

For a pair of spaces the cohomology groups (X, A) are defined as

$$H^n(X, A; R) := H^n(C^*(X, A; R))$$

For every morphism $f : (X, A) \to (Y, B)$ the induced morphism on cohomology groups is denoted by

$$f^* : H^n(Y, B) \to H^n(X, A)$$

All important Properties and results from chain complexes also hold for cochain complexes, i.e.

- Chain homotopic morphisms induce the same morphism in cohomology
- Short exakt sequences of cochain complexes give rise to long exact sequences in cohomology.

Lemma 2.2.6. *1. For every pair of spaces (X, A) the sequence*

$$0 \to C^*(X, A) \to C^*(X) \to C^*(A) \to 0$$

is short exact (like the corresponding sequence in homology). In particular we get a long exact sequence in cohomology and connecting homomorphisms of the form

$$\ldots \xrightarrow{\delta_*} H^n(X, A) \xrightarrow{p^*} H^n(X) \xrightarrow{i^*} H^n(A) \xrightarrow{\delta_*} H^{n+1}(X, A) \xrightarrow{p^*} H^{n+1}(X) \xrightarrow{i^*} H^{n+1}(A) \xrightarrow{\delta_*} \ldots$$

2. For homotpic morphisms $f, g : X \to Y$ the induced morphisms $C^*(Y) \to C^*(X)$ are chain homotopic.

Proof. Lets analyze the sequence:

$$C^*(X, A) = \mathrm{Hom}(C_*(X, A), \mathbb{Z}) = \mathrm{Hom}(C_*(X)/C_*(A), \mathbb{Z}) = \{f : \Delta_n(X) \to \mathbb{Z} \mid f|_{\Delta_n(A)} = 0\}$$

Then we have clearly that

$$C^*(X) = \{f : \Delta_n(X) \to \mathbb{Z}\} \qquad \text{and} \qquad C^*(A) = \{f : \Delta_n(A) \to \mathbb{Z}\}$$

From this description it is easy to see that the sequence is exact.

For the second part note that a chain homotopy between chain complexes A, B is a sequence $D_n : A_n \to B_{n+1}$ such that $\partial D + D\partial = f - g$. Applying the dualizing we get morphisms

$$D_n^* : B^{n+1} \to A^n$$

satisfiying $D^*\delta + \delta D^* = f^* - g^*$. We have thus shown that the dual of a chain homotopy is also a chain homotopy. Now for two homotopic morphisms of topological spaces $f, g : X \to Y$ (or more generally of pairs of spaces) we have shown in Theorem 1.15.7 that the induced morphismsm $C_*(f)$ and $C_*(g)$ are chain homotopic. Thus it follows by the above argument that also $C^*(f)$ and $C^*(g)$ are chain homotopic. □

Definition 2.2.7. A *generalized cohomology theory* consists of a (contravariant) functor

$$E^* : (\mathrm{Top}^2)^{op} \to \mathrm{grAb}$$

together with morphisms

$$\delta : E^n(A) \to E^{n+1}(X, A) \qquad \qquad \big(\text{here we set } E^n(Y) := E^n(Y, \emptyset)\big)$$

which are natural in the pair (A, X) , called connecting homomorphisms, which satisfies the following axioms:

1. (Homotopy invariance) If $f \simeq g : (X, A) \to (Y, B)$ then $f^* = g^* : E^n(Y, B) \to E^n(X, A)$
2. (Long exact sequence) For a pair $i : A \subseteq X$ (with morphism $j : (X, \emptyset) \to (X, A)$ the sequence
$$\ldots \xrightarrow{\delta} E^n(X, A) \to E^n(X) \to E^n(A) \xrightarrow{\delta} E_{n+1}(X, A) \to \ldots$$
is long exact.

3. (Excision) Given (X, A) and $U \subseteq A$ with $\overline{A} \subseteq \mathrm{int}(A)$ then the morphisms
$$E^n(X, A) \to E^n(X \setminus U, A \setminus U)$$
induced by the inclusion $(X \setminus U, A \setminus U) \to (X, A)$ is an isomorphism.

4. (Additivity) For a topological space $X \cong \bigsqcup_{\alpha \in A} X_\alpha$ the canonical morphism
$$E^*(X) \to \prod_{\alpha \in A} E^*(X_\alpha)$$
induced by the inclusions $X_\alpha \to X$ is an isomorphism.

A generalized cohomology theory E is called *ordinary* if it in addition satisfies the dimension axiom
$$E^n(*) = 0 \text{ for all } n \neq 0.$$

Theorem 2.2.8. *The singular cohomology groups $H^*(X, A; R)$ for an abelian group R together with the functoriality and the connecting homomorphisms $\delta : H^n(A) \to H^{n+1}(X, A)$ form a cohomology theory.*

Conversely every cohomology theory E which satisifes the dimension axiom has to agree with singular on CW complexes in the sense that there is a isomophism
$$E^*(X, A) \to H^*(X, A; R)$$
for $R := H^*(pt)$ and for every CW complex X.

Proof. We have to check the axioms. First the naturality of δ and the functoriality are clear by definition and the functoriality of the construction $X \mapsto C^*(X)$. Homotopy invariance follows immediately from part two of Lemma 2.2.6. The long exact sequence follows from the first part of that Lemma. The proof of Excision is a bit more tricky. For this we recall the proof of excision in homology. The key was to construct natural subdivision operators (1.16.3)
$$\Upsilon^k : C_*(X) \to C_*(X)$$
with the property that Υ^k is chain homotopic to the identity functor and with the property that the Υ^k decrese the 'size' of simplices in the following sense: For every open covering U_i of X and every element $c \in C_n(X)$ there exists a natural number k such that $\Upsilon^k(c)$ is $\mathcal{U}$-small, i.e. that every summand of $\Upsilon^k(c)$ is contained entirely in one of the $U_i's$.

By dualizing the Υ^k we get natural subdivision operators
$$\Upsilon^k : C^*(X) \to C^*(X)$$
which are chain homotopic to the identity as well. Now we proceed exactly as before, we proove that the canonical morphism
$$H^*(X, A) \to H^*_{\mathcal{U}}(X, A)$$
is an isomorphism (see Theorem 1.16.7) and then exactly as for homology it follows that excision holds (Theorem 1.16.8).

Additivity finally follows since we have $C_*(X) = \bigoplus C_*(X_\alpha)$ and thus $C^*(X) = \prod C^*(X_\alpha)$ which is easily seen to imply the claim since taking cohomology of a cochain complex commutes with products.

The uniqueness follows then from the descriptions in terms of cellular homology that we will discuss now and the same arguments as for homology, see Theorem 1.19.10. □

Remark 2.2.9.
- The explicit excision argument is in fact superflous. We will see soon independently, that if a morphism $f : (X, A) \to (Y, B)$ induces an isomorphism in homology, i.e. $f_* : H_*(X, A) \to H_*(Y, B)$ is an isomorphism, then also the morphism in cohomology $f^* : H^*(Y, B) \to H^*(X, A)$ is an isomorphism. This then immediately implies excision for cohomology, once we know it for homology.
- One could also attempt to define cohomology just as the dual of Homology, i.e. as

$$H^*_{\text{naive}}(X, A) = \text{Hom}(H_*(X, A), \mathbb{Z})$$

and not dualize on the chain level. It turns out however, that this would <u>not</u> give a cohomology theory, as the long exact sequence axiom would fail. We will investigate the precise relation between cohomology $H^*(X, A)$ and the dual of homology $H^*_{\text{naive}}(X, A)$ soon.

Definition 2.2.10. Let H^* be singular cohomology and X a CW complex. Define the cellular cohomology chain complex $C^*_{cell}(X; R)$ as the dual of the cellular chain complex $C^{cell}_*(X, R)$, i.e. $C^n(X; R) = \text{Hom}(C^{cell}_*(X), R)$. Cellular cohomology is then defined as the cohomology of the cellular chain complex.

If H^* is an arbitrary cohomology theory, then define the cellular chain complex as $C^*_{cell}(X) := H^n(X_n, X_{n-1})$ (cf. as 1.12.4). The differential comes from the triple long exact sequence of (X_{n+1}, X_n, X_{n-1}), more precisely is given by the composite:

$$d : H^n(X_n, X_{n-1}) \to H^n(X_n) \xrightarrow{\delta} H^{n+1}(X_{n+1}, X_n)\ .$$

Theorem 2.2.11. *For every cohomology theory and every CW complex there is a canonical ismorphism*

$$H^n(X; R) \to H^n_{cell}(X; R)$$

which is natural in X (by cellular morphisms) as in Theorem 1.12.6.

Proof. The proof is precisely the same as the proof of Theorem 1.12.6. □

Examples 2.2.12. Let us compute cohomology groups of some spaces using the 'standard' cell structures.

1. For $X = S^n$ the sphere we had

$$C^{cell}_*(S^n) = (... \to 0 \to \mathbb{Z} \to ... \to \mathbb{Z})$$

thus we have

$$C^*_{cell}(S^n; R) = (R \to 0 \to ... \to R \to 0 \to ...)$$

Thus

$$H^*(S^n; R) = \begin{cases} R & \text{for } * = 0, n \\ 0 & \text{else} \end{cases}$$

2. Let $X = \mathbb{C}P^n$ be complex projective space. We have

$$C_*^{cell}(\mathbb{C}P^n) = (... \to 0 \to 0 \to \mathbb{Z} \to ... \to 0 \to \mathbb{Z} \to 0 \to \mathbb{Z})$$

Thus we find that

$$C^*_{cell}(\mathbb{C}P^n; R) = (R \to 0 \to R \to 0 \to R \to ... \to R \to 0 \to 0 \to ...)$$

and therefore

$$H^*(\mathbb{C}P^n; R) = \begin{cases} R & \text{for } * = 0, 2, 4, 6, 2n \\ 0 & \text{else} \end{cases}$$

3. For Lens spaces $L_{p,q}$ we have

$$C_*^{cell}(L_{p,q}) = (... \to 0 \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \xrightarrow{0} \mathbb{Z})$$

Thus we find that

$$C^*_{cell}(L_{p,q}; R) = (R \xrightarrow{0} R \xrightarrow{\cdot p} R \xrightarrow{0} R \to 0 \to ...)$$

We get

$$H^*(Ł_{p,q}; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } * = 0, 3 \\ \mathbb{Z}/p\mathbb{Z} & \text{for } * = 2 \\ 0 & \text{else} \end{cases}$$

4. For the real projective space $\mathbb{R}P^n$ we have the cellular chain complex

$$.... \to \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

Thus we find that the cellular cochain complex is given by

$$R \xrightarrow{0} R \xrightarrow{2} R \xrightarrow{0} R \xrightarrow{2} R \xrightarrow{0} ...$$

which implies

$$H^*(\mathbb{R}P^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } * = 0 \\ \mathbb{Z}/2 & \text{for } * = 2, 4, ..., * \leq n \\ \mathbb{Z} & \text{for } * = n \text{ odd} \\ 0 & \text{else} \end{cases}$$

2.3 Homology versus cohomology

By definition the cochain complex of a space arises from the chain complex by ‘dualizing’. This spots the question whether the cohomology can be computed from homology by dualizing. For example for S^n and $\mathbb{C}P^n$ the respective homology and cohomology groups turn out to by isomorphic as abstract groups. This can not be expected in general (already because they have different variances). A better guess is that $H^n(X; R)$ and $\mathrm{Hom}(H_n(X; \mathbb{Z}), R)$ are isomorphic. Indeed there is a canonical morphism: let C be a chain complex and R be an abelian group. We define a natural morphism

$$\Phi : H^n(\mathrm{Hom}(C, R)) \to \mathrm{Hom}(H_nC, R)$$

by

$$[f : C_n \to R] \mapsto ([c] \mapsto f(c))$$

where $\mathrm{Hom}(C, R)$ denotes the cochain complex which in degree n is given by $\mathrm{Hom}(C_n, R)$ and with the induced differential given by the 'dual' of the differential on C.

Proposition 2.3.1. *The morphism Φ is well defined, additive and natural in C. If C is a levelwise free $\mathbb{Z}$-module, then it admits a section (which is not natural), in particular it is surjective*

Proof. To show that the morphism is well defined let $[c] = [d]$. Then there is x with $\partial x = c - d$. Thus we find that

$$f(c) = f(d + \partial x) = f(d) + (\delta f)(x) = f(d)$$

Thus for every cycle f we get an induced morphism $\mathrm{Hom}(H_nC, R)$. Now let g be another cycle with $[f] = [g]$. Then there is $h : C_{n-1}(X) \to R$ with $dh = f - g$. We find that

$$f(c) = \delta h(c) + g(c) = h(\partial c) + g(c) = g(c).$$

Now we want to construct an additive section $s : \mathrm{Hom}(H_nC, R) \to H^n(\mathrm{Hom}(C, R))$ of Φ. To do this we first choose a retract r of the inclusion

$$Z_nC \to C_n$$

which is possible since the quotient is isomorphic to $B_{n-1}C$ and thus free. Now we define the section s for an element $\gamma \in \mathrm{Hom}(H_nC, R)$ the class $s(\gamma)$ as the class of the composition

$$C_n \xrightarrow{r} Z_n \xrightarrow{\text{quot}} H_nC \xrightarrow{\gamma} R$$

Then we find that in fact $s(\gamma)$ is a cycle, since

$$(ds(\gamma))(x) = s(\gamma)(\partial x) = \gamma(r\partial x) = \gamma(\partial x) = 0$$

We also have that s is a section since

$$\Phi(s(\gamma))(c) = s(\gamma)(c) = \gamma(rc) = \gamma(c).$$

$\square$

We conclude that the morphism

$$\Phi : H^n(X) \to \mathrm{Hom}(H_n(X), \mathbb{Z})$$

is always (split) surjective. Note that the split is not canonical!

Corollary 2.3.2. *For a space X there is a canonical, split surjective homomorphism*

$$H^n(X; R) \to \mathrm{Hom}(H_n(X, \mathbb{Z}), R)$$

Example 2.3.3.

1. One could hope that the homomorphism $H^n(X; R) \to \mathrm{Hom}(H_n(X), R)$ is not only surjective, but an isomorphism. This however turns out to be too optimistic, as the example of Lens spaces shows. We have that $H_2(L_{p,q}) = 0$, hence $\mathrm{Hom}(H_2(X), \mathbb{Z}) = 0$ but $H^2(L_{p,q}) = \mathbb{Z}/p \neq 0$. Note that this is because of the existence of torsion...

The question is whether we can compute the kernel of the morphism Φ. This turns out to be possible in terms of the Homology of C. Thus we can in principle describe the cohomology purely in terms of homology. This description however need some tools from homological algebra which we want to introduce now.

2.4 The functors Tor and Ext

Definition 2.4.1. Let R be a commutative, unital ring. An R-module M is called projective, if for each surjective R-module morphism $p : N \to M$ there exists a R-linear section $s : M \to N$, i.e. $ps = \mathrm{id}_M$.

Example 2.4.2. Over $\mathbb{Z}$ the modules $\mathbb{Z}$, $\mathbb{Z}^n$ and $\bigoplus \mathbb{Z}$ are projective. More generally direct sums of projective modules are projective over every ring R. In particular free R-modules are projective.

The group $\mathbb{Z}/n$ is not projective over $\mathbb{Z}$, since $\mathbb{Z} \to \mathbb{Z}/n$ does not admit a section. Also $\mathbb{Q}$ as a $\mathbb{Z}$-module is not projective. To see this choose a surjective homomorphisms $\bigoplus \mathbb{Z} \to \mathbb{Q}$. This can not admit a section, since every homomorphism $s : \mathbb{Q} \to \bigoplus \mathbb{Z}$ is zero. This is because in $\mathbb{Q}$ all elements are infinitely divisible.

Proposition 2.4.3. *The following are equivalent*

1. *M is projective*
2. *M is a summand of a free module, i.e. there exists N such that $M \oplus N \cong \bigoplus_{i\in I} R$*
3. *For every surjective homomorphism $\varphi : N \to N'$ of R-modules and every homomorphism $f : M \to N'$ there exists $g : M \to N$ such that $\varphi \circ g = f$.*

$$\begin{tikzcd} & N \arrow[d] \\ M \arrow[ur, dashed] \arrow[r, "f"] & N' \end{tikzcd}$$

4. *The functor*
$$\mathrm{Hom}_R(M, -) : \mathrm{Mod}_R \to \mathrm{Ab}$$
sends short exact sequences to short exact sequences

Proof. Only in lecture. □

Example 2.4.4. Over $R = \mathbb{Z}/6$ the modules $\mathbb{Z}/2$ and $\mathbb{Z}/3$ are projective, since $\mathbb{Z}/2 \oplus \mathbb{Z}/3 \cong \mathbb{Z}/6$. But neither of them can be free.

For PIDs (like $\mathbb{Z}$) it is true that submodules of free modules are again free. This means that there are no projective modules that are not free.

Remark 2.4.5. Note that in the proof of Proposition 2.4.3 we have shown more generally that for every short exact sequence $0 \to A \to B \to C \to 0$ and every not necessarily projective module M the induced sequence

$$0 \to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C)$$

is exact. But the right hand map is in general not surjective! We say that the functor $\mathrm{Hom}_R(M, -)$ is left exact. Dually it is easy to see that the sequence

$$A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$$

is exact, i.e. the functor $- \otimes_R M$ is right exact. If M is projective then it also follows that this second sequence is exact (which can be easily seen using description 2 of Proposition 2.4.3 above). But the converse is not true, in fact if this second sequence is exact for each short exact sequence $A \to B \to C$ then M is called flat.

The Tor and Ext groups that we are going to study now measure the deviation of these sequence to be short exact.

Definition 2.4.6. A chain complex C_* is called non-negative if $C_n = 0$ for $n < 0$. A non-negative chain complex is called projective if all C_n are projective. A (projective) resolution of an R-module M consists of a non-negative (projective) chain complex P_* such that $H_n(P_*) = 0$ for $n \geq 1$ together with an isomorphism $\varphi : H_0(P_*) \to M$.

Remark 2.4.7. A projective resolution P_* is the same as an exact sequence

$$... \to P_2 \to P_1 \to P_0 \xrightarrow{\varphi'} M$$

because for such a sequence φ' factors over $H_0(P_*)$. We can also see a projective resolution as a chain complex P_* together with a chain map $P_* \to M[0]$ where $M[0]$ is the chain complexes

$$... \to 0 \to M \to 0 \to 0 \to ...$$

with M concentrated in degree 0. We will frequently use these different descriptions.

Note that every module M admits a projective resolution. To see this use the following construction: choose a free module P_0 with a surjective morphism $d : P_0 \to M$. We can e.g. take $P_0 = \bigoplus_{m \in M} R$. Now choose P_1 also free together with a homomorphism $d : P_1 \to P_0$ with image the kernel of d. The can be iterated inductively to get an exact chain complex

$$... \to P_2 \to P_1 \to P_0 \to M$$

for which all P_i are free, hence projective. Then the chain complexs P_i with the induced isomorphism $\varphi : H_0(P_*) \to M$.

Theorem 2.4.8 (Fundamental lemma of homological algebra)**.** *Let (P_*, φ) be a projective resolution of M and (Q_*, ψ) be any (not necessarily projective) resolution of N.*

- *For every homomorphism $f : M \to N$ there is a chain morphism $P_* \to Q_*$ which induces f on H_0, i.e. f is equal to $M \cong H_0(P_*) \to H_0(Q_*) \cong N$.*
- *Any two chain maps $P_* \to Q_*$ which induce the same homomorphism on H_0 are chain homotopic.*

Another way of stating the result is that under the given assumptions the canonical map $[P_*, Q_*] \to \mathrm{Hom}_R(M, N)$ is a bijection, where $[-, -]$ denotes chain homotopy classes of maps.

Proof. Only in lecture. □

Corollary 2.4.9. *Any two projective resolutions of an R-module M are chain homotopy equivalent.*

Proof. Let P_* and Q_* be two resolutions. We lift the morphism $\mathrm{id} : M \to M$ to a morphism of chain complexes $f : P_* \to Q_*$ and to a morphism $g : Q_* \to P_*$. Then we have that fg is a morphism $P_* \to P_*$ which induces the identity on H_0. Thus it has to be chain homotopic to the identity. The same is true for gf. □

Another way of stating (and proving) the last result is to say that the functor

$$H_0 : \mathrm{Ho}(\mathrm{Ch}_R)^{ppa} \to \mathrm{Mod}_R$$

is an equivalence of subcategories, where $\mathrm{Ho}(\mathrm{Ch}_R)^{ppa} \subseteq \mathrm{Ho}(\mathrm{Ch}_R)$ is the full subcategory consisting of chain complexes that are positive, projective and have vanishing positive homology.

Definition 2.4.10. Let M and N be two R-modules and P_* a projective resolution of M. Define the R-module

$$\mathrm{Tor}_n^R(M,N) := H_n(P_* \otimes_R N)$$

for every natural number n.

Proposition 2.4.11. *1. The groups Tor_n^R are up to (canonical) isomorphism independent of the choice of projective resolution of M.*

2. The assigment $(M,N) \to \mathrm{Tor}_*^R$ can be extended to a functor

$$\mathrm{Tor}_*^R : \mathrm{Mod}_R \times \mathrm{Mod}_R \to \mathrm{grMod}_R$$

3. There is a canonical isomorphism $\mathrm{Tor}_0^R(M,N) \cong M \otimes_R N$.

Proof. 1: Any two resolutions P_* and Q_* are chain homotopy equivalent. Thus also $P_* \otimes M$ and $Q_* \otimes M$ are chain homotopy equivalent and thus have the same homology. The chain homotopy equivalence is itself unique up to chain homotopy which leads to the fact that everything is canonical.

2: There is an essentially unique resolution functor $\mathrm{Mod}_R \to \mathrm{Ho}(\mathrm{Ch}_R)$ which sends every M to some projective resolution. Thus we consider the Tor-functor as a composite

$$\mathrm{Mod}_R \times \mathrm{Mod}_R \to \mathrm{Ho}(\mathrm{Ch}_R) \times \mathrm{Mod}_R \xrightarrow{\otimes} \mathrm{Ho}(\mathrm{Ch}_R) \xrightarrow{H_*} \mathrm{grMod}_R.$$

3: We observe that we have that Tor_0 is given by the 0-th homology of the chain complex

$$... \to P_2 \otimes N \to P_1 \otimes N \to P_0 \otimes N$$

thus by the cokernel of the map

$$P_1 \otimes N \to P_0 \otimes N$$

But tensoring interchanges with cokernels, thus we have that

$$\mathrm{coker}(P_1 \otimes N \to P_0 \otimes N) = \mathrm{coker}(P_1 \to P_0) \otimes N$$

which implies the claim since $\mathrm{coker}(P_1 \to P_0) = M$. □

Example 2.4.12. Let R be a PID. Then we have that $\mathrm{Tor}_n^R(M,N) = 0$ for each $n \geq 2$.

Let M be an R-module. Then we choose a free module F_0 mapping surjectively to M. Then take the kernel F_1 of $F_0 \to M$. It is again free. Thus we find that the following is a free resolution:

$$... \to 0 \to 0 \to F_1 \to F_0 \to M$$

Thus we compute that

$$\mathrm{Tor}_n^R(M,N) = H_n(F_* \otimes M) = (...0 \to F_1 \otimes_R M \to F_0 \otimes_R M) = 0$$

for $n \geq 2$. Note that the key is that every module has a projective resolution of lenght two. This can even happen, if the ring R is not a PID. In that case we say that R has projective dimension 1.

If R is a field then we have $\mathrm{Tor}_n^R(M,N) = 0$ for all $n \geq 1$ since M is itself projective.

For $R = \mathbb{Z}$ we simply write $\mathrm{Tor}(M,N)$ for $\mathrm{Tor}_1^{\mathbb{Z}}(M,N)$. This is the most important case.

Example 2.4.13. We find that

$$\mathrm{Tor}(M,\mathbb{Z}) = 0 \qquad \mathrm{Tor}(\mathbb{Z},N) = 0$$

and

$$\mathrm{Tor}(\mathbb{Z}/n,\mathbb{Z}/m) \cong \mathbb{Z}/\mathrm{gcd}(m,n)$$

and

$$\mathrm{Tor}(M,\mathbb{Q}) = 0.$$

Proof. Consider a projective resolution P_* of M. Then we have that $P_* \otimes \mathbb{Z} \cong P_*$. Thus we get that $\mathrm{Tor}(M,N) = H_1(P_*) = 0$. For $M = \mathbb{Z}$ we have that the chain complex with $\mathbb{Z}$ in degree 0 is a resolution. Thus the result follows.

For $M = \mathbb{Z}/n$ we choose the resolution $... \to 0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n$. Then we have that

$$P_* \otimes \mathbb{Z}/m = (... \to 0 \to \mathbb{Z}/m \xrightarrow{\cdot n} \mathbb{Z}/m)$$

thus $\mathrm{Tor}(\mathbb{Z}/n,\mathbb{Z}/m) \cong \ker(\mathbb{Z}/m \xrightarrow{\cdot n} \mathbb{Z}/m) = \mathbb{Z}/gcd(m,n)$ The last example will be done in the exercises. □

Lemma 2.4.14. *The functor* $\mathrm{Tor}_n^R(-,-)$ *preserves direct sums in both arguments, i.e. the canonical morphisms*

$$\bigoplus_{i,j} \mathrm{Tor}_n^R(A_i,B_j) \to \mathrm{Tor}_n^R\left(\bigoplus A_i, \bigoplus B_j\right)$$

are isomorphisms.

Proof. Clear since $\otimes$ preserves direct sums in both arguments and since we can choose projective resoltuions of a direct sum as the sum of resolutions. □

Remark 2.4.15. Preserving direct sums is a property. For every functor $F : \mathrm{Ab} \to \mathrm{Ab}$ there is a canonical morphism $F(A) \oplus F(B) \to F(A \oplus B)$ obtained using the universal property of the coproduct. This means that it is an isomorphism.

Theorem 2.4.16. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules. Then there are a canonical induced long exact sequence*

$$\ldots \to \mathrm{Tor}_1^R(B,N) \to \mathrm{Tor}_1^R(C,N) \to \mathrm{Tor}_0^R(A,N) \to \mathrm{Tor}_0^R(B,N) \to \mathrm{Tor}_0^R(C,N) \to 0$$

and

$$\ldots \to \mathrm{Tor}_1^R(M,B) \to \mathrm{Tor}_1^R(M,C) \to \mathrm{Tor}_0^R(M,A) \to \mathrm{Tor}_0^R(M,B) \to \mathrm{Tor}_0^R(M,C) \to 0$$

This sequences are natural in M, N and the short exact sequence.

Proof. For the second sequence pick a projective resolution P_* of M and observe that the resulting sequence

$$0 \to P_* \otimes A \to P_* \otimes B \to P_* \otimes C \to 0$$

is short exact (since P_i are projective, thus flat). Thus we get an induced long exact sequence in homology.

For the first long exact sequence we pick projective resoltuons P_* of A and Q_* of C. The claim is that there is a projective resolution $\widetilde{P_* \oplus Q_*}$ of B such that we have a short exact sequence

$$0 \to P_* \to \widetilde{P_* \oplus Q_*} \to Q_* \to 0$$

extending the short exact sequence $0 \to A \to B \to C \to 0$. More precisely we want $\widetilde{P_* \oplus Q_*}_n = P_n \oplus Q_n$. Thus we have to find a commutative diagram

$$\begin{tikzcd}
P_1 \arrow[r] \arrow[d] & P_1 \oplus Q_1 \arrow[r] \arrow[d] & Q_1 \arrow[d] \\
P_0 \arrow[r] \arrow[d] & P_0 \oplus Q_0 \arrow[r] \arrow[d] & Q_0 \arrow[d] \\
A \arrow[r] & B \arrow[r] & C
\end{tikzcd}$$

This is easy by the universal properties of coproduct and the fact that the Q_i are projective. Then the fact that $\widetilde{P_* \oplus Q_*}$ is a resolution follows from the long exact sequence. □

Remark 2.4.17. Recall that an R-module M is called flat if the functor $- \otimes_R M$ is exact. Using the last Theorem this is easily seen to be equivalent to $\mathrm{Tor}_n^R(M,N) = 0$ for each R-module N and each $n \geq 1$. The last theorem makes $\mathrm{Tor}_*^R(-,N)$ and $\mathrm{Tor}_*^R(M,-)$ into "δ-functors".

The next proposition we want to remark, but do not want to proof. It is more or less straightforward, but we do not want to develop the necessary tools here.

Theorem 2.4.18. *Let M and N be R-modules. There is a canonical isomorphism* $\mathrm{Tor}_n^R(M,N) \cong \mathrm{Tor}_n^R(N,M)$. *In particular the Tor-groups can be computed by resolving the N-variable as well.*

Recall that for a chain complex C_* and and R-module M the cochain complex $\mathrm{Hom}_R(C_*, M)$ is defined as the cochain complex with $\mathrm{Hom}_R(C_*, M)^n = \mathrm{Hom}_R(C_n, M)$ and the induced differential.

Definition 2.4.19. Let M and N be R-modules. Let P_* be a projective resolution of M. Then define

$$\mathrm{Ext}^n_R(M,N) := H^n(\mathrm{Hom}_R(P_*,N))$$

The R-modules Ext have very similar properties to the R-modules Tor.

Proposition 2.4.20 (Properties of Ext)**.** *1. The groups Ext^R_n are up to isomorphism independent of the choice of projective resolution of M.*

2. The assigment $(M,N) \to \mathrm{Ext}^R_*$ can be extended to a functor

$$\mathrm{Ext}^R_* : \mathrm{Mod}^{op}_R \times \mathrm{Mod}_R \to \mathrm{grMod}_R$$

3. There is a canonical isomorphism $\mathrm{Ext}^R_0(M,N) \cong \mathrm{Hom}_R(M,N)$.

4. For a PID R we have that $\mathrm{Ext}^i_R(M,N) = 0$ for $i \geq 2$. We write $\mathrm{Ext}(M,N) := \mathrm{Ext}^1_{\mathbb{Z}}(M,N)$. For a field R we have that $\mathrm{Ext}^i_R(M,N) = 0$ for all $i \geq 1$.

5. Ext commutes with direct sums in the first variable and with direct products in the second variable.

Proof. The proof is precisely the same as for Proposition 2.4.11 and Example 2.4.12 and Lemma 2.4.14. □

Theorem 2.4.21. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules. Then there are canonical induced long exact sequences*

$$0 \to \mathrm{Ext}^0_R(C,N) \to \mathrm{Ext}^0_R(B,N) \to \mathrm{Ext}^0_R(A,N) \to \mathrm{Ext}^1_R(C,N) \to \mathrm{Ext}^1_R(B,N) \to ...$$

and

$$0 \to \mathrm{Ext}^0_R(M,A) \to \mathrm{Ext}^0_R(M,B) \to \mathrm{Ext}^0_R(M,C) \to \mathrm{Ext}^1_R(M,A) \to \mathrm{Ext}^1_R(M,B) \to ...$$

These sequences are natural in M, N and the short exact sequence.

Proof. Exactly as Theorem 2.4.16. □

Example 2.4.22. We have that

$$\mathrm{Ext}(\mathbb{Z}, M) = 0$$

and

$$\mathrm{Ext}(\mathbb{Z}/n, M) = \mathrm{coker}(M \xrightarrow{\cdot n} M)$$

in particular

$$\mathrm{Ext}(\mathbb{Z}/n,\mathbb{Z}) = \mathbb{Z}/n \qquad \text{and} \qquad \mathrm{Ext}(\mathbb{Z}/n,\mathbb{Z}/m) = \mathbb{Z}/gcd(m,n)$$

Proof. Choose the standard resolution $\mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n$. We get that

$$\mathrm{Hom}_R(P_*, M) = \left(M \xrightarrow{n} M \to 0 \to ...\right)$$

which implies the non-trivial part of the claim. □

2.5 Universal coefficient theorems

In this section we explain how to compute homology with R-coefficients in terms of homology with $\mathbb{Z}$ coefficients and cohomology in terms of homology.

Theorem 2.5.1. *Let R be a PID. Let C_* be a projective (hence free) R chain complex and M an R-module. Then there is a canonical short exact sequences*

$$0 \to H_n(C_*) \otimes M \xrightarrow{\alpha} H_n(C_* \otimes_R M) \to \mathrm{Tor}_1^R(H_{n-1}(C_*), M) \to 0$$

This sequence is non-canonically split.

Proof. We first construct the map $\alpha : H_n(C_*) \otimes M \xrightarrow{\alpha} H_n(C_* \otimes_R M)$ and see that it admits a non-canonical retract. This is dual to Proposition 2.3.1 so that we will be somewhat brief.

We define α by $\alpha([c] \otimes m) = [c \otimes m]$ (more precisely as the addtive map corresponding to the bilinear map $([c], m) \mapsto [c \otimes m]$. We first have to check that this is well-defined. To this extend observe that $[c \otimes m]$ is a cycle since $\partial(c \otimes m) = \partial c \otimes m = 0$. Moreover it is independent of the choice of representative $[c]$ since for $c' = c + \partial d$ we have that $c' \otimes m = (c + \partial d) \otimes m = c \otimes m + \partial(d \otimes m)$ thus $[c \otimes m] = [c' \otimes m]$.

To construct the retract of α we choose a retract r of the inclusion

$$Z_n C \to C_n$$

which is possible since the quotient is isomorphic to $B_{n-1}C$ and thus free (similar to the argument in Proposition 2.3.1). Now we consider the map

$$Z_n(C \otimes_R M) \to C \otimes_R M \xrightarrow{r \otimes \mathrm{id}} Z_n C \otimes_R M \to H_n(C) \otimes_R M$$

and claim that this descends to the required retract. This is straightforward to verify and similar to Proposition 2.3.1.

Now we want to construct the short exact sequence. We consider the short exact sequence of chain complexes

$$0 \to Z_* C \to C_* \to B_{*-1} C \xrightarrow{\partial} 0$$

where the first and the latter are chain complexes when equipped with the zero differential. All three chain complexes are projective and thus the sequence is levelwise split. It follows that the induced sequence

$$0 \to Z_* C \otimes_R M \to C_* \otimes_R M \to B_{*-1} C \otimes_R M \to 0$$

is also short exact and we get therefore a long exact sequence

$$\ldots \to B_n C \otimes_R M \to Z_n C \otimes_R M \to H_n(C_* \otimes_R M) \to B_{n-1} C \otimes_R M \to Z_{n-1} C \otimes_R M \to \ldots$$

and one verifies directly from the defintion of the connecting homomorphism (see Theorem 1.5.8) that the first and last maps are give by the obvious inclusions. It follows that we get a short exact sequence

$$0 \to \mathrm{coker}(B_n C \otimes_R M \to Z_n C \otimes_R M) \to H_n(C_* \otimes_R M) \to \ker(B_{n-1} C \otimes_R M \to Z_{n-1} C \otimes_R M) \to 0$$

The first term is $H_n(C_*)$ and the third term is $\mathrm{Tor}_1^R(H_{n-1}(C_*), M)$, which follows from the long exact Tor sequence associated with the short exact sequence $B_{n-1}C \to Z_{n-1}C \to H_{n-1}C$ (since Z_{n-1} is projective and thus has vanishing Tor). $\square$

Corollary 2.5.2. *Let M be an abelian group and (X, A) be a pair of topological spaces. Then there is a short exact sequence*

$$0 \to H_n(X, A) \otimes M \to H_n(X, A; M) \to \operatorname{Tor}(H_{n-1}(X, A), M) \to 0$$

which is natural in M and the pair and which admits a non-canonical splitting, i.e. we have an abstract, non-canonical isomorphism

$$H_n(X, A; M) \cong H_n(X, A) \otimes M \oplus \operatorname{Tor}(H_{n-1}(X, A), M)$$

Example 2.5.3. We can compute the homology of $\mathbb{R}P^n$ with coefficients in $\mathbb{Z}/2$ and $\mathbb{Q}$ using this result:

$$H_*(\mathbb{R}P^n; \mathbb{Z}/2) \cong H_*(\mathbb{R}P^n) \otimes \mathbb{Z}/2 \oplus \operatorname{Tor}(H_{*-1}(\mathbb{R}P^n), \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & 0 \leq * \leq n \\ 0 & \text{else} \end{cases}$$

and

$$H_*(\mathbb{R}P^n; \mathbb{Q}) \cong H_*(\mathbb{R}P^n) \otimes \mathbb{Q} \oplus \operatorname{Tor}(H_{*-1}(\mathbb{R}P^n), \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & * = 0 \text{ or } * = n \text{ for } n \text{ odd} \\ 0 & \text{else} \end{cases}$$

Corollary 2.5.4. *There is a canonical isomorphism*

$$H_*(X) \otimes \mathbb{Q} \xrightarrow{\sim} H_*(X; \mathbb{Q})$$

for every space X. In particular the morphism $H_*(X) \to H_*(X; \mathbb{Q})$ (induced by the inclusion $\mathbb{Z} \to \mathbb{Q}$) has kernel the torsion subgroup of $H_*(X)$

$$TH_*(X) = \{m \in H_*(X) \mid \exists n \geq 1, nm = 0\}$$

and its image is a full latice in the vector space $H_*(X; \mathbb{Q})$. The same is true for $\mathbb{R}$ and $\mathbb{C}$ coefficients instead of $\mathbb{Q}$.

Corollary 2.5.5. *We have $\operatorname{Tor}(M, \mathbb{Q}/\mathbb{Z}) \cong TM$. In particular there is a split short exact sequence*

$$0 \to H_n(X, A) \otimes \mathbb{Q}/\mathbb{Z} \to H_n(X, A; \mathbb{Q}/\mathbb{Z}) \to TH_{n-1}(X, A) \to 0$$

Proof. Exercises. □

Theorem 2.5.6. *Let R be a PID and let C_* be a projective (hence free) R chain complex and M an R-module. Then there is a canonical short exact sequences*

$$0 \to \operatorname{Ext}^1_R(H_{n-1}(C), M) \to H^n(\operatorname{Hom}_R(C_*, M)) \xrightarrow{\Phi} \operatorname{Hom}(H_n(C_*), M) \to 0$$

where Φ is the obvious generalization of the morphism from Proposition 2.3.1. This sequence is non-canonically split.

Proof. Similar to Theorem 2.5.1. □

Corollary 2.5.7. *Let X be a space, R a PID and M an R-module. Then there is a canonical short exact sequences*

$$0 \to \mathrm{Ext}_R(H_{n-1}(X,A;R),M) \to H^n(X,A;M) \xrightarrow{\Phi} \mathrm{Hom}_R(H_n(X,A;R),M) \to 0$$

where Φ is the morphism from Proposition 2.3.1. This sequence is non-canonically split, i.e. we have an isomorphism

$$H^n(X,A;M) \cong \mathrm{Hom}_R(H_n(X,A;R),M) \oplus \mathrm{Ext}_R(H_{n-1}(X,A;R),M)$$

Corollary 2.5.8. *If $H_{n-1}(X,A)$ and $H_n(X,A)$ are finitely generated then so is $H^n(X,A)$. More precisely we have*

$$H^n(X,A) \cong F_n \oplus T_{n-1}$$

where F_n is the free part of $H_n(X,A)$ and T_{n-1} is the torsion part of $H_{n-1}(X,A)$. [1]

Proof. The only thing that has to be shown is that $\mathrm{Hom}(H_n(X.A),\mathbb{Z}) \cong F_n$ and $\mathrm{Ext}(H_{n-1}(X,A),\mathbb{Z}) = T_{n-1}$ both follows from the classification of finitely generated abelian groups. □

Corollary 2.5.9. *Let M be a compact topological manifold. Then the cohomology groups $H^n(M)$ are finitely generated in each degree and we have $H^n(M) = 0$ for $n > \dim M$. The same is true for coefficients in a PID R.*

Proof. We first use that the homology groups of a compact manifold are finitely generated according to Proposition 1.17.2. For $n \leq \dim M$ we find that

$$H^n(M) = F_n \oplus T_{n-1}$$

Clearly F_n is zero since $H_n(M)$ is zero (Theorem 2.10.1) and since the top homology does not contain any torsion (its either $\mathbb{Z}$ or 0). □

Example 2.5.10. Consider $\mathbb{R}P^n$. Then we find that

$$H^*(\mathbb{R}P^n) \cong \mathrm{Hom}(H_*(\mathbb{R}P^n),\mathbb{Z}) \oplus \mathrm{Ext}(H_{*-1}(\mathbb{R}P^n),\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } * = 0, * = n \text{ odd} \\ \mathbb{Z}/2 & \text{for } * = \text{ even }, * \leq n \\ 0 & \text{else} \end{cases}$$

Example 2.5.11. For all spaces X we have that $H_0(X,\mathbb{Z})$ is free. Thus we get $H^1(X) \cong \mathrm{Hom}(H_1(X),\mathbb{Z}) \cong \mathrm{Hom}(\pi_1(X),\mathbb{Z})$. In particular $H^1(X)$ is always torsion free.

Corollary 2.5.12. *If a morphism $f : (X,A) \to (Y,B)$ induces an isomorphism in homology*

$$f_* : H_*(X,A) \to H_*(Y,B)$$

then f also induces an isomorphism in cohomology

$$f^* : H^*(Y,B;M) \to H^*(X,A;M)$$

and in homology with M-coefficients

$$f_* : H_*(X,A;M) \to H^*(Y,B;M)$$

[1] These identifications are not natural at all. It is for example better to think of the dual of F_n

Proof. Consider the obvious maps of short exact seuqences and conclude by the 5-lemma. □

Its now an immediate corollary of the generalized Jordan curve theorem 1.9.6 that:

Example 2.5.13. Let $S^r \to S^n$ be an embedding. Then we have

$$H^*(S^n \setminus S^r) = \begin{cases} \mathbb{Z} & \text{for } * = 0, n-r-1 \\ 0 & \text{else} \end{cases}$$

Corollary 2.5.14. *Let k be a field. Then the morphism*

$$H^n(X, A; k) \to \mathrm{Hom}_k(H_n(X, A, k), k)$$

is an isomorphism. Moreover for the rationals $\mathbb{Q}$ (and $\mathbb{R}$, $\mathbb{C}$) we have

$$H^n(X, A; \mathbb{Q}) \xrightarrow{\sim} \mathrm{Hom}(H_n(X, A), \mathbb{Q})$$

Proof. We use the universal coefficients theorem 2.5.6 for the ring k and the free module being the singular chain complex $C_*(X, A)$ to obtain a short exact sequence

$$0 \to \mathrm{Ext}(H_n(X, A; k), k) \to H^n(X, A; k) \to \mathrm{Hom}_k(H_n(X, A; k), k) \to 0$$

But the *Ext*-term vanishes since $H_{n-1}(X, A; k)$ is free over k. Thus we get the result. The second result now follows since we already know that $H_n(X, A; \mathbb{Q}) \cong H_n(X, A) \otimes \mathbb{Q}$ by Corollary 2.5.4 and since $\mathrm{Hom}_{\mathbb{Q}}(A \otimes \mathbb{Q}; V) \to \mathrm{Hom}(A; V)$ for every $\mathbb{Q}$-vector space V. □

Note that the second result shows that $\mathrm{Ext}(H_{n-1}(X, A); \mathbb{Q}) = 0$ since there is also the sequence

$$0 \to \mathrm{Ext}(H_n(X, A), \mathbb{Q}) \to H^n(X, A; \mathbb{Q}) \to \mathrm{Hom}(H_n(X, A), \mathbb{Q}) \to 0$$

This is true for all M instead of $H_{n-1}(X, A)$ and relies on the fact that $\mathbb{Q}$ is 'injective' over $\mathbb{Z}$. This is not true for $\mathbb{F}_p$. While we still have

$$H^n(X, \mathbb{F}_p) \cong \mathrm{Hom}(H_n(X, \mathbb{F}_p), \mathbb{F}_p)$$

it is not true that the latter term is isomorphic to $\mathrm{Hom}(H_n(X), \mathbb{F}_p))$.

Example 2.5.15.

$$H^*(\mathbb{R}P^\infty; \mathbb{Z}/2) \cong \begin{cases} \mathbb{Z}/2 & \text{for } * \geq 0 \\ 0 & \text{else} \end{cases}$$

In degree one we have

$$\mathbb{Z}/2 \cong H^1(\mathbb{R}P^\infty; \mathbb{Z}/2) \cong \mathrm{Hom}(H_1(\mathbb{R}P^\infty), \mathbb{Z}/2) \cong \mathrm{Hom}(\pi_1(\mathbb{R}P^\infty), \mathbb{Z}/2)$$

The generator can be described as the classifying morphism of the universal cover $S^\infty \to \mathbb{R}P^\infty$ (which is a 2-1 cover) and is denoted by $w_1 \in H^1(\mathbb{R}P^\infty; \mathbb{Z}/2)$. It is the universal first Stiefel-Whitney class for line bundles.

2.6 Direct and inverse limits

Recall the definition direct limit and inverse limits. We will here specialize to the case of the natural numbers and call them sequential, i.e. we consider diagrams of the form

$$M_0 \to M_1 \to M_2 \to ...$$

and denote the colimit by $\varinjlim M_i$. Similarly we consider diagrams of the form

$$... \to M_2 \to M_1 \to M_0$$

and denote the limit by $\varprojlim M_i$. Recall that in abelian groups the direct and inverse limits are preserved by the forgetful functor to sets, i.e.

$$\varinjlim M_i = \coprod_i M_i / \sim \qquad m_i \sim f(m_i)$$

and

$$\varprojlim M_i = \{(m_i) \mid m_i = f(m_{i+1}\} \subseteq \prod_i M_i$$

Example 2.6.1. 1. The Prüfer group is defined as

$$\mathbb{Z}(p^\infty) := \varinjlim \big(\mathbb{Z}/p \to \mathbb{Z}/p^2 \to Z/p^3 \to ...\big)$$

where the morphismsm $\mathbb{Z}/p^n \to \mathbb{Z}/p^{n+1}$ are given by $k \mapsto pk$. It is isomorphic to

$$\mathbb{Z}(p^\infty) \cong \mathbb{Z}[1/p]/\mathbb{Z} \cong \mathbb{Q}/\mathbb{Z}_{(p)}$$

where $\mathbb{Z}[1/p] \subseteq \mathbb{Q}$ is the set of rationals $\frac{m}{n}$ where n is a power of p and $\mathbb{Z}_{(p)} \subseteq \mathbb{Q}$ is the set of rationals $\frac{m}{n}$ where n is not divisible by p. These isomorphisms are induced by the inclusion $\mathbb{Z}/p^n \to \mathbb{Q}$ given by $k \mapsto \frac{k}{p^n}$ and the universal property of the direct limit. Also the prüfer group can be described as the sylow p-subgroup of $\mathbb{Q}/\mathbb{Z}$.

2. The p-adic numbers $\mathbb{Z}_p$ (also written as $\mathbb{Z}_p$) are defined as the limit

$$\mathbb{Z}_p := \varprojlim(\mathbb{Z}/p \leftarrow \mathbb{Z}/p^2 \leftarrow \mathbb{Z}/p^3 \leftarrow ...)$$

where the maps send $[k] \in \mathbb{Z}/p^n \mapsto [k] \in \mathbb{Z}/p^{n-1}$. Note that $\mathbb{Z}_p$ inherits the structure of a commutative ring, since all these maps are ring maps. Despite the fact that $\mathbb{Z}/p$ have a lot of torsion, it turns out that $\mathbb{Z}_p$ is torsion free, a domain and even a principal ideal domain (its in fact a DVR). Its uncountable. Using the universal property it is also not hard to check that $\mathbb{Z}_p$ is the endomorphism ring of $\mathbb{Z}(p^\infty)$. Moreover the ring $\mathbb{Z}_p$ contains $\mathbb{Z}_{(p)}$ as a subring.

A short exact sequence of direct systems is a diagram of the form

$$0 \to M_i \to N_i \to P_i \to 0$$

in which all rows are exact.

Lemma 2.6.2. *For a direct system M_i, the map*

$$\tau : \bigoplus_i M_i \to \bigoplus_i M_i$$

sending with $\tau(x)_i = x_i - f(x_{i-1})$ is injective with cokernel $\varinjlim M_i$.

Proof. Assume (x_i) is in the kernel. Assume $(x_i) \neq 0$. Let n be minimal such that $x_n \neq 0$. Then $\tau(x)_n = x_n - f(x_{n-1}) = x_n$, thus $x_n = 0$, contradiction.

Let C denote the cokernel. Thus, maps out of C are the same as maps out of $\bigoplus_i M_i$ which vanish on the image of τ. In other words, a map $C \to N$ is a tuple of maps $g_i : M_i \to N$ with $g_n - f \circ g_{n-1} = 0$, thus the same as a system of maps out of the M_i compatible with the $f : M_{i-1} \to M_i$. So $C = \varinjlim M_i$. $\square$

Lemma 2.6.3. *The functor $\varinjlim$ is exact, i.e. for a short exact sequence of direct systems $0 \to M_i \to N_i \to P_i \to 0$ the induced sequence $0 \to \varinjlim M_i \to \varinjlim N_i \to \varinjlim P_i \to 0$ is also short exact.*

Proof. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_i M_i & \longrightarrow & \bigoplus_i N_i & \longrightarrow & \bigoplus_i P_i & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} & & \\
0 & \longrightarrow & \bigoplus_i M_i & \longrightarrow & \bigoplus_i N_i & \longrightarrow & \bigoplus_i P_i & \longrightarrow & 0,
\end{array}$$

then apply the long exact sequence of homology, where we consider the diagram as a short exact sequence of 2-term chain complexes. $\square$

Lemma 2.6.4. *Let C^i_* be a direct system of chain complexes. Then the canonical morphism*

$$\varinjlim_i H_n\left(C^i_*\right) \to H_n(\varinjlim C^i_*)$$

is an isomorphism.

Proof. This is generally true for any exact functor, since one can characterize homology groups of a chain complex in terms of some short exact sequences. But here we can give a direct proof: Consider the short exact sequence of chain complexes

$$0 \to \bigoplus C_i \xrightarrow{\tau} \bigoplus C_i \to \varinjlim C_i \to 0.$$

and pass to homology. Since homology commutes with direct sums, the left-hand map on H_n still admits the description

$$\tau_* : \bigoplus_i H_n(C^i_*) \to \bigoplus_i H_n(C^i_*)$$

with $\tau_*(x)_i = x_i - f_*(x_{i-1})$. Thus, Lemma 2.6.2 implies that all those maps are injective. The long exact sequence thus decomposes into short exact sequences

$$0 \to \bigoplus_i H_n(C_i) \xrightarrow{\tau_*} \bigoplus_i H_n(C_i) \to H_n(\varinjlim C^i_*) \to 0$$

which furthermore imply that $H_n(\varinjlim C^i_*) \simeq \varinjlim_i H_n\left(C^i_*\right)$. $\square$

The next theorem we have already seen (Lemma 1.9.1) , but lets reprove it using our new methods.

Theorem 2.6.5. *Let*

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$$

be a sequence of subspaces of a space X with $X = \cup X_i$ such that either each X_i is open in X or such that each $X_i \to X_{i+1}$ is a CW inclusion. Then the canonical morphism

$$\varinjlim_i H_n(X_i) \to H_n(\varinjlim_i X_i)$$

is an isomorphism.

Proof. First case: If all the maps $X_i \to X_{i+1}$ (and therefore also $X_i \to X$) are injective, the singular chain complexes $C_*(X_i)$ are an ascending sequence of subcomplexes of $C_*(X)$, spanned by the singular simplices in X_i. It is easy to see that a sequential colimit of an ascending sequence of subgroups is their union (universal property), thus we need to check that any singular simplex $\Delta^n \to X$ factors through some X_i. Since the X_i are all open subsets of X, and thus an open cover, this follows from compactness of Δ^n.

For the second case, use the cellular chain complex instead. Then $C_*^{\text{cell}}(X_i) \subseteq C_*^{\text{cell}}(X)$ is spanned by the cells contained in X_i, and since every cell of X lies in some X_i, the claim follows again. □

We now try to apply this to sequential limits. However, the analogue of Lemma 2.6.2 is wrong: We can write $\varprojlim M_i$ as a kernel, but the map is not generally surjective:

Definition 2.6.6. Let $M^0 \leftarrow M^1 \leftarrow M^2 \leftarrow M^3 \leftarrow \dots$ be an inverse system of R-modules. Then we define the group $\varprojlim^1(M^i)$ to be the cokernel of the morphism

$$\mu : \prod M^i \to \prod M^i \qquad (x_i) \mapsto x_i - f(x_{i+1})$$

Note that the kernel of μ is the inverse limit. It is easy to see that $\varprojlim^1$ is functorial in morphisms of inverse systems (i.e. ladder diagrams).

Lemma 2.6.7. *For a short exact sequence of inverse systems $0 \to M^i \to N^i \to P^i \to 0$ we get an induced long exact sequence*

$$0 \to \varprojlim(M^i) \to \varprojlim(N^i) \to \varprojlim(P^i) \to \varprojlim{}^1(M^i) \to \varprojlim{}^1(N^i) \to \varprojlim{}^1(P^i) \to 0$$

Proof. This is the long exact sequence associated with the short exact sequence of chain complexes

$$\begin{tikzcd}
\prod M^i \arrow[r] \arrow[d, "\mu"] & \prod N^i \arrow[r] \arrow[d, "\mu"] & \prod P^i \arrow[d, "\mu"] \\
\prod M^i \arrow[r] & \prod N^i \arrow[r] & \prod P^i
\end{tikzcd} .$$

□

The last lemma in particular shows that the functor $\varprojlim$ is left exact but in general not right exact. This is why the $\varprojlim^1$ correction terms come up.

Now one wants to study situations in which the $\lim^1$-groups vanish. Therefore the following condition turns out to be crucial.

Definition 2.6.8. An inverse system of R-modules (or even of sets)

$$M^0 \leftarrow M^1 \leftarrow M^2 \leftarrow M^3 \leftarrow ...$$

satisfies the Mittag-Leffler condition if for each n the decreasing family

$$\text{im } (M^{n+k} \to M^n) \subseteq M_n$$

stabilizes as k tends to ∞, i.e. for each n there exists k_0 such that for each $k > k_0$ we have

$$\text{im } (M^{n+k} \to M^n) = \text{im } (M^{n+k_0} \to M^n)$$

Example 2.6.9.

1. The inverse system of the p-adic numbers is Mittag-Leffler, since
$$\mathbb{Z}/p^n \to \mathbb{Z}/p^{n-1}$$
is surjective. More generally if all morphisms are surjective, then the system is Mittag-Leffler.

2. Every inverse system $M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow ...$ in which all M_i are finite abelian groups are Mittag-Leffler. Also all inverse systems of finite dimensional $\mathbb{Q}$-vector spaces.

Lemma 2.6.10. *If M^i is Mittag-Leffler, then $\varprojlim^1(M^i) = 0$.*

Proof. By considering the sequence $\overline{M}^i := \bigcap_k \text{im } (M^{i+k} \to M^i)$ we obtain a short exact sequence

$$0 \to \overline{M}^i \to M^i \to M^i/\overline{M}^i \to 0$$

with the following properties: in $\overline{M}^i$, all the restriction maps are surjective. Indeed, f : im $(M^{i+k} \to M^{i+1}) \to$ im $(M^{i+k} \to M^i)$ is surjective, and if we choose k large enough, by Mittag-Leffler, the sides agree with $\overline{M}^{i+1}$ and $\overline{M}^i$. In $M^i/\overline{M}^i$, the maps are eventually trivial: For any i, there exists k such that $f_{(i+k,i)} : M^{i+k}/\overline{M}^{i+k} \to M^i/\overline{M}^i$ is zero. From the long exact $\varprojlim$ sequence, it is now sufficient to prove the vanishing of $\varprojlim^1$ in these two cases.

First case: Let M^i wlog have the property that all restriction maps are surjective. Then let $x_i \in \prod M_i$ be any sequence, we want y_i to be a preimage. We need $x_i = y_i - f(y_{i+1})$ for all i, since all the f are surjective, we can inductively define y_{i+1} to be any value that satisfies $f(y_{i+1}) = y_i - x_i$.

Second case: Let M^i wlog have the property that all restriction maps are eventually trivial. Then we try to find a preimage (y_i) for a sequence (x_i) by writing

$$y_i = x_i + f_{(i+1,i)}x_{i+1} + f_{(i+2,i)}x_{i+2} + \ldots,$$

which makes sense precisely because from some k on, $f_{(i+k,i)}$ is zero. Now

$$\mu(y)_i = (x_i - f_{(i+1,i)}x_{i+1}) + (f_{(i+1,i)}x_{i+1} - f_{(i+2,i)}x_{i+1}) + \ldots = x_i.$$

$\square$

Lemma 2.6.11. *Let $(C^*)_i$ be an inverse system of cochain complexes such that for each n the inverse system $(C^n)_i$ satisfies the Mittag-Leffer condition. Then there is a natural short exact sequence*

$$0 \to \varprojlim{}^1 H^{n-1}((C^*)_i) \to H^n(\varprojlim(C^*)_i) \to \varprojlim H^n((C^*)_i) \to 0$$

Proof. By assumption, $\varprojlim^1 (C^*)_i$ vanishes, so we have a short exact sequence of chain complexes

$$0 \to \varprojlim (C^*)_i \to \prod (C^*)_i \xrightarrow{\mu} \prod (C^*)_i \to 0.$$

Now since homology commutes with products, we get a long exact sequence

$$\prod_i H^{n-1}((C^*)_i) \xrightarrow{\mu_*} \prod_i H^{n-1}((C^*)_i) \longrightarrow H^n(\varprojlim (C^*)_i) \longrightarrow \prod_i H^n((C^*)_i) \xrightarrow{\mu_*} \prod_i H^n((C^*)_i)$$

The map μ_* still sends a sequence (x_i) to the sequence $\mu_*(x)_i = x_i - f(x_{i+1})$, so its cokernel on the left is $\varprojlim^1 H^{n-1}((C^*)_i)$ and its kernel on the right is $\varprojlim H^n((C^*)_i)$. Thus,

$$0 \to \varprojlim^1 H^{n-1}((C^*)_i) \to H^n(\varprojlim (C^*)_i) \to \varprojlim H^n((C^*)_i) \to 0.$$

□

Theorem 2.6.12. *Let*

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq ...$$

be a sequence of subspaces of a space X with $X = \cup X_i$ such that either each X_i is open in X or such that each $X_i \to X_{i+1}$ is a CW inclusion. Then there is a short exact sequence

$$0 \to \varprojlim^1 H^{n-1}(X_i; M) \to H^n(X; M) \to \varprojlim H^n(X_i, M) \to 0$$

Proof. Like in the proof of Theorem **??**, we do both cases separately, with the simplicial or the cellular chain complex. In both cases, the dual cochain complex will have restriction maps surjective, in particular satisfies Mittag-Leffler. Thus, we can apply the previous Lemma to obtain the desired sequence. □

Proposition 2.6.13. *For coefficients in a field the morphism and under the conditions of Theorem 2.6.12 the canonical morphism*

$$H^n(X; k) \to \varprojlim H^n(X_i, k)$$

is an isomorphism, i.e. the $\varprojlim^1$-term vanishes.

Proof. We have shown in Corollary 2.5.14 that over a field, $\mathrm{Hom}_k(H_n(X; k), k) = H^n(X; k)$. Thus, from

$$H_n(X; k) = \varinjlim H_n(X_i; k),$$

we obtain the desired equality by dualizing. □

Warning: it is not generally true that $\varprojlim^1$-over a field vanishes. There are algebraic counterexamples. But the last proof shows that those sequences coming up as cohomology groups have vanishing $\varprojlim^1$-terms.

2.7 Künneth theorems

Let us start recalling some conventions. Let A_*, B_*, C_* and D_* be graded abelian groups. A morphism of degree d from A_* to B_* is a sequence of morphisms $f : A_* \to B_{*+d}$. We also write $f : A_* \to B_*$ and say that it is of degree d. A chain complex is this a graded abelian group A_* with a morphism $d : A_* \to A_*$ of degree -1 such that $d^2 = 0$.

The tensor product of graded groups is defined as

$$(A_* \otimes B_*)_n := \bigotimes_{p+q=n} A_p \otimes B_q$$

For $f : A_* \to C_*$ and $g : B_* \to D_*$ (possibly of non-zero degree) we define an induced morphism $f_* \otimes g_* : A_* \otimes B_* \to C_* \otimes D_*$ as

$$(f \otimes g)(a \otimes b) := (-1)^{|a||g|} f(a) \otimes g(b)$$

If g is of degree zero, then the sign vanishes. The rule is that interchanging elements induces a sign. Given complexes (A_*, d) and (B_*, d) we get an induced differential

$$d_\otimes := d \otimes \mathrm{id} + \mathrm{id} \otimes d : A_* \otimes B_* \to A_* \otimes B_*$$

This is given concretely as

$$d_\otimes(a \otimes b) = da \otimes b + (-1)^{|a|} a \otimes db$$

We will drop the subscript in $d_\otimes$.

Theorem 2.7.1 (Algebraic Künneth theorem)**.** *Let R be a PID. Let C_* and D_* be projective (hence free) R chain complexes. Then there is a canonical short exact sequences*

$$0 \to (H_*(C_*) \otimes_R H_*(D_*))_n \xrightarrow{\alpha} H_n(C_* \otimes_R D_*) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_1^R(H_p(C_*), H_q(D_*)) \to 0$$

where the morphism α sends $[c], [d]$ to $[c \otimes d]$. This sequence is non-canonically split.

Proof. The proof is a straightforward generalization of the proof of the UCT 2.5.1. Instead of tensoring with a module M, we tensor with the chain complex D_*, but the formula for the retract and the long exact sequence

$$H_n(B_*C \otimes D_*) \to H_n(Z_*C \otimes D_*) \to H_n(C_* \otimes D_*) \to H_{n-1}(B_*C \otimes D_*) \to H_{n-1}(B_*C \otimes D_*)$$

still work as before. Finally, since Z_*C and B_*C have zero differential and are levelwise free, one easily sees $H_n(B_*C \otimes D_*) = B_*C \otimes H_n(D_*)$ etc. □

Remark 2.7.2.

1. Only one of the chain complexes needs to be free for the above result.
2. The universal coefficients sequence in homology is a special case of the Künneth theorem for $D_* = M[0]$.

Lemma 2.7.3. *Let X and Y be CW complexes and assume one of the them is locally compact. Then the cellular chain complex $C_*^{cell}(X \times Y)$ is naturally isomorphic to the tensor product $C_*^{cell}(X) \otimes C_*^{cell}(Y)$.*

Proof. Under these conditions, the filtration given on $X \times Y$ by $(X \times Y)^n = \bigcup_{i+j=n} X^i \times Y^j$ is a CW filtration with each n-cell given by the product of an i-cell in X and a j-cell in Y for all (i,j) with $i+j=n$. This shows the claim additively, to compute the differentials the idea is roughly to describe the attaching map

$$\partial(D^i \times D^j) \to (X \times Y)^{i+j-1}$$

in terms of attaching maps in X and in Y by decomposing $\partial(D^i \times D^j)$ into $\partial D^i \times D^j \cup_{\partial D^i \times \partial D^j} D^i \times \partial D^j$. The attaching map thus factors through the subspace $X^{i-1} \times Y^j \cup_{X^{i-1} \times Y^{j-1}} X^i \times Y^{j-1}$, and after passing to the quotient $(X \times Y)^{i+j-1}/(X \times Y)^{i+j-2}$, factors as

$$\begin{gathered} D^i \times \partial D^j \cup_{\partial D^i \times \partial D^j} \partial D^i \times D^j \\ \downarrow \\ (\partial D^i \wedge D^j/\partial D^j) \vee (D^i/\partial D^i \wedge \partial D^j) \\ \downarrow \\ (X^i/X^{i-1} \wedge Y^{j-1}/Y^{j-2}) \vee (X^{i-1}/X^{i-2} \wedge Y^j/Y^{j-1}) \\ \downarrow \\ (X \times Y)^{i+j-1}/(X \times Y)^{i+j-2}. \end{gathered}$$

If $D^i/\partial D^i \to H_i(X^i/X^{i-1})$ represents x and $D^j/\partial D^j \to H_j(Y^j/Y^{j-1})$ represents y, one sees how this composite suggests $\partial x \times y \pm x \times \partial y \in H_{i+j-1}((X \times Y)^{i+j-1}/(X \times Y)^{i+j-2})$. To determine the signs, one needs to fix conventions for generators of the homology groups and determine the map

$$\begin{aligned} H_{i+j-1}(\partial(D^i \times D^j)) &\to H_{i+j-1}((\partial D^i \wedge D^j/\partial D^j) \vee (D^i/\partial D^i \wedge \partial D^j)) \\ &= H_{i+j-1}(\partial D^i \wedge D^j/\partial D^j) \oplus H_{i+j-1}(D^i/\partial D^i \wedge \partial D^j), \end{aligned}$$

which we won't do. □

Corollary 2.7.4. *For CW complexes X and Y with one of them locally compact, there is a canonical short exact sequence*

$$0 \to (H_*^{cell}(X) \otimes H_*^{cell}(Y))_n \xrightarrow{\alpha} H_n^{cell}(X \times Y) \to \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p^{cell}(X), H_q^{cell}(Y)) \to 0$$

This sequence is non-canonically split.

Example 2.7.5.

- Lets compute the homology of $S^m \times S^m$. Thus all relevant homology groups are $\mathbb{Z}$. Thus all Tor terms vanish. As a result we have:

$$H_*(S^n \times S^m) \cong H_*(S^n) \otimes H_*(S^m)$$

i.e.

$$H_*(S^n \times S^m) \cong \begin{cases} \mathbb{Z} & \text{for } * = 0, n+m \\ \mathbb{Z} & \text{for } * = n, m \text{ if } n \neq m \\ \mathbb{Z} \oplus \mathbb{Z} & \text{for } * = n \text{ if } n = m \\ 0 & \text{else} \end{cases}$$

- The same argument shows that for a space X we get

$$H_*(X \times S^n) \cong H_*(X) \oplus H_{*-n}(X)$$

- For $\mathbb{R}P^2 \times \mathbb{R}P^2$ we get:

$$H_*(\mathbb{R}P^2 \times \mathbb{R}P^2) \cong \begin{cases} \mathbb{Z} & \text{for } * = 0 \\ \mathbb{Z}/2 \oplus \mathbb{Z}/2 & \text{for } * = 1 \\ \mathbb{Z}/2 & \text{for } * = 2 \\ \mathbb{Z}/2 & \text{for } * = 3 \\ 0 & \text{else} \end{cases}$$

  where the (only) Tor term is in degree 3.

Recall from Proposition 1.15.4 the cross product

$$\times : C_*(X) \times C_*(Y) \to C_*(X \times Y)$$

which induces the morphism

$$\times : H_*(X) \times H_*(Y) \to H_*(X \times Y)$$

It was constructed such that it precisely gives a chain morphism

$$C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$$

because we had

$$\partial(a \times b) = \partial a \times b + (-1)^{\deg a} a \times \partial b.$$

More generally for pairs (X, A) and (Y, B) we get a map

$$C_*(X, A) \otimes C_*(Y, B) \xrightarrow{\times} C_*(X \times Y, A \times Y \cup X \times B)$$

Note that we can use the cross product to describe the isomorphism of Lemma 2.7.3 as follows:

$$\begin{aligned} &C_n^{cell}(X) \otimes C_m^{cell}(Y) \\ &= H_*(X_n, X_{n-1}) \otimes H_*(Y_n, Y_{n-1}) \xrightarrow{\times} H_*(X_n \times Y_n, X_n \times Y_{n-1} \cup X_{n-1} \times Y_n) \\ &\qquad \to C_n^{cell}(X \times Y) \end{aligned}$$

In general we can not expect that $C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$ is an isomorphism. But we want to show that it is in fact a chain homotopy equivalence.

Lemma 2.7.6. *If X and Y are contractible. Then $C_*(X) \otimes C_*(Y)$ has homology zero in positive degrees and $\mathbb{Z}$ in degree 0 generated by $[x_0 \otimes y_0]$ for any point $x_0 \in X$, $y_0 \in Y$.*

Proof. It follows directly from the algebraic Künneth theorem that the Homology of the chain complexes is isomorphic to $\mathbb{Z}$ in degree zero. □

Lemma 2.7.7. *There exists a natural chain morphism*

$$\theta : C_*(X \times Y) \to C_*(X) \otimes C_*(Y),$$

which is in degree 0 the canonical morphism $(x, y) \mapsto (x \otimes y)$ (recall that x is the 0-simplex $\Delta^0 \to X$ given by the point $x \in X$).

Proof. Given any map $\sigma : \Delta^n \to X \times Y$, we obtain the projections $\sigma_X : \Delta^n \to X$ and $\sigma_Y : \Delta^n \to Y$, and a commutative diagram

$$\begin{tikzcd} \Delta^n \arrow[r, "\Delta"] \arrow[dr, "\sigma"'] & \Delta^n \times \Delta^n \arrow[d, "\sigma_X \times \sigma_Y"] \\ & X \times Y \end{tikzcd}$$

Thus, the element $\sigma \in C_n(X \times Y)$ is the image under $(\sigma_X \times \sigma_Y)_*$ of the element $\Delta \in C_n(\Delta^n \times \Delta^n)$. By naturality, defining a natural map

$$C_n(X \times Y) \to \bigoplus_{i+j=n} C_i(X) \otimes C_j(Y)$$

is thus the same as defining its value on $\Delta \in C_n(\Delta^n \times \Delta^n)$, which is just a choice of element in $\bigoplus_{i+j=n} C_i(\Delta^n) \otimes C_j(\Delta^n)$.

Now assume we have defined θ in degrees smaller than n. In order to define θ_n with the compatibility

$$\begin{tikzcd} C_n(X \times Y) \arrow[r, "\theta_n"] \arrow[d, "\partial"] & \bigoplus_{i+j=n} C_i(X) \otimes C_j(Y) \arrow[d, "\partial \times \mathrm{id} \pm \mathrm{id} \times \partial"] \\ C_{n-1}(X \times Y) \arrow[r, "\theta_{n-1}"] & \bigoplus_{i+j=n-1} C_i(X) \otimes C_j(Y) \end{tikzcd}$$

we just have to do it for $\Delta \in C_n(\Delta^n \times \Delta^n)$ and extend naturally. So we can take $\theta_n(\Delta) \in \bigoplus_{i+j=n} C_i(\Delta^n) \otimes C_j(\Delta^n)$ to be any element whose boundary is the cycle $\theta_{n-1}(\partial\Delta)$, which we can do as $C_*(\Delta^n) \otimes C_*(\Delta^n)$ is acyclic by Lemma 2.7.6. (For $n = 1$, observe that $\theta_0(\partial\Delta) = \theta_0(1, 1) - \theta_0(0, 0) = [1] \otimes [1] - [0] \otimes [0]$ is also zero in homology.)

□

Remark 2.7.8. One can write down a completly explict version of θ if needed. We don't want to to do this at the moment but will give it later.

Lemma 2.7.9. *Any two natural chain morphisms $f, g : C_*(X \times Y) \to C_*(X \times Y)$ which are equal to the canonical isomorphisms in degree 0 for $X = Y = pt$ are naturally chain homotopic. The same holds if we replace the source and/or the target of the morphism by $C_*(X) \otimes C_*(Y)$*

Proof. The proof again proceeds by defining a natural map inductively. In the first case, we want a chain homotopy $H : C_n(X \times Y) \to C_{n+1}(X \times Y)$, with $\partial H - H\partial = f - g$. We are again reduced to choosing a value $H(\Delta) \in C_{n+1}(\Delta^n \times \Delta^n)$ for $\Delta : \Delta^n \to \Delta^n \times \Delta^n$ the diagonal map, having inductively defined H on all $C_{n-1}(X \times Y)$. The compatibility now requires us to pick $H(\Delta)$ in such a way that

$$\partial H(\Delta) = H(\partial\Delta) + f(\Delta) - g(\Delta),$$

where the right hand element is already defined and a cycle in $C_n(\Delta^n \times \Delta^n)$, and since $\Delta^n \times \Delta^n$ is contractible, a boundary. For $n = 0$, we need to check that $f[(0,0)] - g[(0,0)]$ is indeed zero in homology, but this follows from the assumptions.

In the second case, we want a chain homotopy $H : (C_*(X) \otimes C_*(Y))_n \to (C_*(X) \otimes C_*(Y))_{n+1}$. It suffices to define it on $C_i(X) \otimes C_j(Y)$ one pair (i,j) after another, and by naturality we are reduced to $\Delta^i \otimes \Delta^j \in C_i(\Delta^i) \otimes C_j(\Delta^j)$. Again one inductively assume it is defined already for lower degrees, and have to write down an element in $(C_*(\Delta^i) \otimes C_*(\Delta^j))_{i+j+1}$ whose boundary is a suitable cycle. By Lemma 2.7.6 we can do this. □

This immediately implies:

Theorem 2.7.10 (Eilenberg-Zilber theorem)**.** *The chain maps*

$$\theta : C_*(X \times Y) \to C_*(X) \otimes C_*(Y)$$

and

$$\times : C_*(X) \otimes C_*(Y) \to C_*(X \times Y)$$

are naturally homotopy inverses of each other. In particular both are homotopy equivalences.

Observe that the Eilenberg-Zilber theorem implies that also the chain complexes $C_*(X \times Y; R)$ and $C_*(X; R) \otimes_R C_*(Y; R)$ are chain homotopy equivalent over R for every commutative ring R since $C_*(X; R) = C_*(X) \otimes R$ and therefore

$$C_*(X; R) \otimes_R C_*(Y; R) = (C_*(X) \otimes C_*(Y)) \otimes R \simeq C_*(X \times Y) \otimes R)$$

Theorem 2.7.11 (Geometric Künneth theorem)**.** *For spaces X and Y and R a PID there is a natural short exact sequences*

$$0 \to (H_*(X; R) \otimes_R H_*(Y; R))_n \xrightarrow{\times} H_n(X \times Y; R) \to \bigoplus_{p+q=n-1} \mathrm{Tor}_R(H_p(X; R), H_q(Y; R)) \to 0$$

which is natural in the spaces X and Y. This sequence is non-canonically split.

Proof. Apply the algebraic Künneth theorem 2.7.1. The fact that the morphism is given by the cross product is immediate from the construction. □

Corollary 2.7.12. *For field coefficients we have a canonical isomorphism*

$$H_*(X; k) \otimes_k H_*(Y; k) \xrightarrow{\times} H_*(X \times Y; k)$$

We can now improve a result about the Euler characteristic that we have obtained in Proposition 1.14.9 by counting cells.

Corollary 2.7.13. *For X and Y are homologically finite over $\mathbb{Q}$ then $X \times Y$ is also homologically finite over $\mathbb{Q}$ and we have*

$$\chi(X \times Y) = \chi(X) \times \chi(Y)$$

Proof. We compute the homology $H_*(X \times Y; \mathbb{Q}) \cong H_*(X; \mathbb{Q}) \otimes_{\mathbb{Q}} H_*(Y; \mathbb{Q})$. In particular we find

$$\chi(X \times Y) = \sum (-1)^n \dim H_n(X \times Y; \mathbb{Q}) = \sum (-1)^n \sum_{p+q=n} \dim H_p(X; \mathbb{Q}) \cdot \dim H_q(Y; \mathbb{Q}) = \chi(X) \cdot \chi(Y)$$

□

Now we want to consider cohomology with coefficients in a commutative ring with unit R. Recall that we have $\theta : C_*(X \times Y; R) \to C_*(X; R) \otimes_R C_*(Y; R)$ and $\times : C_*(X; R) \otimes_R C_*(Y; R) \to C_*(X \times Y; R)$. We now want to define a cohomology cross product by dualizing θ.

In order for this we need to be slightly careful with signs. Up until now, we have defined the differential on the cochain complex $C^*(X)$ by

$$(df)(\alpha) = f(\partial\alpha),$$

i.e. as the dual map. However, we see that two symbols switched places here, d and f. Since d has odd degree, in order to be consistent with the Koszul sign rule we introduced earlier (for the tensor product of maps between chain complexes), we need to define

$$(df)(\alpha) = (-1)^{|f|} f(\partial\alpha).$$

So from now on $C^*(X)$ is defined taking this sign into account. This change of sign does not really affect anything we did up until now, since the cohomology is unaffected (as ker and im do not change).

Definition 2.7.14. The cohomology cross product

$$\times : \quad C^*(X; R) \otimes_R C^*(Y; R) \to C^*(X \times Y; R)$$

is defined as follows: it maps $f \otimes g \in C^p(X; R) \otimes_R C^q(X; R)$ to

$$\left(C_{p+q}(X \times Y; R) \xrightarrow{\theta} (C_*(X; R) \otimes_R C_*(Y; R)_{p+q} \xrightarrow{f \otimes g} R \otimes_R R \xrightarrow{\mu} R\right)$$

where $f \otimes g$ is interpreted to be zero on the summands which are not $C_p(X; R) \otimes_R C_q(Y; R)$.

Remark 2.7.15.

1. One can describe the procedure slightly more conceptual as follows: first we introduce the tensor product of cochain complexes as for chain complexes. Now for a chain complex C_* denote the 'dual' cochain complex by $(C_*)^\vee$. Thus $(C_*)^\vee$ is in degree n given by $\mathrm{Hom}_R(C_n; R)$. Then there is a canonical morphism of chain complexes

   $$C_*^\vee \otimes_R D_*^\vee \to (C_* \otimes_R D_*)^\vee$$

   given by

   $$f \otimes g \mapsto (f \otimes g)$$

   This is a chain morphism. To see this, we recall that $(f \otimes g)(\alpha \otimes \beta) = (-1)^{|g||\alpha|} f(\alpha) g(\beta)$. Now

   $$\begin{aligned}
   (d(f \otimes g))(\alpha \otimes \beta) &= (-1)^{|f|+|g|}(f \otimes g)(\partial(\alpha \otimes \beta)) \\
   &= (-1)^{|f|+|g|}(f \otimes g)(\partial\alpha \otimes \beta + (-1)^{|\alpha|}\alpha \otimes \partial\beta) \\
   &= (-1)^{|f|+|g|+|g|(|\alpha|+1)} f(\partial\alpha)g(\beta) + (-1)^{|f|+|g|+(|g|+1)|\alpha|} f(\alpha)g(\partial\beta) \\
   &= (-1)^{|g||\alpha|}(df)(\alpha)g(\beta) + (-1)^{|f|+(|g|+1)|\alpha|} f(\alpha)(dg)(\beta) \\
   &= (df \otimes g)(\alpha \otimes \beta) + (-1)^{|f|}(f \otimes dg)(\alpha \otimes \beta) \\
   &= (df \otimes g + (-1)^{|f|} f \otimes dg)(\alpha \otimes \beta).
   \end{aligned}$$

2. Some authors here take a slightly different convention. Be careful with signs!

Lemma 2.7.16. *The morphism $\times$ is a chain map.*

Proof. This is precisely the work we did in Remark 2.7.15, together with the fact that the dual of a chain map is a chain map. $\square$

The last lemma in particular implies that the cross product induces a well defined morphism

$$\begin{aligned} \times : \quad & H^*(X;R) \otimes_R H^*(Y;R) \to H^*(X \times Y;R) \\ & [p] \otimes [q] \mapsto [p \times q] = [(p \otimes q) \circ \theta] \end{aligned}$$

Definition 2.7.17. The Kronecker product is the pairing

$$H^p(X;R) \otimes H_q(X;R) \to R$$

given by $\langle [f],[c] \rangle := \Phi([f])([c]) = f(c)$. It is of course zero unless $p = q$, but it is convenient to define it for all p and q. The same pairing exists as

$$H^p(X;R) \otimes_R H_q(X;R) \to R$$

For any space X let $1 \in H^0(X)$ be the class represented by the cocycle $1 : C_0(X) \to \mathbb{Z}$ which takes every element $x \in X$ to 1. Let $\tau : X \times Y \to Y \times X$ be the flip map, that sends $(x,y) \mapsto (y,x)$.

Proposition 2.7.18 (Properties of cross product)**.**

1. *For $\alpha, \beta \in H_*(X)$ and $f, g \in H^*(X)$ we have*

$$\langle f \times g, \alpha \times \beta \rangle = (-1)^{|\alpha||g|} \langle f, \alpha \rangle \cdot \langle g, \beta \rangle$$

2. *The cross product is natural in X and Y.*

3. *For the product projection $p : X \times Y \to X$ and $\alpha \in H^k(X)$ we have*

$$\alpha \times 1 = p^*(\alpha)$$

   Similarly we have $1 \times g = q^* g$ where $q : X \times Y \to Y$ is the other projection.

4. *For $\alpha \in H^k(X)$ and $\beta \in H^n(Y)$ we have*

$$\alpha \times \beta = (-1)^{|\alpha||\beta|} \tau^*(\beta \times \alpha)$$

5. *For $\alpha \in H^*(X), \beta \in H^*(Y)$ and $\gamma \in H_*(Z)$ we have*

$$(x \times y) \times z = x \times (y \times z)$$

   in $H^*(X \times Y \times Z)$.

Proof. For the first point, represent f, g, α and β by actual cocycles and cycles (which we denote by the same letters for simplicity). Then by definition of the cross product,

$$\langle f \times g, \alpha \times \beta\rangle = (f \otimes g)(\theta(\alpha \times \beta)) = (f \otimes g)(\alpha \otimes \beta) = (-1)^{|g||\alpha|} f(\alpha) g(\beta) = (-1)^{|g||\alpha|}\langle f, \alpha\rangle\langle g, \beta\rangle$$

Here the key step is the second equality, which uses that θ and $\times$ are homotopy inverses and the value of the cocycle $f \otimes g$ on a cycle only depends on its homology class.

The second point is clear. For the third point, observe that it is enough to do it for $Y = \mathrm{pt}$ because of naturality. For $Y = \mathrm{pt}$ we have to show that $\alpha \times 1 \in H^*(X \times \mathrm{pt})$ corresponds to α under the natural isomorphism (by the variant of Lemma 2.7.9 for $C_*(X)$) . This follows from the fact that the map $C_*(X) \otimes C_*(\mathrm{pt}) \to C_*(X \times \mathrm{pt})$ inducing the homology cross product is the canonical isomorphism.

For the fourth point, let $\tau^{\mathrm{alg}} : C_*(X) \otimes C_*(Y)$ be the map $\tau(a \otimes b) = (-1)^{|a||b|} b \otimes a$. The sign is necessary to make this a chain map! Now the composite $\tau \circ (\times) \circ \tau^{\mathrm{alg}}$ agrees with $\times$ on degree 0 terms, so by the uniqueness of $\times$ is chain homotopic to $\times$. We thus have a chain homotopy commutative diagram

$$\begin{tikzcd}
C_*(X) \otimes C_*(Y) \arrow[r, "\times"] \arrow[d, "\tau^{\mathrm{alg}}"] & C_*(X \times Y) \arrow[d, "\tau"] \\
C_*(Y) \otimes C_*(X) \arrow[r, "\times"] & C_*(Y \times X)
\end{tikzcd}$$

and thus get the same chain homotopy commutative diagram in the other direction with θ. By dualizing we get the desired compatibility for the cohomology cross product. □

Recall that we say that a space X is homologically of finite type over a PID R if all homology groups $H_*(X; R)$ are finitely generated. This is for example the case if X is a CW complex of finite type. Similarly a chain complex C is homologically of finite type over a PID R if if is bounded below, i.e. $C_k = 0$ for $k \leq\leq 0$, and all homology groups $H_*(C))$ are finitely generated. For general R there is a similar condition, but finitely generated is not sufficient and we abstain from giving it here.

Lemma 2.7.19. *Let C_* be a finite type chain complex that is levelwise free over R. Then C_* is chain homotopy equivalent to a bounded below chain complex that is finite and free in each degree. For another bounded below chain complex D_* t that is levelwise free over R the canonical morphism of cochain complexes*

$$(C_*)^\vee \otimes (D_*)^\vee \to (C_* \otimes D_*)^\vee$$

is a chain homotopy equivalence.

Proof. We will show the slightly weaker statement that the map is a homology iso and that we can replace C_* be a finite chain complex that is isomorphic in homology. In fact it can be abstractly shown that if a map between projective, bounded R-chain complexes is a homology iso, then it is already a chain homotopy equivalence.

We consider the sequence

$$0 \to Z_*C \to C_* \to B_{*-1}C \xrightarrow{\partial} 0$$

Because C_* is free and R is a PID it follows that Z_*C and $B_{*-1}C$ are also free in every degree. In particular the short exact sequence above splits giving rise to isomorphisms $C_n = Z_nC \oplus B_{n-1}C$. Under these identifications the differential is given by the inclusion of $B_{n-1}C$ into $Z_{n-1}C$, so that the chain complex C_* is isomorphic to the chain complex

$$C_* \cong (... \to Z_{n+1}C \oplus B_nC \to Z_nC \oplus B_{n-1}C \to Z_{n-1}C \oplus B_{n-2}C \to ...)$$

with the maps the inclusions on the last summand into the first summand of the next term. All the terms are free.

Now we use that $H_nC = Z_nC/B_nC$ is finitely generated to find a finitely generated subgroup $Z_n^{\mathrm{fin}} \subseteq Z_n$ for every n with the property that the map $Z_n^{\mathrm{fin}} \to Z_n \to H_nC$ is surjective (for example by lifting generators). We consider the intersection $B_n^{\mathrm{fin}} := Z_n^{\mathrm{fin}} \cap B_n$ which is also finite free and get the subcomplex

$$C_*^{\mathrm{fin}} = \big(... \to Z_{n+1}^{\mathrm{fin}}C \oplus B_n^{\mathrm{fin}}C \to Z_n^{\mathrm{fin}}C \oplus B_{n-1}^{\mathrm{fin}}C \to Z_{n-1}^{\mathrm{fin}}C \oplus B_{n-2}^{\mathrm{fin}}C \to ...\big)$$

The inclusion $C_*^{\mathrm{fin}} \subseteq C_*$ is a homology iso by construction showing the first part.

This implies that $C_*^\vee \to (C_*^{\mathrm{fin}})^\vee$ is also a homology iso by UCT and thus also $C_*^\vee \otimes D_*^\vee \to (C_*^{\mathrm{fin}})^\vee \otimes D_*^\vee$ by Künneth . Also the map $(C_*^{\mathrm{fin}}) \otimes D_* \to C_* \otimes D_*$ is a homology iso by the algebraic Künneth and it follows that the dual map $(C_* \otimes D_*)^\vee \to ((C_*^{\mathrm{fin}}) \otimes D_*)^\vee$ is a homology iso again by UCT. We can now look at the commutative square

$$\begin{tikzcd}
C_*^\vee \otimes D_*^\vee \arrow[r] \arrow[d] & (C_*^{\mathrm{fin}})^\vee \otimes D_*^\vee \arrow[d] \\
(C_* \otimes D_*)^\vee \arrow[r] & ((C_*^{\mathrm{fin}}) \otimes D_*)^\vee
\end{tikzcd}$$

in which the horizontal maps are homology isos. We want to show that the left hand map is a homology iso which is therefore equivalent to showing it for the right hand map. But the right hand map is even an isomorphism since C_*^{fin} is finite free.

□

Theorem 2.7.20. *Let R be a PID. If one of the spaces X and Y is homologically finite over R then there is a short exact sequence*

$$0 \to (H^*(X;R) \otimes_R H^*(Y;R))_n \xrightarrow{\times} H^n(X \times Y;R) \to \bigoplus_{p+q=n+1} \mathrm{Tor}_R(H^p(X), H^q(Y)) \to 0$$

which is natural in the spaces X and Y. This sequence is non-canonically split.

Proof. The dual of Eilenberg-Zilber allows us to identify $C^*(X \times Y)$ with the dual complex $(C_*(X) \otimes C_*(Y))^\vee$. This is not generally the same as the tensor product of duals. However, in our case, we can replace $C_*(X)$ with C_*, and get

$$\mathrm{Hom}(C_* \otimes C_*(Y), R) \cong \mathrm{Hom}(C_*, C_*(Y)^\vee) \cong \mathrm{Hom}(C_*, R) \otimes C_*(Y)^\vee = C_*^\vee \otimes C_*(Y)^\vee,$$

Since $C_*^\vee$ is furthermore still degreewise projective, we can apply the algebraic Künneth theorem to $C_*^\vee \otimes C_*(Y)^\vee$. □

2.8 Cup and cap product

Definition 2.8.1. Let $d : X \to X \times X$ be the diagonal map $dx = (x, x)$. Then the cup product is the homomorphism

$$\cup : H^p(X; R) \otimes H^q(X; R) \to H^{p+q}(X; R)$$

defined by $\alpha \cup \beta := \Delta^*(\alpha \times \beta)$ (where R is commutative with unit).

We will sometimes only write $\alpha\beta$ or $\alpha \cdot \beta$ for $\alpha \cup \beta$.

Proposition 2.8.2 (Properties of the cup product)**.** *We have*

1. *For $\alpha \in H^*(X)$ and $\beta \in H^*(Y)$ we have $\alpha \times \beta = p^*(\alpha) \cup q^*(\beta)$ where $p : X \times Y \to X$ and $q : X \times Y \to Y$ are the projections.*
2. *For $\alpha, \beta \in H^*(X)$ we have $\alpha \cup \beta = (-1)^{|\alpha||\beta|}\beta \cup \alpha$*
3. $(\alpha \cup \beta) \cup \gamma = \alpha \cup (\beta \cup \gamma)$
4. $\alpha \cup 1 = 1 \cup \alpha = \alpha$
5. *Cup is natural, i.e. for a morphism $f : X \to Y$ we have $f^*\alpha \cup f^*\beta = f^*(\alpha \cup \beta)$*
6. $(\alpha \times \beta) \cup (\gamma \times \delta) = (-1)^{|\beta||\gamma|}(\alpha \cup \gamma) \times (\beta \cup \delta)$.

Proof. (1) Consider the map $\pi_{1,4} : X \times Y \times X \times Y \to X \times Y$. Then by naturality of $\times$ we have that

$$p^*\alpha \times q^*\beta = \pi_{1,4}^*(\alpha \times \beta)$$

so that we get

$$p^*(\alpha) \cup q^*(\beta) = \Delta^*(p^*\alpha \times q^*\beta) = \Delta^*(\pi_{1,4}^*(\alpha \times \beta) = \alpha \times \beta$$

(2) We have

$$\alpha \cup \beta = \Delta^*(\alpha \times \beta) = \Delta^*((-1)^{|\alpha||\beta|}\tau^*(\beta \times \alpha)) = (-1)^{|\alpha||\beta|}\beta \cup \alpha$$

since the diagram

$$\begin{tikzcd} & X \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\ X \times X \arrow[rr, "\tau"] & & X \times X \end{tikzcd}$$

commutes. (3) Follows from associativity of cross product
(4)

$$\alpha \cup 1 = \Delta^*(\alpha \times 1) = \Delta^* p^* \alpha = (p\Delta)^*\alpha = \alpha$$

(5) Clear
(6)

$$\begin{aligned}(\alpha \times \beta) \cup (\gamma \times \delta) &= d^*(\alpha \times \beta \times \gamma \times \delta) \\ &= (-1)^{|\beta||\gamma|}\Delta^*(\alpha \times \gamma \times \beta \times \delta) \\ &= (-1)^{|\beta||\gamma|}\Delta^*(\alpha \times \gamma) \times \Delta^*(\beta \times \delta) \\ &= (-1)^{|\beta||\gamma|}(\alpha \cup \gamma) \times (\beta \cup \delta)\end{aligned}$$

□

Among other things, these properties guarantee that

$$H^*(X) := \bigoplus_{n\in\mathbb{N}} H^n(X)$$

is a graded commutative ring, i.e. it is a ring together with an associative multiplication such that we have the graded commutativity. For every morphism $f : X \to Y$ the induced morphism $f^* : H^*(Y) \to H^*(X)$ is a morphism of graded rings. In particular if f is a homotopy equivalence then the rings $H^*(Y)$ and $H^*(X)$ are isomorphic. Thus the ring $H^*(X)$ is an invariant of the homotopy type of X.

Note that property (6) says that the morphism

$$(H^*(X;R) \otimes_R H^*(Y;R)) \xrightarrow{\times} H^*(X \times Y;R)$$

in the cohomological Künneth theorem (2.7.20) is a graded ring homomorphism where the graded commutative ring on the left gets the obvious multiplication. More generally for two graded commutative rings S^* and T^* the tensor product is the new graded commutative ring $S^* \otimes T^*$ with

$$(s \otimes t) \cdot (s' \otimes t') := (-1)^{|t||s'|}(s \cdot s') \otimes (t \cdot t')$$

Example 2.8.3. For the n-sphere we find the cohomology ring as:

$$H^*(S^n) = \mathbb{Z}[x]/x^2 \qquad |x| = n$$

For a torus $S^1 \times S^1$ we therefore get using the Künneth theorem

$$H^*(S^1 \times S^1) \cong H^*(S^1) \otimes H^*(S^1) \cong \Lambda(x_1, x_2) \qquad |x_i| = 1$$

More generally we get by induction for an n-torus

$$H^*(S^1 \times ... \times S^1) \cong H^*(S^1) \otimes ... \otimes H^*(S^1) \cong \Lambda(x_1, ..., x_n) \qquad |x_i| = 1$$

where

$$\Lambda(x_1, ..., x_n) := \frac{T(x_1, ..., x_n)}{x_i x_j = -x_j x_i, x_i^2 = 0}$$

is the extrior algebra (here $T(x_1, ..., x_n)$ is the tensor algebra).

Now there are relative versions of all our constructions. As we have already seen earlier, the cross product in Homology descends to a morphism

$$H_p(X, A) \times H_q(Y, B) \to H_{p+q}(X \times Y, X \times B \cup A \times Y)$$

The same is true for the cup and cross product in cohomology.

Proposition 2.8.4. *The cohomology cross product as well as the cup product descend to morphisms*

$$\times : H^p(X, A) \times H^q(Y, B) \to H^{p+q}(X \times Y, X \times B \cup A \times Y)$$

given by $[f] \times [b] = [f \times g]$ *and*

$$\cup : H^p(X, A) \times H^q(X, B) \to H^{p+q}(X, A \cup B)$$

as $\Delta^*(\alpha \times \beta)$.

Proof. First follows from naturality and the second from the fact that diagonal is a respective morphism. ☐

Again the cross product can be reconstructed from the cup product. So when axiomatizing we could use any of the two.

Proposition 2.8.5. *The cup product endows the cohomology theory (H^*, δ) with a multiplicative structure, i.e. we have multiplications*

$$\cup : H^p(X, A) \times H^q(X, B) \to H^{p+q}(X, A \cup B)$$

for a space X with subspaces A and B and a unit element $1 \in H^0(X)$ such that the following axioms are satisified:

1. *The cup is natural in morphisms $f : X \to X'$ which send $f(A) \subseteq A'$ and $f(B) \subseteq B'$.*
2. *It is graded commutative*
3. *It is associative*
4. 1 *is a unit*
5. *We have the following compatibility with the connecting homomorphism: for the inclusion $i : A \to X$ and $\alpha \in H^p(A)$ and $\beta \in H^q(X)$ we have*

$$\delta(\alpha) \cup \beta = \delta(\alpha \cup i^*(\beta))$$

Proof. Everything works as before, except for (5) for which we shall give an argument. We realize the cup product as induced from a natural chain map

$$\cup : C^*(X) \otimes C^*(X) \to C^*(X) \ .$$

Naturality tell us that for any lift $\alpha' \in C^*(X)$ of $\alpha \in C^*(A)$ (that is $i^*(\alpha') = \alpha$) we have that $i^*(\alpha' \cup \beta) = \alpha \cup i^*\beta$ where we abusively confuse α and β with cycle representatives. By construction of the connecting homomorphism it follows that we can compute $\delta(\alpha \cup i^*(\beta))$ as the class of $d(\alpha' \cup \beta)$ in $H^*(X, A)$ which is given by

$$d(\alpha' \cup \beta) = d\alpha' \cup \beta \pm \alpha' \cup d\beta = d\alpha' \cup \beta$$

which verifies the claim as $[d\alpha']$ represents $\delta(\alpha)$. ☐

Note that the cup product for $B = \emptyset$ makes $H^*(X, A)$ into a (graded) module over $H^*(X)$. Then the connecting homomorphism $\delta : H^*(A) \to H^{*+1}(X, A)$ is an $H^*(X)$-linear morphism.

Lemma 2.8.6. *Let $X = U \cup V$ be an open covering of a space X. The connecting homomorphism in the Mayer-Vietoris sequence*

$$\ldots \to H^*(U) \oplus H^*(V) \to H^*(U \cap V) \xrightarrow{\delta} H^{*+1}(X) \to H^*(U) \oplus H^*(V) \to H^{*+1}(U \cap V) \to \ldots$$

is a $H^*(X)$-module homomorphism, i.e. for $\alpha \in H^*(U \cap V))$ and $\beta \in H^*(X)$

$$\delta(\alpha \cup i^*(\beta)) = \delta(\alpha) \cup \beta$$

where $i : U \cap V \to X$.

Proof. We recall that the Mayer-Vietoris sequence arose from the diagram

$$\begin{tikzcd}
\dots \arrow[r] & H^*(U) \arrow[r] \arrow[d] & H^{*+1}(X,U) \arrow[r] \arrow[d, "\sim"] & H^{*+1}(X) \arrow[r] \arrow[d] & H^{*+1}(U) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & H^*(U\cap V) \arrow[r, "\delta"] & H^{*+1}(V, U\cap V) \arrow[r] & H^{*+1}(V) \arrow[r] & H^{*+1}(U\cap V) \arrow[r] & \dots
\end{tikzcd}$$

The connecting homomorphism in the Mayer-Vietoris sequence arises as the composite of the leftmost lower horizontal map, the inverse vertical isomorphism and the middle upper horizontal map.

All of these groups are modules over $H^*(X)$ via the various restriction maps, and all of the maps are module maps (the connecting homomorphism in the lower row by part 5 of Proposition 2.8.5), so the same follows for the composite. $\square$

Proposition 2.8.7. *Suppose that $X = U \cup V$ where U and V are open, acyclic sets. Then $\alpha \cup \beta = 0$ for all cohomology classes of positive degree in X. More generally if X can be covered by n-acyclic open sets, then all n-fold cup products in positive degree vanish.*

Proof. We generally have the following commutative diagram:

$$\begin{tikzcd}
H^*(X,U)\otimes H^*(X,V) \arrow[r, "\cup"] \arrow[d] & H^*(X, U\cup V) \arrow[d] \\
H^*(X)\otimes H^*(X) \arrow[r, "\cup"] & H^*(X)
\end{tikzcd}$$

So the cup product of two elements in the image of $H^*(X,U) \to H^*(X)$ and $H^*(X,V) \to H^*(X)$ lies in the image of the map $H^*(X, U\cup V) \to H^*(X)$. Now if U is acyclic, the map $H^*(X,U) \to H^*(X)$ is surjective in positive degrees by the long exact sequence, same for V. So the cup product of any two positive-degree elements lies in the image of $0 = H^*(X,X) = H^*(X, U\cup V) \to H^*(X)$ and thus vanishes. The statement for n opens follows analogously by inductively observing that the product of n elements in the images of $H^*(X, U_i) \to H^*(X)$ is in the image of $H^*\left(X, \bigcup U_i\right)$. $\square$

Note that it would have been sufficient to ask that the map $H_*(X) \to H_*(U)$ is zero in positive degrees (and same for V) instead of asking them to be acyclic.

Definition 2.8.8. The maximal number of elements in $H^*(X)$ ($H^*(X;R)$) of positive degree such that the product is non-zero is called the cup length of X (with respect to R-coefficients). It can also be 0 or ∞. It is an invariant of X.

Example 2.8.9. The torus $S^1 \times ... \times S^1$ has cup length n since the cohomology ring is $\Lambda(x_1, ..., x_n)$. A maximal product is given by $x_1 \cup ... \cup x_n$. Thus we need at least $n+1$-contractible open sets to cover $S^1 \times ... \times S^1$. For the two ordinary torus we thus need at least 3 open sets.

Corollary 2.8.10. *The suspension SX and the reduced suspension ΣX of a space X (resp. a pointed space X, x) have trivial cup products in positive degrees. In particular the cup lenghts are* 1.

Proof. Both can be covered by two open sets. □

Theorem 2.8.11. *For n even the sphere S^n is not an H-space, i.e. there is no continuous morphism*

$$\mu : S^n \times S^n \to S^n$$

such that $\mu \circ i_1 \simeq \mathrm{id}$ and $i_2 \circ \mu \simeq \mathrm{id}$ where $i_1(x) = (x, 1)$ and $i_2(x) = (1, x)$.

Proof. Suppose we have such a map. Recall that $H^*(S^n \times S^n) = \Lambda(x_1, x_2)$ as a ring, where $x_1 = x \times 1$ and $x_2 = 1 \times x$ for a generator x of $H^n(S^n)$, and equivalently $x_1 = p_1^*(x)$ and $x_2 = p_2^*(x)$. In particular (as $p_k \circ i_l$ is the identity if $k = l$ and constant otherwise) we have

$$i_1^* x_1 = x, \quad i_2^* x_2 = x, \quad i_1^* x_2 = 0, \quad i_2^* x_1 = 0.$$

Now suppose $\mu^* x = a x_1 + b x_2$ for some coefficients a, b. Then because $\mu \circ i_1 \simeq \mathrm{id}$, we have

$$ax = i_1^*(a x_1 + b x_2) = i_1^* \mu^*(x) = x$$

and so $a = 1$. Analogously $b = 1$. Now $x^2 = 0$ in $H^*(S^n)$ and the fact that μ^* is a ring homomorphism shows that

$$0 = \mu^*(x^2) = (x_1 + x_2)^2 = x_1 x_2 + x_2 x_1 = (1 + (-1)^n) x_1 x_2$$

Since $x_1 x_2$ is a generator of $H^{2n}(S^n \times S^n)$ we see that $1 + (-1)^n = 0$ and so n is odd. □

Definition 2.8.12. A diagonal approximation is a natural chain map

$$\Delta : C_*(X) \to C_*(X) \otimes C_*(X)$$

such that on 0-simplices $\Delta(x) = x \otimes x$.

Clearly such a map is determined by what it does on simplices, i.e. its an association of an element in $\bigoplus_{p+q=n} C_p(X) \otimes C_q(X)$ for any n-simplex in X with certain properties.

Proposition 2.8.13. *There exists a diagonal approximation and any two diagonal approximations are chain homotopic.*

Proof. For the existence just use the composition

$$C_*(X) \xrightarrow{\Delta_*} C_*(X \times X) \xrightarrow{\theta} C_*(X) \otimes C_*(X)$$

The uniqueness follows similarly as in 2.7.9. □

For any choice of diagonal approximation we get an explict formula for the cup product as

$$\alpha \cup \beta := (\alpha \otimes \beta) \circ \Delta : C_*(X) \to R$$

for $\alpha, \beta \in C^*(X)$.

Now we want to construct a more explict diagonal approximation. For an n-simplex $\Delta^n \to X$ with $n = p + q$ we denote by σ_p the front face of σ, that is the composition with the inclusion

$$\Delta^p \to \Delta^n \qquad (x_0, ..., x_p) \mapsto (x_0, ..., x_p, 0, ..., 0)$$

Similarly we denote by σ_q the back face of σ, that is the composition of σ with the inclusion

$$\Delta^q \to \Delta^n \qquad (x_0, ..., x_q) \mapsto (0, ..., 0, x_0, ..., x_q)$$

Note that the two have one element in common.

Definition 2.8.14. The Alexander-Whitney diagonal approximation is defined by

$$\Delta(\sigma) := \sum_{p+q=n} \sigma_p \otimes \sigma_q \in C_*(X) \otimes C_*(X)$$

for $\sigma : \Delta^n \to X$.

Proposition 2.8.15. *The Alexander-Whitney diagonal approximation is a diagonal approximation. The induced cup product*

$$\begin{aligned} \cup : \quad & C^p(X;R) \otimes_R C^q(X;R) \to C^{p+q}(X;R) \\ & (f \cup g)(\sigma) = (f \otimes g)(\Delta(\sigma)) = (-1)^{pq} f(\sigma_p) g(\sigma_q) \end{aligned}$$

endows the cochain complex $C^*(X;R)$ with the structure of a differential graded algebra, i.e. we have the following:

1. *Its a chain map, i.e. $d(\alpha \cup \beta) = d\alpha \cup \beta + (-1)^{|\alpha|} \alpha \cup d\beta$*
2. *It is associative $\alpha \cup (\beta \cup \gamma) = (\alpha \cup \beta) \cup \gamma$*
3. *Its unital, i.e. $\alpha \cup 1 = 1 \cup \alpha = \alpha$*

It moreover defines a functor

$$\mathrm{Top}^{op} \to \mathrm{DGA}$$

where DGA *is the category of differential graded algebras.*

Proof. The main thing to check is that the Alexander-Whitney diagonal approximation is a chain morphism. Indeed, for an n-simplex $\sigma \in C_*(X)$ we have

$$\begin{aligned} \partial\Delta(\sigma) &= \sum_{p+q=n} \partial(\sigma_p \otimes \sigma_q) \\ &= \sum_{p+q=n} \Big(\partial(\sigma_p) \otimes \sigma_q + (-1)^p \sigma_p \otimes \partial\sigma_q \Big) \\ &= \sum_{p+q=n} \Big(\sum_{i=0}^{p} (-1)^i (\sigma_p)^{(i)} \otimes \sigma_q + \sum_{i=0}^{q} (-1)^{p+i} \sigma_p \otimes (\sigma_q)^{(i)} \Big) \\ &= \sum_{p+q=n} \Big(\sum_{i=0}^{p} (-1)^i (\sigma_p)^{(i)} \otimes \sigma_q + \sum_{i=p}^{n} (-1)^i \sigma_p \otimes (\sigma_q)^{(i-p)} \Big) \\ &= \sum_{p+q=n} \Big(\sum_{i=0}^{p} (-1)^i (\sigma^{(i)})_p \otimes (\sigma^{(i)})_q + \sum_{i=p}^{n} (-1)^i \sigma_p \otimes (\sigma_q)^{(i-p)} \Big) \\ &= \sum_{i=0}^{n-1} (-1)^i \sum_{p+q=n-1} (\sigma^{(i)})_p \otimes (\sigma^{(i)})_q \\ &= \Delta(\partial\sigma) \end{aligned}$$

Then it is clear that in degree zero it does the right thing since for a zero-simplex, there is just one pair of front-face and back-face and both are just again that 0-simplex. For associativity, we check that both composites $(\Delta\otimes\mathrm{id})\circ\Delta$ and $(\mathrm{id}\otimes\Delta)\circ\Delta$ from $C_*(X)\to C_*(X)^{\otimes 3}$ send

$$\sigma\mapsto\sum_{p+q+r=n}\sigma_p\otimes\sigma_q\otimes\sigma_r,$$

where for a triple (p,q,r) we have σ_p, σ_q, σ_r denote the precomposites of σ with the three maps

$$\begin{aligned}\Delta^p\to\Delta,\quad &(x_0,\dots,x_p)\mapsto(x_0,\dots,x_p,0,\dots,0)\\ \Delta^q\to\Delta,\quad &(x_0,\dots,x_q)\mapsto(0\dots,0,x_0,\dots,x_p,0,\dots,0)\\ \Delta^r\to\Delta,\quad &(x_0,\dots,x_r)\mapsto(0,\dots,0,x_0,\dots,x_r)\end{aligned}$$

again overlapping in one index respectively.

Unitality follows from the fact that the front-face, back-face splitting corresponding to $(0,n)$ is given by $\sigma(0)\otimes\sigma$. Since 1 is the cochain that sends every 0-simplex to 1, we thus see that

$$((1\otimes\alpha)\circ\Delta)(\sigma)=\alpha(\sigma)$$

and analogously the other way around. □

Warning 2.8.16. Note that the cup product is still not commutative on the level of cochains i.e. $f\cup g\neq\pm g\cup f$, although it is commutative on the level of cohomology $[f]\cup[g]=\pm[g]\cup[f]$. This failure of commutativity is not because we have not written down a better formula, but its a fundamental fact. It has some important consequences that will be studied later.

Proposition 2.8.17. *The cohomology ring of $H^*(\mathbb{R}P^2;\mathbb{F}_2)$ is given by*

$$H^*(\mathbb{R}P^2;\mathbb{Z}/2)\cong\mathbb{Z}/2[\alpha]/\alpha^3\qquad|\alpha|=1$$

Proof. Consider the generator $\alpha\in H^1(\mathbb{R}P^2;\mathbb{Z}/2)\cong\mathrm{Hom}(H_1(\mathbb{R}P^2);\mathbb{Z}/2)$. Then choose a loop $\lambda:S^1\to\mathbb{R}P^2$ to represent the generator of $\pi_1(\mathbb{R}P^2)$. Since λ generates $H_1(\mathbb{R}P^2;\mathbb{F}_2)$ and $H^1(\mathbb{R}P^2;\mathbb{F}_2)\to\mathrm{Hom}_{\mathbb{F}_2}(H_1(\mathbb{R}P^2;\mathbb{F}_2),\mathbb{F}_2)$ is an isomorphism, we see that $\alpha(\lambda)=1$.

Define a 2-simplex σ as follows: it maps the edges $(0,1)$ and $(1,2)$ along λ and $(0,2)$ along a constant map. This can be done since 2λ is nullhomotopic. Now, in $C_*(X)$, $\partial\sigma=2\lambda-\mathrm{const}^1_*$, where const^1_* denotes the 1-simplex which is constant at $*$. We also have $\partial\mathrm{const}^2_*=\mathrm{const}^1_*-\mathrm{const}^1_*+\mathrm{const}^1_*=\mathrm{const}^1_*$. So in $C_*(X;\mathbb{F}_2)$, we have

$$\partial(\sigma+\mathrm{const}^2_*)=0.$$

Now we have

$$(\alpha\cup\alpha)(\sigma)=\alpha(\lambda)\cdot\alpha(\lambda)=1$$

and for the constant 2-simplex we have

$$(\alpha\cup\alpha)(\mathrm{const}^2_*)=\alpha(\mathrm{const}^1_*)\cdot\alpha(\mathrm{const}^1_*)=0$$

since $\alpha(\mathrm{const}^1_*)=\alpha(\partial\mathrm{const}^2_*)(\delta\alpha)(\mathrm{const}^2_*)=0$. Now $\alpha\cup\alpha$ is a cocycle, $\sigma+\mathrm{const}^2_*$ is a cycle, and

$$(\alpha\cup\alpha)(\sigma+\mathrm{const}^2_*)=1,$$

so both are nonzero in (co)homology since the pairing descends to a well-defined pairing there.

□

Example 2.8.18. The integral cohomology ring of $\mathbb{R}P^2$ is given by

$$H^*(\mathbb{R}P^2;\mathbb{Z}) \cong \mathbb{Z}[\beta]/(\beta^2 = 0, 2\beta = 0) \qquad |\beta| = 2$$

The reduction morphism

$$H^*(\mathbb{R}P^2;\mathbb{Z}) \to H^*(\mathbb{R}P^2;\mathbb{Z}/2)$$

is the ring morphism

$$\mathbb{Z}[\beta]/(\beta^2 = 0, 2\beta = 0) \to \mathbb{Z}/2[\alpha]/\alpha^3 \qquad \beta \mapsto \alpha^2$$

The integral cohomology ring of $\mathbb{R}P^3$ is given by by

$$H^*(\mathbb{R}P^3;\mathbb{Z}) \cong \mathbb{Z}[\beta,\nu]/(\beta^2 = 0, 2\beta = 0, \nu^2 = 0) \qquad |\beta| = 2, |\nu| = 3$$

Corollary 2.8.19. *The space $\mathbb{R}P^2$ can not be written as the union of two open, acyclic sets.*

We now fix a diagonal approximation. Every diagonal approximation will do, but for concreteness we think about the Alexander-Whitney map

$$\Delta : C_*(X) \to C_*(X) \otimes C_*(X)$$

We will now define the rest of this section for homology and cohomology with values in R but we will supress the R.

Definition 2.8.20. We define the (chain level) cap product as the homomorphism

$$\cap : C^q(X) \otimes C_{p+q}(X) \to C_p(X)$$

as

$$f \cap c := (\mathrm{id} \otimes f)(\Delta(c))$$

where again $\mathrm{id} \otimes f$ is set to be zero on the summands of

$$(C_*(X) \otimes C_*(X))_{p+q} = \bigoplus_{i+j=p+q} C_i(X) \otimes C_j(X)$$

where $(i,j) \neq (p,q)$ and our usual sign rule applies.

Using the Alexander-Whitney approximation we get the formula

$$f \cap \sigma = (-1)^{pq} f(\sigma_q)\sigma_p$$

for $f \in C^q(X)$ and $\sigma \in C_{p+q}(X)$ (first backface then frontface).

Remark 2.8.21. Note that one can also define a ‘right’ cup product $\sigma \cap f = f(\sigma_p)\sigma_q$ with first frontface then backface and without signs. This would canonically give a right module structure (as opposed to a left, see next Proposition). But we prefer to work with left modules.

Proposition 2.8.22. *Let $1 : C_0(X) \to \mathbb{Z}$ (also written as ε) be the augmentation which we consider as an element $1 \in C^0(X)$. Then we have*

1. $1 \cap c = c$ *for* $c \in C_q(X)$ *and*
$$1(f \cap c) = \langle 1, f \cap c\rangle = f(c) = \langle f, c\rangle$$
if $|f| = |c|$.

2. $(f \cup g) \cap c = f \cap (g \cap c)$

3. *For* $\varphi : X \to Y$ *and* $f \in C^q(Y)$ *and* $c \in C_{p+q}(X)$ *we have*
$$\varphi_*(\varphi^*(f) \cap c) = f \cap \varphi_*(c)$$

4. *For* $f \in C^*(X)$ *and* $g \in C_*(X)$ *we have*
$$\partial(f \cap c) = df \cap c + (-1)^{|c||f|} f \cap \partial c$$

Proof. 1) We have $1 \cap \sigma = 1(\sigma_0) \cdot \sigma = \sigma$. Similarly for $|f| = |\sigma|$ we find
$$f \cap \sigma = f(\sigma) \cdot \sigma_{|f|}$$
which shows the second part.
2) For $|f| = p$ and $|g| = q$ and $|\sigma| = r + p + q$ we have
$$\begin{aligned}(f \cup g) \cap \sigma &= (-1)^{(p+q)r}(f \cup g)(\sigma_{p+q})\sigma_r \\ &= (-1)^{(p+q)r}(-1)^{pq} f(\sigma_p)g(\sigma_q)\sigma_r \qquad \text{middle, back, first}\end{aligned}$$
Similarly we have that $g \cap \sigma = (-1)^{q(r+p)} g(\sigma_q)\sigma_{r+p}$ and thus find that
$$\begin{aligned}f \cap (g \cap \sigma) &= (-1)^{q(r+p)} g(\sigma_q)(f \cap \sigma_{r+p}) \\ &= (-1)^{q(r+p)} g(\sigma_q)(-1)^{pr} f(\sigma_p)\sigma_r\end{aligned}$$
3) We again assume $c = \sigma$ and get
$$\begin{aligned}\varphi_*(\varphi^*(f) \cap \sigma) &= (-1)^{pq}\varphi_*(\varphi^*(f)(\sigma_q)\sigma_p) \\ &= (-1)^{pq}(\varphi^*(f)(\sigma_q))\varphi_*(\sigma_p) \\ &= (-1)^{pq}(f(\varphi_*(\sigma_q))\varphi_*(\sigma_p) = f \cap \varphi_*(\sigma)\end{aligned}$$
4) We find that
$$\partial(f \cap \sigma) =$$
□

The last boundary formula shows that we have an induced morphism
$$\cap : H^q(X) \otimes H_{p+q}(X) \to H_p(X)$$
which is well defined, since e.g. for f and c cycles it follows that $f \cap c$ is a cycle.

Proposition 2.8.23. *The cap product has the following properties:*

1. *(Unitality)* $1 \cap \gamma = \gamma$
2. *If* $|\alpha| = |\gamma|$ *then* $1(\alpha \cap \gamma) = \langle \alpha, \gamma \rangle$.
3. $(\alpha \cup \beta) \cap \gamma = \alpha \cap (\beta \cap \gamma)$
4. $f : X \to Y$, $\alpha \in H^*(Y)$ *and* $\beta \in H_*(X)$ *then* $f_*(f^*(\alpha) \cap \beta) = \alpha \cap f_*(\beta)$

Proof. Immediate from Proposition 2.8.22. □

The last proposition shows that $H_*(X)$ is a (graded) module over $H^*(X)$ and that $f_* : H_*(X) \to H_*(Y)$ is a $H^*(Y)$-linear map.

Definition 2.8.24. If R^* is a (cohomologically) graded ring. Then a (homologically) graded module M_* is a graded group together with action maps

$$R^p \otimes M_{p+q} \to M_q$$

which are unital and associative. Similarly a differential graded module over a DGA (A^*, d) is a chain complex (M_*, d) together with a chain map

$$(M_*, d) \otimes \Phi(A^*, d)$$

that is associative and unital in the obvious sense.

In general the homology of a differential graded module (A^*, d)-module is a graded $H^*(A)$-module.

Corollary 2.8.25. *The Kronecker product* $\langle -, - \rangle : H^p(X) \otimes H_p(X) \to R$ *satisfies:*

1. $\langle \alpha \cup \beta, \gamma \rangle = \langle \alpha, \beta \cap \gamma \rangle$
2. $\langle f^*(\alpha), \gamma \rangle = \langle \alpha, f_*(\gamma) \rangle$.

Moreover we have

$$(\alpha \times \beta) \cap (\gamma \times \delta) = (-1)^{|\beta||\gamma|} (\alpha \cap \gamma) \times (\beta \cap \delta)$$

Proof. Follows directly from the fact that the Kronecker pairing is a special case of the cap product (by Proposition 2.8.22) and the relation of the cap product:

$$\langle \alpha \cup \beta, \gamma \rangle = \langle 1, (\alpha \cup \beta) \cap \gamma \rangle = \langle 1, \alpha \cap (\beta \cap \gamma) \rangle = \langle \alpha, \beta \cap \gamma \rangle$$

and

$$\langle f^*(\alpha), \gamma \rangle = \langle 1_X, f^*(\alpha) \cap \gamma \rangle = \langle 1_Y, \alpha \cap f_*(\gamma) \rangle = \langle \alpha, f_*(\gamma) \rangle$$

And

$$(\alpha \times \beta) \cap (\gamma \times \delta) = ...$$

□

Example 2.8.26. For a sphere S^n we denote the generator in homology of degree n by $[S^n]$ and the generator in cohomology by α. Then we get that

$$\alpha \cap [S^n] = 1$$

where $1 \in \mathbb{Z} = H_0(S^n)$ is the basepoint. Thus. as a module, $H_*(S^n)$ over $H^*(S^n) = \mathbb{Z}[\alpha]/\alpha^2$, the homology is isomorphic to the cohomology and the isomorphism is implemented by

$$- \cap [S^n] : H^k(S^n) \to H_{n-k}(S^n)$$

We will see that this is is true for every oriented manifold (Poincaré duality).

Finally let us remark that there is also a relative version of the cap product for $A, B \subseteq X$ we have

$$\cap : H^p(X, A) \otimes H_{p+q}(X, A \cup B) \to H_q(X, B)$$

which has the same properties as the cap product. In particular for $A = \emptyset$ this shows that $H_*(X, B)$ is a module over $H^*(X)$.

2.9 Orientations revisited

Recall from Topoiogy I that an orientation of a manifold M is defined to be an (equivalence class) oriented atlas.

> Question: How many orientations exist on M? How can we know that a manifold is not orientable?

The idea is to translate oriented charts into homological data.

Remark 2.9.1. A local orientation of a top. manifold M in $m \in M$ is a choice of a generator $\mathfrak{o}_m$ of $H_n(M, M \setminus \{m\})$. Every chart $\varphi : U \to \mathbb{R}^n$ around m induces such a local orientation $\mathfrak{o}_m(\varphi)$ since it induces an isomorphisms

$$H_n(M, M \setminus \{m\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{\varphi(m)\}) \cong \mathbb{Z}$$

Then $\mathfrak{o}_M(\varphi)$ is the element corresponding to $1 \in \mathbb{Z}$ under this isomorphism. Conversely every local orientation $\mathfrak{o}_m$ singles out a class of charts around m which we call oriented, namely those φ with $\mathfrak{o}_m(\varphi) = \mathfrak{o}_m$.

We want to assemble these charts together to an oriented atlas. Therefore we need a local orientation in every point. But these local orientations have to depend continuously on the point, they can not be allowed to jump. We want to make this precise now.

Definition 2.9.2. Let M be a topological manifold of dimension n and G an abelian group. For $x \in M$ set $\Theta_x \otimes G := H_n(M, M \setminus \{x\}; G) \cong H_n(M, M \setminus \{x\}; \mathbb{Z}) \otimes G$. Set

$$\Theta_M \otimes G := \coprod_{x \in M} \Theta_x \otimes G$$

and let $p : \Theta_M \otimes G \to M$ be the map that takes $\Theta_x \otimes G$ to $\{x\}$. For $G = \mathbb{Z}$ we just write Θ. We call Θ the orientation bundle (with values in G).

Give $\Theta_M \otimes G$ the following topology: Let $U \subseteq M$ be open and $\alpha \in H_n(M, M \setminus U)$. Assembling together the images under

$$H_n(M, M \setminus U) \to H_n(M, M \setminus \{x\})$$

defines a mapping $s_\alpha : U \to \Theta_M \otimes G$. We give $\Theta_M \otimes G$ the finest topology such that all s_α (for all choices of α) become continuous.

Proposition 2.9.3. *The images $U_\alpha := s_\alpha(U)$ for all $U \in M$ and $\alpha \in H_n(M, M \setminus U; G)$ form a basis of the topology. Moreover $p : \Theta \otimes G \to M$ is a covering map and fibrewise addition is continuous.*

Proof. Let $U \subseteq M$ be a convex ball in some coordinate neighborhood. Then by excision and the long exact sequence the morphism

$$\varphi_x : H_n(M, M \setminus U) \to H_n(M, M \setminus \{x\})$$

is an isomorphism for every $x \in U$. Thus we have an isomorphism

$$\begin{tikzcd} U \times H_n(M, M \setminus U; G) \arrow[rr, "\varphi"] \arrow[dr] & & p^{-1}(U) \arrow[dl] \\ & U & \end{tikzcd}$$

We equip $H_n(M, M \setminus U; G)$ with the discrete topology. Then this morphism is by definition of the topology continuous. It is also bijective and it is straightforward (but a bit annoying) to check that it is a homeomorphism.

Since every point $x \in M$ is contained in some convex coordinate ball this shows that the morphism $p : \Theta_M \to M$ is locally a product with a discrete sets (namely the set $\mathbb{Z}$) and therefore a covering map. The statement about the basis of the topology can easily be derived from that. The fibrewise addition corresponds to the addition on the product factor and is therefore also continuous. $\square$

Definition 2.9.4. The orientation cover of M is the subspace $\hat{M} \subset \Theta$ consisting of all elements $H_n(M, M \setminus x)$ that are generators.

Corollary 2.9.5. *The orientation cover $\hat{M} \to M$ is a 2-sheeted covering of M (with not necessarily connected total space). Hence for connected M it is classified by a group homomorphism $w_1 : \pi_1(M) \to \mathbb{Z}/2$ which we call the orientation character.*

Proof. Use the same proof as the the last Proposition to see that locally it is reduced to the generators. For the covering statement note that in general 2 sheeted coverings are described by a index 2-subgroup of the fundamental group which in turn is the same as a group homomorphism to $\mathbb{Z}/2$ (since all index 2 subgroups are normal). $\square$

Remark 2.9.6. Note that the orientation character can equivalently be described as a group homomorphism $w_1 : H_1(M) \to \mathbb{Z}/2$ since the group $\mathbb{Z}/2$ is abelian and hence by the universal property of the abelianization. This is also the same as a cohomology class $H^1(M; \mathbb{Z}/2)$. Also note that there is unique non-trivial covering transformations $\tau : \hat{M} \to \hat{M}$.

Definition 2.9.7. A section of a covering $p : N \to M$ is a morphism $s : M \to N$ such that $ps = \mathrm{id}$. For a closed subspace $A \subset M$ we denote by

$$\Gamma(A, \Theta_M \otimes G) = \{s : A \to \Theta_M \otimes G \mid ps(a) = a\}$$

the groups of sections of the orientation bundle (with pointwise addition) and by

$$\Gamma_c(A, \Theta_M \otimes G) \subset \Gamma(A, \Theta_M \otimes G)$$

the subgroup of sections with compact support, i.e. those s for which the set $\overline{\{x \in A \mid s(x) \neq 0\}}$ is compact.

Lemma 2.9.8. *For a connected manifold M the following are equivalent :*

1. *The orientation character w_1 is trivial.*
2. *The orientation cover $\hat{M}$ admits a section.*
3. *We have a homeomorphism $\hat{M} \to M \sqcup M$ that commutes with the maps to M.*
4. *We have a homeomorphism $\Theta_M \cong M \times \mathbb{Z}$ that commutes with the morphisms to M and the group structures.*
5. *For every abelian group G we have a homeomorphism $\Theta_M \otimes G \cong M \times G$ that commutes with the morphisms to M and the group structures.*

Proof. The first three assertions are equivalent for every 2-fold covering by standard covering theory. To see the equivalence of (3) and (4) note that if we have a homeomorphism

$$\phi : \hat{M} \cong M \amalg M \cong M \times \{-1, 1\}$$

we can extend it to a homeomorphism

$$\Phi : \Theta_M \to M \times \mathbb{Z}$$

by setting $\Phi(x) = \Phi(ny) = n\phi(y)$ for a generator $y \in H_n(M, M \setminus \{m\})$. This is easily seen to be well-defined and commutes with the group structures. Conversely assume we have such a Φ. Then its restriction to the subset $\hat{M} \subset \Theta_M$ has image $M \times \{-1, 1\}$ (since these are the generators). Thus it induces the desired equivalence.

To go from (4) to (5) we tensor the equivalence fibrewise with G. Conversely (4) is clearly a special case of (5). □

Theorem 2.9.9. *There is a 1-1 correspondence between orientations on M and sections of the orientation cover.*

Proof. We describe two explicit maps

$$\{\text{Oriented atlases of } M\} \longleftrightarrow \{\text{sections of } \hat{M} \to M\}$$

Given an orientation $[\mathcal{A}]$ send it to the section $s_{\mathcal{A}}$ defined as follows: for $m \in M$ choose a chart $\varphi : U \to \mathbb{R}^n$ from $\mathcal{A}$ and set $s_{\mathcal{A}}(m) := \mathfrak{o}_M(\varphi) \in H_n(M, M \setminus \{m\})$. This is well-defined since for another chart φ' the transition function is orientation preserving and thus leads to the same generator $\mathfrak{o}_M(\varphi')$. Now we have to show that $s_{\mathcal{A}}$ is continuous. For this it sufficed to proof this locally. But for a chart $\varphi : U \to \mathbb{R}^n$ choose an open $U' \subset U$ such that $\overline{U'} \subset U$ and such that $\varphi(U')$ is a ball. Then we can describe the restriction $s_{\mathcal{A}} |_{U'}$ as the section $s_\alpha : U' \to \hat{M} \subset \Theta_M$ corresponding to the element $\alpha \in H_n(M, M \setminus U')$ which maps to 1 under the isomorphism

$$H_n(M, M \setminus U') \cong H_n(U, U \setminus U') \cong H_n(U, U \setminus U') \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus B) \cong \mathbb{Z}$$

Thus by definition of the topology on Θ_M the restriction $s_{\mathcal{A}} |_{U'}$ is continuous and therefore a section of $\hat{M} \to M$.

The converse construction works as follows: for a given section $s : M \to \hat{M}$ we define an atlas $\mathcal{A}_s$ a follows: $\mathcal{A}_s$ contains all charts

$$\varphi : U \to \mathbb{R}^n$$

such that U is connected and for one $m \in U$ we have that $\mathfrak{o}_m(\varphi) = s(m)$. First note that this already implies that we have $\mathfrak{o}_m(\varphi) = s(m)$ for all $m \in U$ since U is connected and since both sides are sections of a covering. We have to check that $\mathcal{A}_s$ defines an atlas. Thus for every $m \in M$ there has to be a chart in $\mathcal{A}_s$ around m. Pick an arbitrary chart φ around m of the manifold M (which is not necessarily in $\mathcal{A}_s$). Make a new chart φ' by postcomposing with an orientation reversing homeomorphism $\mathbb{R}^n \to \mathbb{R}^n$. Then we have $\mathfrak{o}_m(\varphi') \neq \mathfrak{o}(\varphi)$. But since the group $H_n(M, M \setminus \{m\})$ has only two generators one of the two elements has to agree with $s(m)$ and hence the corresponding chart is in $\mathcal{A}_s$. To see that the atlas $\mathcal{A}_s$ is oriented we use a similar straightforward argument.

Now it remains to show that the two constructions are inverse to each other. First to see that $s_{\mathcal{A}_s} = s$ we reduce this to a local chart, where it follows immediately from the local descriptions given in the first part of the proof. For the converse, i.e. that $\mathcal{A}_{s_{\mathcal{A}}}$ is equivalent to $\mathcal{A}$ one has to check that the transition functions are orientable, which is immediate unwinding the definitions. □

Corollary 2.9.10. *If a manifold M is orientable and connected, then it has precisely two orientations. Moreover if two orientations agree in one point (meaning that the induced local orientations in that point agree) then they agree globally. In other words: an orientation is uniquely determined by a single chart resp. a single local orientation.*

Example 2.9.11. We have already stated that the genus g-surface is orientable (we will see an independent proof soon). Recall that the genus g-surface was defined as a quotient $P_g/\sim$ of the regular 4g-gon. This gives as a canonical chart

$$\mathrm{int} P_g \to \Sigma_g$$

which we want to be oriented. This uniquely fixes the orientation.

Corollary 2.9.12. *If a manifold M is simply connected or if $H_1(M, \mathbb{Z})$ contains no subgroups of index 2 then it is orientable.*

Proof. By Lemma 2.9.8 and Theorem 2.9.9 M is orientable precisely if $w_1 : H_1(M) \to \mathbb{Z}/2$ is trivial, which is automatically the case if there is no subgroup of index 2. □

Example 2.9.13. We can now directly see (without having to write down an oriented atlas) that the following manifolds are all orientable: S^n for $n \geq 2$, $\mathbb{C}P^n$, $L(p,q)$ for p not divisible by 2. Recall that we have $\pi_1(L_{p,q}) = H_1(L_{p,q}) = \mathbb{Z}/p$.

Again all of these spaces have preferred orientations which are described using (one) preferred chart. Also products of these manifolds are orientable, e.g. $S^2 \times S^3$.

The case of $\mathbb{R}P^n$ (at least for n odd) still remains open (since $H_1(\mathbb{R}P^n) \cong \mathbb{Z}/2$).

Corollary 2.9.14. *The total space of the orientation cover $\hat{M}$ is a manifold that is canonically oriented. The involution τ is orientation reversing.*

Proof. Exercise. □

2.10 Fundamental class I

For $A \subseteq M$ closed define a homomorphism

$$J_A : H_n(M, M \setminus \{A\}; G) \to \Gamma_c(A, \Theta_M \otimes G) \tag{2.1}$$

as before (it sends $\alpha \in H_n(M, M \setminus \{A\}; G)$ to the section $s_\alpha : A \to \Theta_M \times G$). We must show that this section is continuous and has compact support. Continuity can be checked locally i.e. for opens $U \subset A$ with $U = V \cap A$. Then $\alpha\mid_U \in H_n(M, M \setminus U)$ can be extended to $H_n(M, M \setminus V)$ for U and V sufficiently small. Thus we have that s_α is the restriction of the section induced form the extension. The fact that the section has compact support follows from the fact that every representative of α in $C_n(M, M \setminus A)$ factors through some compact subset $B \subset M$ since Δ^n is compact. Then the section s_α has also support in B.

Theorem 2.10.1. *Let M be a topological manifold of dimension n. Then the homology groups $H_k(M; G)$ are zero for $k > n$ and the canonical morphism 2.1 is a group isomorphism*

$$H_n(M; G) \cong \Gamma_c(M, \Theta_M \otimes G)$$

Proof. We give the proof for $G = \mathbb{Z}$ and the other cases works exactly similar. For a topological manifold M with a closed subset A we will prove the following statement:

> $S(M, A)$: $H_k(M, M \setminus A) = 0$ for all $k > \dim M$ and $J_A : H_n(M, M \setminus \{A\}) \to \Gamma_c(A, \Theta_M)$ is an isomorphism.

The statement of the theorem that we want to prove is $S(M, M)$. We observe that we have:

1. $S(M, A)$ holds for A compact and convex in some coordinate chart of M.
2. If $S(M, A_1), S(M, A_2)$ and $S(M, A_1 \cap A_2)$ hold, then also $S(M, A_1 \cup A_2)$.
3. For $A_1 \supseteq A_2 \supseteq A_3 \supseteq ...$ compact subsets of M such that $S(M, A_i)$ holds, then also $S(M, \cap A_i)$.
4. If $A_i \subseteq M$ are compact with disjoint open neighborhoods such that $S(M, A_i)$ holds, then also $S(M, \cup A_i)$ holds.

5. If $S(A \cap N, N)$ holds for all $N \subseteq M$ open relatively compact[2] then also $S(A, M)$.

Let us prove these statements and note that these statements together with the following Lemma (Bootstrap Lemma) will imply our Theorem.

ad1) We have that

$$H_n(M, M \setminus A) = H_n(\mathbb{R}^n, \mathbb{R}^n \setminus A) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{a\})$$

for some $a \in A$ where the last statement follows since A is convex and thus we can simply find a homotopy equivalence. But this map just evaluates the section $J_A(-)$ at the point a which is an isomorphism

ad2) We consider the commutative diagram

$$\begin{tikzcd}
H_{n+1}(M, M \setminus (A \cap B)) \arrow[r] \arrow[d] & H_n(M, M \setminus (A \cup B)) \arrow[r] \arrow[d] & H_n(M, M \setminus A) \oplus H_n(M, M \setminus B) \arrow[r] \arrow[d] & H_n(M, M \\
0 \arrow[r] & \Gamma_c(A \cup B, \Theta_M) \arrow[r] & \Gamma_c(A, \Theta_M) \oplus \Gamma_c(B, \Theta_M) \arrow[r] & \Gamma_c(A \cap
\end{tikzcd}$$

where the upper row is the Mayer-Vietoris sequence and the lower row is the obvious sequence. Both are exact so the conclusion follows by the 5-Lemma (also the higher cases).

ad3) We have that

$$H_n(M, \cap A_i) = \operatorname{colim} H_n(M, A_i)$$

and

$$\Gamma_c(A, \Theta_M) = \operatorname{colim} \Gamma_c(A_i, \Theta_M)$$

which together imply the claim.

ad4) We choose disjoint neighborhoods $A_i \subseteq U_i$ and find that

$$H_*(M, M \setminus \cup A_i) = H_*(\coprod U_i, \coprod U_i \setminus \coprod A_i) = \oplus H_*(U_i, \coprod A_i)$$

The analogous statement also holds for Γ_c.

ad5) We have that

$$H_p(M, M \setminus A) = \operatorname{colim} H_p(N, N \setminus (N \cap A))$$

and similarly for Γ_c. □

Lemma 2.10.2. *(Bootstrap Lemma) Let $S(M; A)$ be a statement about pairs (A, M) where M is an n-manifold and A a closed subset. If (1)-(5) hold then $S(M, A)$ holds for all A and M.*

Proof. From (1)-(2) we conclude that $S(M, A)$ holds for all finite unions of convex neighborhoods contained in some given coordinate chart (by induction using (2) and the formula $A_1 \cap (A_2 \cup .. \cup A_n) = (A_1 \cap A_2) \cup ...(A_1 \cap A_n)$). Then every compact subset in that given chart can be written as the intersection of such unions $A_1 \supseteq A_2 \supseteq A_3 \supseteq ...$ so that by (3) the statement follows for every compact A which lies in a coordinate neighborhood. Now we claim that any given compact subset A can be written as a finite union of such: this can be seen by covering A by open set $U_i \subseteq M$ that have the property that the $\overline{U_i}$ are still contained

[2] That is, subspaces with compact closure

in a coordinate neighborhood of M. For example take coordinate neigborhoods U_i' and let U_i be open balls in those. Then A can by compactness be covered by finitely many of those. The intersections of A with $\overline{U_i}$ are still compact and finitely many cover A.

Now assume that M is second countable (or equivalently, separable, i.e. there exists a dense countable subset). In that case there exists a map $f : M \to \mathbb{R}_{\geq 0}$ which is proper, i.e. preimages of compact subsets are compact. This follows from metrizability results, but more elementarily from playing with partitions of unity. We let $A_i := A \cap f^{-1}[2i-1, 2i]$ and $B_i := A \cap f^{-1}[2i-2, 2i-1]$. Now the A_i's are compact and satisfy (4) so that the result follows for the union A and similar for B. It also holds for the intersection $A \cap B$ which is the union of $A_i \cap B_j$. Thus by an application of (2) the statement $S(A, M)$ follows for $A \subseteq M$ closed.

Finally, we observe that if $N \subseteq M$ is open and relatively compact (i.e. has compact closure), it is separable, and thus by the previous observation, $S(A \cap N, N)$ holds for arbitrary closed $A \subseteq M$. So (5) shows $S(A, M)$ for arbitrary M.

□

This implies all the results about orientability we had before, e.g Theorem 1.17.16 which states that

$$H_n(M, G) \cong \begin{cases} G & \text{if } M \text{ is compact and orientable} \\ G[2] & \text{if } M \text{ is compact and not orientable} \\ 0 & \text{of } M \text{ is not compact} \end{cases}$$

for a connected manifold and its consequences (such as invariance of dimension and non-orientability for $\mathbb{R}P^n$ with n odd. Now we can also make Definition 1.17.19 more rigorous: for a compact oriented manifold M the class $[M] \in H_n(M, \mathbb{Z})$ corresponding to the orientation section $\Gamma(M, \Theta_M) = \Gamma(M, \Theta_M)$ is called the fundamental class.

Also note that every compact connected manifold admits a $\mathbb{Z}/2$ fundamental class, since $\Theta_M \otimes \mathbb{Z}/2$ is necessarily trivial (pick nonzero point in each fiber).

We will more generally talk about R-oriented manifolds in what follows. Here an R-orientation is a section of $\Theta_M \otimes R$. Typically R will be $\mathbb{Z}$ or $\mathbb{Z}/2$. A $\mathbb{Z}$-orientation is an orientation as before, and any manifold has a unique $\mathbb{Z}/2$-orientation.

2.11 Poincaré duality and Applications

Our next task is to understand Poincaré duality. Now that we have the fundamental class we can state the result:

Theorem 2.11.1 (Poincaré duality). *Let M be a compact, connected, R-oriented manifold. Then for every R-module G the map*

$$- \cap [M] : H^*(M; G) \to H_{n-*}(M; G)$$

is an isomorphism.

For nonorientable manifolds, we still have a version

For $* = 0$ and $* = n$ this in particular implies that $H_n(M) \cong \mathbb{Z}$ and $H^n(M) \cong \mathbb{Z}$. Our strategy of proof will be to first generalize this to a statement that holds also for non-compact manifolds and then prove it similar to the proof of 2.10.1. This will be implemented in the next section, but first we want to collect some important consequences.

Proposition 2.11.2 (Poincare). *There is a compact 3-dimensional manifold which has the same homology groups as the 3-sphere but which is not simply connected, in particular not homotopy equivalent to* S^3.

Proof. Consider the group $G \subseteq SO(3)$ of rotation symmetries of the regular icosahedron.This is isomorphic to the group A_5, the alternating group on 5 letters which is a simple group. Thus $[G, G] = G$. Consider the universal covering $SU(2) \to SO(3)$ which is a 2-1 covering (think of $SU(2)$ as the unit quaternions and let them act on the imaginary part). Then we consider the preimage $G' \subseteq SU(2) = S^3$ of G under the projection. We have a short exact sequence

$$\{\pm 1\} \to G \to G'$$

We claim that minus the identity matrix -1 is contained in the commutator subgroup of G'. This follows for example from the observation that along $SU(2) \to SO(3)$, *every* lift of any order 2 element has order 4 and squares to $-1 \in SU(2)$. In particular, writing a π-rotation in G as a commutator, we can write $-1 \in SU(2)$ as the square of a commutator. We see that $[G', G'] = G'$. Define the manifold

$$M := SU(2)/G' = S^3/G'$$

Then we get that $H_1(M) = \pi_1(M)^{ab} = (G')^{ab} = 0$ and thus by UCT $H^1 = 0$ which by Poincaré dualtiy implies the claim. $\square$

Remark 2.11.3. The first version of the Poincare conjecture was that each 3-manifold which has the homology of the 3-sphere is already homeomorphic to the 3-sphere. The above example shows that this can not be true. Poincare conjecture: each compact simply connected, 3-manifold is homeomorphic to the 3-sphere. Note that if M is such a simply connected, compact 3-manifold we first conclude that $H_1(M) = 0$ and $H^1(M) = 0$ and that M is orientable. Then by Poincare duality it follows that $H_2(M) = H^2(M) = 0$. Thus M is a homology sphere, i.e. it has the same homology as the sphere. It is also not hard to see that it is already homotopy equivalent to the 3-sphere. But the hard part is that it is homeomorphic to the 3-sphere. Thats a deep theorem of Perelman.

It is even true more generally that every compact, connected manifold M of dimension n that is homotopy equivalent to S^n (equivalently $\pi_1(M) = ... = \pi_{\lfloor n/2 \rfloor}(M) = 0$) is already homeomorphic to S^n. This is called the generalized Poincare conjecture. This was proven for $n = 4$ by Freedman and for $n \geq 5$ by Smale.

It is sometimes desirable to have a version of Poincare duality entirely in terms of cohomology.

Definition 2.11.4. Let M be a compact R-oriented n-dimensional manifold. The bilinear form

$$\cup : H^k(M; R) \otimes H^{n-k}(M; R) \to R$$

given by $\alpha \otimes \beta \mapsto \langle \alpha \cup \beta, [M] \rangle$ is called the *intersection form.*

Recall R a PID we have $TH^k(M;R) \subseteq H^k(M;R)$ torsion submodule, set $\overline{H}^k(M;R) := H^k(M;R)/TH^k(M;R)$ torsion free part. For a field R the torsion submodule is zero, thus everyting is free. For M a compact manifold we have a canonical isomorphism

$$\overline{H}^k(M;R) \cong \mathrm{Hom}(\overline{H}_k(M);R)$$

as a consequence of the Universal coefficents theorem

$$H^n(M,R) \cong \mathrm{Ext}_R(H_{n-1}(M);R) \oplus \mathrm{Hom}(H_n(M);R)$$

A bilinear pairing $M \otimes_R N \to R$ is called perfect, if the induced morphisms

$$M \to \mathrm{Hom}_R(N;R) \qquad N \to \mathrm{Hom}_R(M;R)$$

are isomorphisms.

Theorem 2.11.5. *For a PID, let M be a compact, connected, R-oriented n-manifold . Then the pairing*

$$\cup : \overline{H}^k(M;R) \otimes \overline{H}^{n-k}(M;R) \to R$$

induced by the intersection form is perfect.

Proof. By properties of the cap product,

$$\langle \alpha \cup \beta, [M] \rangle = \langle \alpha, \beta \cap [M] \rangle,$$

so the pairing

$$\overline{H}^k(M;R) \otimes \overline{H}^{n-k}(M;R) \to R$$

corresponds to the pairing

$$\overline{H}^k(M;R) \otimes \overline{H}_k(M;R) \to R,$$

which we saw to be perfect above. □

Corollary 2.11.6. *For a closed (i.e. compact without boundary) manifold the pairing*

$$\cup : H^k(M,\mathbb{Z}/2) \otimes H^{n-k}(M;\mathbb{Z}/2) \to \mathbb{Z}/2$$

is perfect. If M is oriented then

$$\cup : H^k(M,\mathbb{Q}) \otimes H^{n-k}(M;\mathbb{Q}) \to \mathbb{Q}$$

is perfect. If M in addition has torsion free homology then

$$\cup : H^k(M) \otimes H^{n-k}(M) \to \mathbb{Z}$$

is perfect.

Remark 2.11.7. If we compose the intersection form with the Poincare duality isomorphism then we obtain the *homological intersection pairing*

$$\bullet : H_{n-k}(M;R) \otimes H_k(M;R) \to R$$

given by

$$\alpha \otimes \beta \mapsto \langle ((- \cap [M])^{-1}\alpha \cup (- \cap [M])^{-1}\beta, [M]\rangle = \langle (- \cap [M])^{-1}(\alpha), \beta\rangle$$

Historically this pairing came before the cup product. It has the following geometric interpretation and that is why it is called intersection pairing: assume M is smooth and the classes α and β are the fundamental classes of oriented manifolds $f : N_\alpha \to M$ and $g : N_\beta \to M$. Here we also assume that f and g are smooth, i.e. $\alpha = f_*(N_\alpha)$ and $\beta = g_*(N_\beta)$. We get that $\dim N_\alpha + \dim N_\beta = \dim M$. Now assume that f and g are transversal to each other which means that $f(N_\alpha) \cap g(N_\beta)$ consists of finitely many points and that for each $x = f(y) = g(z)$ we have that $T_y N_\alpha \oplus T_z N_\beta \simeq T_x M$. We define $\epsilon(X)$ to be 1 if that isomorphism is compatible with orientations and -1 if it is not. Then we have

$$\alpha \bullet \beta = \sum_{x \in f(N_\alpha) \cap g(N_\beta)} \epsilon(x)$$

Theorem 2.11.8. *The cohomology rings of the projective spaces are given as follows (where $n = \infty$ is allowed))*

$$\begin{aligned} H^*(\mathbb{R}P^n; \mathbb{Z}/2) &\cong \mathbb{Z}/2[\alpha]/\alpha^{n+1} && |\alpha| = 1 \\ H^*(\mathbb{C}P^n; \mathbb{Z}) &\cong \mathbb{Z}[x]/x^{n+1} && |x| = 2 \\ H^*(\mathbb{H}P^n; \mathbb{Z}) &\cong \mathbb{Z}[y]/y^{n+1} && |y| = 4 \end{aligned}$$

Proof. We do $\mathbb{C}P^n$, the rest works equivalently. We have to show that for each k, x^k is a generator of $H^{2k}(\mathbb{C}P^n; \mathbb{Z})$. For $k < n$, this follows inductively by restricting along the map

$$H^{2k}(\mathbb{C}P^n; \mathbb{Z}) \to H^{2k}(\mathbb{C}P^{n-1}; \mathbb{Z}),$$

which is an isomorphism by cellular cohomology. For $k = n$, we use that the cohomology intersection pairing is nondegenerate to see that $x^{n-1} \cup x$ is a generator of $H^{2n}(\mathbb{C}P^n)$. □

Corollary 2.11.9. *The space $\mathbb{C}P^2$ is not homotopy equivalent to $S^2 \vee S^4$. In particular the attaching map $\eta : S^3 \to \mathbb{C}P^1 \cong S^2$ is not homotopic to the trivial map.*

This is the first element in $\pi_3(S^2) = [S^3, S^2]_*$ which is not trivial. This is somewhat surprising since by analogy with homology one could expect this homotopy group to be zero. Similarly we non-nullhomotopic elements in $\pi_1(S^1)$ and $\pi_7(S^4)$ as the attaching maps of $\mathbb{R}P^2$ and $\mathbb{H}P^2$.

We can try to imitate this construction.

Definition 2.11.10. Let $\varphi : S^{2n-1} \to S^n$ be a continuous map for $n \geq 2$. We consider the CW complex

$$C_\varphi := S^n \cup_\varphi D^{2n}$$

(i.e. it has one n-cell and one $2n$-cell attached through φ). We have $H^*(C_\varphi) = \mathbb{Z}$ for $* = 0, n, 2n$ and zero else. We denote the generator in degree n be α and in degree $2n$ by β. The *Hopf* invariant of φ is the integer $h = h(\varphi)$ such that $\alpha \cup \alpha = h\beta$.

Note that for n odd the Hopf-invariant is zero. In particular none of these CW complexes C_φ can not be (homotopy equivalent to) manifolds.

Remark 2.11.11.

1. One can show that h is a group homomorphism $\pi_{2n-1}(S^n) \to \mathbb{Z}$ which for n even contains $2\mathbb{Z}$ in the image.

2. It is a famous problem in algebraic topology which maps can have Hopf invariant one (the Hopf invariant 1 problem). The answer is as proved first by Adams (1960) that this can only happen for the $n = 2, 4, 8$ and the attaching maps of the complex, quaternionic and octonionic projective planes (and $n = 1$ if sufficiently interpreted).

Proposition 2.11.12. *Suppose M and N are closed topological manifolds of dimension n and $f\colon M \to N$ is a map of degree ± 1. Then the map*

$$H^*(N;\mathbb{Z}) \xrightarrow{f^*} H^*(M;\mathbb{Z})$$

is split injective and the map

$$H_*(M;\mathbb{Z}) \xrightarrow{f_*} H_*(N;\mathbb{Z})$$

is split surjective.

Proof. We have $f_*[M] = [N]$. So

$$f_*(f^*(\beta) \cap [M]) = \beta \cap [N].$$

We get the diagram

$$\begin{tikzcd} H^*(M) \arrow[d, "-\cap[M]"] & H^*(N) \arrow[l, "f^*"'] \arrow[d, "-\cap[N]"] \\ H_{n-*}(M) \arrow[r, "f_*"] & H_{n-*}(N) \end{tikzcd}$$

In particular, since the right vertical map is an isomorphism, the claim follows. ☐

Corollary 2.11.13. *Suppose $f\colon S^n \to M$ is a map of degree 1. Then $M \simeq S^n$.*

Definition 2.11.14. Let M be a closed connected oriented topological manifold of dimension $4n$. Recall that Poincaré duality implies that

$$\begin{aligned} H^{2n}(M;\mathbb{Z})/\mathrm{tors} \otimes H^{2n}(M;\mathbb{Z})/\mathrm{tors} &\longrightarrow \mathbb{Z} \\ x \otimes y &\longmapsto \langle x \cup y, [M]\rangle \end{aligned}$$

is a symmetric unimodular (also called perfect or non-degenerate) bilinear form which is called the *intersection form* the manifold M and denoted by λ_M. We can tensor this up to $\mathbb{R}$ to obtain a symmetric unimodular form $\lambda_M \otimes \mathbb{R}$ over $\mathbb{R}$. By Sylvester's theorem any such form is equivalent to a form described by a diagonal matrix with only 1's and -1's as non-zero entries (the number of 1's equals the number of positive eigenvalues of $\lambda_M \otimes \mathbb{R}$ and the number of -1's equals the number of negative eigenvalues of $\lambda_M \otimes \mathbb{R}$.) We define the *signature* of a symmetric unimodular form λ over $\mathbb{R}$ to be the integer

$$\sigma(\lambda) = \sharp \text{ positive eigenvalues} - \sharp \text{ negative eigenvalues}$$

and the signature of a manifold M of dimension $4n$ by

$$\sigma(M) = \sigma(\lambda_M \otimes \mathbb{R}) \in \mathbb{Z}.$$

In order to compute the signature of a manifold we thus a priori need to know about the intersection form.

Example 2.11.15.

$$\begin{aligned}
&\sigma(\mathbb{C}P^{2n}) = 1\\
&\sigma(S^{4n}) = 0\\
&\sigma(S^{2n} \times S^{2n}) = 0.
\end{aligned}$$

Lemma 2.11.16. *Suppose $f\colon M \to N$ is a homotopy equivalence between oriented closed $4n$-manifolds. Then $\sigma(M) = \sigma(N)$ if f is orientation preserving (i.e. of degree $+1$), and $\sigma(M) = -\sigma(N)$ if f is orientation reversing (i.e. of degree -1).*

Proof. Clear □

Proposition 2.11.17. *Let M and N be oriented and compact as always. Then*

$$\lambda_{M \,\sharp\, N} = \lambda_M \oplus \lambda_N$$

in particular we have

$$\sigma(M \sharp N) = \sigma(M) + \sigma(N)$$

Proof. The map $M \sharp N \to M \vee N$ induces an isomorphism on cohomology of the middle dimension $2n$ and the canonical surjection $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ in degree $4n$. The result follows. □

Corollary 2.11.18. $\mathbb{C}P^2 \sharp \mathbb{C}P^2$ *is not homotopy equivalent to* $\mathbb{C}P^2 \sharp \overline{\mathbb{C}P^2}$.

We will now see that in general to compute the signature one needs to know the intersection form. But if one only wants to show that the signature is zero, one can make arguments without computing the full intersection form:

Proposition 2.11.19. *Suppose that there exists a self-map $\varphi\colon M \to M$ with* $\deg(\varphi) = -1$. *Then $\sigma(M) = 0$.*

Proof. This is an orientation-preserving equivalence between M and $\overline{M}$. Thus, $\sigma(M) = \sigma(\overline{M}) = -\sigma(M)$. □

Corollary 2.11.20. *We have that $\sigma(T^{4n}) = 0$.*

In general we have the following

Lemma 2.11.21. *Let λ be a symmetric non-degenerate form on an $\mathbb{R}$-module V. Then $\sigma(\lambda) = 0$ if and only if there exists a* Lagrangian *for λ, i.e. a submodule $L \subseteq V$ such that $\lambda|_L = 0$ and $2 \cdot \dim(L) = \dim(V)$.*

Proof. Pick an orthogonal decomposition $V_+ \oplus V_-$ into spaces where λ is negative definite resp. positive definite. Now decompose a $v \in L$ accordingly as $v_+ + v_-$.

$$0 = \lambda(v, v) = \lambda(v_+, v_+) + \lambda(v_-, v_-).$$

If $v_+ = 0$, then this shows that $v_- = 0$. So the projections $L \to V_+$ and $L \to V_-$ are both injective, and $\dim(L) \le \min\{\dim V_+, \dim V_-\} \le \frac{1}{2} \dim V$, with equality only possible if $\sigma(\lambda) = \dim V_+ - \dim V_- = 0$. Vice versa, if $\dim V_+ = \dim V_-$, there is an isometry φ between the inner product spaces (V_+, λ_+) and $(V_-, -\lambda_-)$, and we let L be its graph $\{(x, \varphi(x)\}$. Now

$$\lambda((x, \varphi(x)), (y, \varphi(y))) = \lambda_+(x, y) + \lambda_-(\varphi(x), \varphi(y)) = 0.$$

□

Using this we can produce nice topological criteria for vanishinig signatures:

Theorem 2.11.22. *Suppose M is a $4n-1$-dimensional closed oriented manifold and $\varphi\colon M \to M$ is an orientation preserving homeomorphism. Then we denote by T_φ the mapping torus of φ. It is an $4n$-dimensional closed orientable manifold and $\sigma(T_\varphi) = 0$.*

Proof. Here, we consider the long exact sequence associated to the pair (T_φ, M). It takes the form

$$\dots \longrightarrow H^{2n-1}(M) \xrightarrow{\delta} H^{2n}(T_\varphi, M) \longrightarrow H^{2n}(T_\varphi) \longrightarrow H^{2n}(M) \xrightarrow{\delta} H^{2n+1}(T_\varphi, M) \longrightarrow \dots$$

We can identify the maps and groups by considering the map of pairs $(M \times [0,1], M \times \{0,1\}) \to (T_\varphi, M)$. It induces the diagram

$$\begin{tikzcd}
H^k(M) \arrow[r, "\delta"] \arrow[d, "(\mathrm{id},\varphi^*)"] & H^{k+1}(T_\varphi, M) \arrow[d, "\cong"] \\
H^k(M \times \{0,1\}) \arrow[r, "\delta"] \arrow[d, "\cong"] & H^{k+1}(M \times [0,1], M \times \{0,1\}) \\
H^k(M) \oplus H^k(M) \arrow[r, "(1,-1)"] & H^k(M) \arrow[u, "\cong"']
\end{tikzcd}$$

which allows us to identify the maps labelled δ in the long exact sequence by

$$\mathrm{id} - \phi^* : H^{2n-1}(M) \to H^{2n-1}(M)$$

and

$$\mathrm{id} - \phi^* : H^{2n}(M) \to H^{2n}(M).$$

Now we work with $\mathbb{R}$-coefficients. The first map is dual to the map $\mathrm{id} - \phi_* : H_{2n-1}(M;\mathbb{R}) \to H_{2n-1}(M;\mathbb{R})$, which, under Poincaré-duality gets identified with the second map. Thus, the second map is dual to the first under the nondegenerate pairing between H^{2n-1} and H^{2n}. In particular, the dimension of the cokernel of the first agrees with the dimension of the kernel of the second. From the exact sequence, it follows that

$$\dim \operatorname{img}(H^{2n}(T_\varphi, M;\mathbb{R}) \to H^{2n}(T_\varphi;\mathbb{R})) = \frac{1}{2} \dim H^{2n}(T_\varphi;\mathbb{R}).$$

Also, the isomorphism $H^*(T_\varphi, M;\mathbb{R}) = H^*(M \times [0,1], M \times \{0,1\};\mathbb{R})$ shows that all cup squares in $H^*(T_\varphi, M;\mathbb{R})$ vanish. So the image forms a Lagrangian and the result follows. □

The second result we want to discuss is that the signature of boundaries vanishes. For that we will need a version of Poincaré duality for manifolds with boundary:

Theorem 2.11.23. *Let W be an oriented n-manifold with boundary. Then there is an orientation class $[W] \in H_n(W, \partial W)$. We get cap product maps*

$$(- \cap [W]) : H^k(W, \partial W) \to H_{n-k}(W),$$
$$(- \cap [W]) : H^k(W) \to H_{n-k}(W, \partial W),$$

and both of these are isomorphisms.

In fact, we shall consider a compact manifold W of dimension n with boundary. Then the boundary ∂W is a manifold of dimension $(n-1)$ and we claim that this is also orientable. One can determine this by geometric considerations, but we will just determine the orientation using the long exact sequence

$$H_n(W) \to H_n(W, \partial W) \to H_{n-1}(\partial W) \to H_{n-1}(W) \to H_{n-1}(\partial W)$$

We infer from Poincaé duality that $H_n(W) = H^0(W, \partial W) = 0$ and that $H_{n-1}(W) = H^1(W, \partial W)$ is torsion free. Since ∂W is a manifold the top homology can only be $\mathbb{Z}$ or 0. The long exact seqeuence now shows that it has to be $\mathbb{Z}$ and that the map $H_n(W, \partial W) \to H_{n-1}(\partial W)$ is an isomorphism (otherwise the cokernel would be torsion). We given ∂M the induced orientation.

Theorem 2.11.24. *Let W be a $4n+1$-dimensional compact oriented manifold with boundary $\partial W = M$. Then $\sigma(M) = 0$.*

Proof. All cohomology groups with real coefficients.

The Lagrangian is given by the image of $H^{2n}(W; \mathbb{R}) \to H^{2n}(M; \mathbb{R})$. The fact that it is a Lagrangian follows from the fact that

$$H^{4n}(W; \mathbb{R}) \to H^{4n}(M; \mathbb{R}) \to H^{4n+1}(W, M; \mathbb{R})$$

is exact, and the right hand side map is an isomorphism (relative orientation classes are compatible with connecting homomorphisms), so the left hand map is zero.

The fact that it is of half dimension follows by examining the exact sequence more closely. It takes the form

$$H^{2n}(W, M) \xrightarrow{i^*} H^{2n}(W) \to H^{2n}(M) \to H^{2n+1}(W, M) \xrightarrow{i^*} H^{2n+1}(W)$$

With $\mathbb{R}$-coefficients, $i^* : H^{2n}(W, M; \mathbb{R}) \to H^{2n}(W; \mathbb{R})$ is dual to $i_* : H_{2n}(W; \mathbb{R}) \to H_{2n}(W, M; \mathbb{R})$, which, under Poincaré-Lefschetz duality, agrees with $i^* : H^{2n+1}(W, M; \mathbb{R}) \to H^{2n+1}(W; \mathbb{R})$. So under the perfect cup pairings

$$\begin{aligned} H^{2n}(W, M; \mathbb{R}) \otimes H^{2n+1}(W; \mathbb{R}) \to \mathbb{R} \\ H^{2n+1}(W, M; \mathbb{R}) \otimes H^{2n}(W; \mathbb{R}) \to \mathbb{R} \end{aligned}$$

the two induced maps $i^* : H^{2n}(W, M; \mathbb{R}) \to H^{2n}(W; \mathbb{R})$ and $i^* : H^{2n+1}(W, M; \mathbb{R}) \to H^{2n+1}(W; \mathbb{R})$ are dual to each other. Thus, the cokernel of the first has the same dimension as the kernel of the second, and thus the image of $H^{2n}(W; \mathbb{R}) \to H^{2n}(M; \mathbb{R})$ has half the dimension. □

Note that this proves that for example $\mathbb{C}P^2$ cannot be the boundary of an oriented manifold. More generally, one can infer from this that if $\partial W = M \sqcup \overline{N}$ (M and N are called cobordant), $\sigma(M) = \sigma(N)$, so σ is a *cobordism invariant.*

At last we want to argue how for odd dimensional manifolds the cup-product structure can distinguish spaces that we could not distinguish before, namely 3-dimensional lens spaces.

Theorem 2.11.25. *Fix a number $p \in \mathbb{N}$ and $q, q' \in \mathbb{Z}$ coprime to p. Then*

$$L_{p,q} \simeq L_{p,q'} \Rightarrow q \cdot q' = \pm n^2 \bmod p.$$

Proof. In $\mathbb{F}_p$-cohomology, $H^*(L_{p,q};\mathbb{F}_p)$ is $\mathbb{F}_p$ in each degree. By working with the universal coefficient theorem, one sees that $\beta : H^1(L_{p,q};\mathbb{F}_p) \to H^2(L_{p,q};\mathbb{F}_p)$ is an isomorphism. Thus, the bilinear form on $H^1(L_{p,q};\mathbb{F}_p)$ given by

$$\lambda(x,y) = \langle x \cup \beta(y), [L_{p,q}]\rangle$$

is nondegenerate. This is generally defined for $2n+1$-dimensional manifolds on H^n and is called the *linking form.*

From the fact that β is a derivation, one can check that $\lambda(x,y) = (-1)^{n+1}\lambda(y,x)$, i.e. the linking form is symmetric in our case. A nondegenerate symmetric bilinear $\mathbb{F}_p$-valued form on $\mathbb{F}_p$ is given by a single coefficient, and change of basis allows us to multiply the coefficient precisely by squares. Now one can check that the linking form of $L_{p,q}$ is given by the form

$$\lambda(x,y) = qxy,$$

and thus the linking forms for $L_{p,q}$ and $L_{p,q'}$ agree if and only if $q \cdot (q')^{-1}$ is a square, or equivalently, $q \cdot q'$ is a square. Thus there can only be an orientation preserving equivalence between $L_{p,q}$ and $L_{p,q'}$ if $qq' = n^2$ modulo p for some n. Passing to arbitrary equivalences introduces a sign. □

Corollary 2.11.26. *The lens spaces $L_{5,1}$ and $L_{5,2}$ are not homotopy equivalent.*

Remark 2.11.27. Unimodular symmetric forms over $\mathbb{Z}$ are complicated and closely related to low-dimensional topology:

(1) two simply connected topological 4-manifolds are homeomorphic if and only if their intersection forms agree (Freedman);

(2) Any perfect symmetric form over $\mathbb{Z}$ arises as the intersection form of a (simply connected) topological 4-manifold;

(3) indefinite forms over $\mathbb{Z}$ are classified by their rank, signature and type (even or odd) (Serre);

(4) A form is called even if $\lambda(x,x) = 0 \bmod 2$ for all x. If λ is even, then $\sigma(\lambda) = 0 \bmod 8$;

(5) definite forms are complicated and not well understood;

(6) If a *smooth* 4-manifold has definite intersection form, then it is equivalent to the standard form (notice that the similar statement for topological manifolds is wrong, due to (1) and (2));

(7) smooth *spin* 4-manifolds have even intersection form, thus for such M we have $\sigma(M) = 0 \bmod 8$. A theorem of Rokhlin says that indeed one has $\sigma(M) = 0 \bmod 16$.

2.12 Twisted (co)homology

In this section and the next section we want to introduce variants of singular (co)homology which are central to generalizing and proving Poincaré duality. Note that if M is a manifold of dimension n then there we have seen in Theorem 2.10.1 that $H_n(M,\mathbb{Z}) = \Gamma_c(M,\Theta)$. So in order for M to admit a fundamental class $[M] \in H_n(M)$ we need to things to be satisfied: the cover Θ must be trival i.e. admit a section, which is equivalent to the assertion that M is orientable. Secondly we need that we have a section with compact support, which is equivalent to M being compact. The idea here now is to alter the Definition of singular homology $H_n(M)$ so that every manifold admits a fundamental class

More precisely, we shall introduce homology and cohomology with local coefficients

$$H_*(X, \mathcal{A}) \qquad H^*(X, \mathcal{A})$$

where X is a topological space and $\mathcal{A}$ a local system of abelian groups. A local system is a covering space with a fibrewise structure of abelian groups. For example the orientation bundle θ_M of a manifold M will be an example. The rough idea is that the chains/cochains that we allow take values in these varying abelian groups.

Then we also introduce locally finite homology

$$H_*^{\mathrm{lf}}(X, \mathcal{A})$$

which is in some sense an enlargement of $H_*(X, M)$ where we allow infinite linear combinations of simplices, but which are locally finite. For a compact space X we have $H_*^{\mathrm{lf}}(X, M) = H_*(X, M)$. We also discuss cohomology with compact support

$$H_c^*(X, \mathcal{A})$$

which is smaller version of cohomology where we require that the cochain have compact support in X.

Theorem 2.12.1. *For every n-dimensional topological manifold M (not necessarily compact or oriented) we have a canonical isomorphism*

$$H_n^{\mathrm{lf}}(M, \Theta_M) \xrightarrow{\simeq} \mathbb{Z}$$

and in particular a canonical fundamental class $[M] \in H_n^{\mathrm{lf}}(M, \mathbb{Z}^\theta)$. Then capping with this class induces isomorphism

$$\cap[M] : H^k(M, \mathcal{A}) \to H_{*-k}^{\mathrm{lf}}(M, \mathcal{A} \otimes \Theta)$$
$$\cap[M] : H_c^k(M, \mathcal{A}) \to H_{*-k}(M, \mathcal{A} \otimes \Theta)$$

In the case that M is compact and oriented we will see that this statement reduces to Poincaré duality as stated in 2.11.1.

Definition 2.12.2. A local system of abelian groups $\mathcal{A}$ on a topological space X is a covering space together with a fibrewise abelian group structure.

In particular we have for every point $x \in X$ an abelian group $\mathcal{A}_x$ given as the fibre over x. For every path $\gamma : x \to y$ in X we get an induced group homomorphism $\mathcal{A}_\gamma : \mathcal{A}_x \to \mathcal{A}_y$. This homomorphism is an invariant of the homotopy class of the path and for compositions of paths we get the composition of maps. By covering theory the local system is for (good) spaces X determined by these assignments. In particular if X is connected and locally path connected then local systems are classified by an abelian group A together with an action of $\pi_1(X)$ through group automorphisms. For what follows we shall only need the groups $(\mathcal{A}_x)_{x \in X}$ together with the maps $\mathcal{A}(\gamma)$.

Example 2.12.3. For every abelian group A we have the constant local system $A \times X \to X$. For every n-dimensional manifold M we have the orientation bundle θ_M.

Now we want to define (co)homology with coefficients in a local system $\mathcal{A}$ on X. To this end we define the singular chains

$$C_n(X;\mathcal{A}) := \bigoplus_{\sigma:\Delta^n \to X} \mathcal{A}_{\sigma_0}$$

where σ_0 is the point obtain by the 0-th vertex of Δ^n. Now lets define a differential

$$d : C_n(X;\mathcal{A}) \to C_{n-1}(X;\mathcal{A})$$

on this chain complex as follows. We will write an element $m \in \mathcal{A}_{\sigma_0}$ as a pair (σ, m) for simplicity. Also recall that the i-th face maps $d^i : \Delta^{n-1} \to \Delta^n$ is given by the unique affine linear map which sends the vertices $v_0, ..., v_{i-1}$ to $v_0, ..., v_{i-1}$ and $v_i, .., v_{n-1}$ to $v_{i+1}, .., v_n$. With this terminology the differential in the chain complex is defined as

$$d(\sigma, m) = \big(\sigma \circ d^0, \left(\sigma_{0,1}\right)_*(m)) + \sum_{i=1}^{n} (-1)^i (\sigma \circ d^i, m)$$

where $d^i : \Delta^{n-1} \to \Delta^n$ is the i-th face map and $\sigma_{0,1}$ is the composition $\Delta^1 \to \Delta^n \to X$ where the first map is the straight line connecting the 0-th vertex of Δ^n and the first vertex of Δ^n. Note that all the faces $\sigma \circ d^i$ have the same 0-th vertex as σ.

Lemma 2.12.4. *This actually defines a chain complex, i.e. the differential squares to zero.*

Proof. TODO □

Moreover its obvious that in the case of the constant local coefficient system this chain complex is just the ordinary singular chain complex of X. Clearly $C_*(X, \mathcal{A})$ is functorial in both variables: if there is a map of local systems $\mathcal{A} \to \mathcal{B}$ then we get an induced map $C_*(X, \mathcal{A}) \to C_*(X, \mathcal{B})$. For a map of spaces $f : X \to X'$ we get a pullback local system $f^*\mathcal{A}$ and a map $C_*(X, f^*\mathcal{A}) \to C_*(X, \mathcal{A})$.

Definition 2.12.5. For a local system $\mathcal{A}$ on X we define $H_*(X, \mathcal{A})$ as the homology of the chain complex $C_\bullet(X, \mathcal{A})$.

We now similary want to define the cochain complex $C^\bullet(X; \mathcal{A})$ for a local system $\mathcal{A}$. This is defined as

$$C^n(X;\mathcal{A}) = \prod_{\sigma:\Delta^n \to X} \mathcal{A}_{\sigma_0} \ .$$

We think of an element in this product as a map f which assigns to every simplex $\sigma : \Delta^n \to X$ an element in $f(\sigma) \in \mathcal{A}_{\sigma_0}$. Then we can define the differential $d : C^n(X;\mathcal{A}) \to C^{n+1}(X;\mathcal{A})$ as

$$(df)(\sigma) = \mathcal{A}(\sigma_{0,1})^{-1}\Big(f(d^0\sigma)\Big) + \sum_{i=1}^{n} (-1)^i f(d^i\sigma)$$

Definition 2.12.6. Cohomology with local coefficients is defined as $H^*(X, \mathcal{A}) := H^*C^\bullet(X, \mathcal{A})$.

Now this has similar properties as ordinary cohomology which we leave to the reader to work out. We will however indicate, how this already solves one of our problems with the existence of fundamental classes.

Example 2.12.7. Recall that before we had seen that the relative homology $H_n(\mathbb{R}^n, \mathbb{R}^n \setminus 0)$ is isomorphic to the integers. This isomorphism however requires a choice of isomorhism on $\mathbb{R}^n$, in fact is is equivalent to chosing an orientation. We have made such a choice implicitly all the time. For a given homoeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(0) = 0$ we get an induced map

$$\mathbb{Z} \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \to H_n(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \cong \mathbb{Z}$$

and this map is given by multiplication $+1$ or -1 depending on whether f is orientation preserving or not.

Also note that the orientation cover of $\mathbb{R}^n$ is trivial, since $\mathbb{R}^n$ is orientable. That is, we have an isomorphism $\Theta_{\mathbb{R}^n} \cong \mathbb{R}^n \times \mathbb{Z}$. But this choice of isomorphism is the same as an orientation of $\mathbb{R}^n$ as well and thus also not natural in self-homeomorphisms. More precisely for a given map $\mathbb{R}^n \to \mathbb{R}^n$ as avove we get an induced diagram

and it follows immediately that the induced map $\mathbb{R}^n \times \mathbb{Z} \to \mathbb{R}^n \times \mathbb{Z}$ is given by $(\mathrm{id}, \pm 1)$ depending on whether f is oriented or not.

The key point is that these two sources of not being canonical cancel each other in the sense that we have a fully canonical isomorphism

$$H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \Theta_{\mathbb{R}^n}) \cong \mathbb{Z}$$

To see this just chose an orientation on $\mathbb{R}^n$ (e.g. the prefered choice that we have used all the time) and we get the induced isomorphism

$$H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \Theta_{\mathbb{R}^n}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \cong \mathbb{Z} \ .$$

If we had chosen the other orientation both isomorphisms above would transform with a sign, so that the composite stays the same. More generally for every homeomorphis $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(0) = 0$ we have the diagram

$$\begin{tikzcd}
\mathbb{R}^n \times \mathbb{Z} \arrow[r, "\cong"] & \Theta_{\mathbb{R}^n} \arrow[r, "\Theta_f"] \arrow[d] & \Theta_{\mathbb{R}^n} \arrow[r, "\cong"] \arrow[d] & \mathbb{R}^n \times \mathbb{Z} \\
& \mathbb{R}^n \arrow[r, "f"] & \mathbb{R}^n &
\end{tikzcd}$$

In particular for every homeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ which takes 0 to 0 the induced map

$$H_n(M, M \setminus \{0\}, \Theta_M) \to H_n(M, M \setminus \{0\}, \Theta_M)$$

from the map f is given under the isomorphism with $\mathbb{Z}$ by the identity.

We let M be an n-dimensional topological manifold and consider the group $H_n(M; \Theta_M)$. For every point $m \in M$ we get a canonical map

$$H_n(M; \Theta_M) \to H_n(M, M \setminus \{m\}); \Theta_M) \cong \mathbb{Z}$$

where the latter isomorphism follows by excision using the last example. If we group all the values for given points $m \in M$ together we get for every class $\alpha \in H_n(M; \Theta_M)$ a map $M \to \mathbb{Z}$. It is easy to see (as before) that this map has compact support and is continuous. Thus we get a map

$$J : H_n(M; \Theta_M) \to \mathrm{Hom}_c(M, \mathbb{Z})$$

Theorem 2.12.8. *The map $J : H_n(M; \Theta_M) \to \mathrm{Hom}_c(M, \mathbb{Z})$ is an isomorphism.*

Proof. The proof more or less varbatim follows the proof of Theorem 2.10.1 above. We first defined the more general map

$$J_A : H_n(M, A; \Theta_M) \to \mathrm{Hom}_c(A, \mathbb{Z})$$

and claim that it is an isomorphism. This we show using the Bootstrap Lemma 2.10.2 by verifying the assumptions, which works precisely as in the proof of Theorem 2.10.1. □

Corollary 2.12.9. *For a connected, n-dimensional manifold M we have*

$$H_n(M; \Theta_M) = \begin{cases} \mathbb{Z} & \text{if } M \text{ is compact} \\ 0 & \text{if } M \text{ is not compact} \end{cases}$$

2.13 Locally finite homology

Now we will introduce the other (co)homology groups. We will do this with values in the abelian group $\mathbb{Z}$, but everything works with coefficients in an arbitrary abelian group, in fact even with values in local system.

Why don't non-compact manifolds have fundamental classes? Draw a picture showing that one needs infinite traingulations.

Definition 2.13.1. Let X be a locally compact topological space. We define the locally finite chains on X to be

$$C_*^{\mathrm{lf}}(X) \subseteq \prod_{\sigma:\Delta^n \to X} \mathbb{Z}$$

to be the subset consisting of all those formal linear combinations $\sum a_\sigma \cdot \sigma$ with $a_\sigma \in \mathbb{Z}$ such that for every compact subset $K \subseteq X$ we have that the set

$$\{\sigma \mid \sigma(\Delta^n) \cap K \neq \emptyset, a_\sigma \neq 0\}$$

is finite. Thus informally these are those infinite linear combinations of chains that are locally finite.

We observe that $C_*^{\mathrm{lf}}(X)$ forms a chain complex as before (since every boundary is again locally finite) and thus we can define the locally finite Homology of X to be the abelian group

$$H_n^{\mathrm{lf}}(X) := H_n(C_*^{\mathrm{lf}}(X); \mathbb{Z}) \ .$$

This is also sometimes called Borel-Moore homology. We warn the reader that locally finite homology is not functorial in all maps. It is functorial in proper maps $f : X \to Y$, i.e. those compact maps for which preimages of compact subsets are compact. Then we get a map

$$H_n^{\mathrm{lf}}(X) \to H_n^{\mathrm{lf}}(Y)$$

defined exactly as in Homology and we observe that for a locally finite chain $c \in C_*^{\mathrm{lf}}(X)$ the induced chain f_*c is also locally finite. Thus locally finite homology gives a functor from locally compact topological spaces and proper maps to abelian groups.

Lemma 2.13.2. *The canonical morphism*

$$C_*^{\mathrm{lf}}(X) \to \varprojlim_{K\subseteq X\mathrm{compact}} C_*(X, X\setminus K)$$

is an isomorphism.

We also want to define a relative version $C_*^{\mathrm{lf}}(X,A)$ for a subset $A\subseteq X$. To this end we require $X\setminus A$ to be locally compact and define

$$C_*^{\mathrm{lf}}(X,A) := \varprojlim_{K\subseteq(X\setminus A)\mathrm{compact}} C_*(X, X\setminus K).$$

One can think of this as formal linear combinations $\sum a_\sigma\cdot\sigma$ which are finite on each compactum in the complement of A. Again this is not functorial in all maps $f:(X,A)\to(Y,B)$ but only in those maps of pairs f such that $f^{-1}(K)$ is compact in $X\setminus A$ for each $K\subseteq Y\setminus B$ compact. We warn the reader that one does not get long exact sequences for a pair in general! We also warn the reader that this definition of relative locally finite homology is not entirely standard.

We have the following very strong form of excision.

Lemma 2.13.3. *For $U\subseteq X$ open and Hausdorff, the morphism $H_*^{\mathrm{lf}}(U)\to H_*^{\mathrm{lf}}(X,X\setminus U)$ is an isomorphism.*

Proof. Apply excision to deduce that $C_*(U)\to C_*(X,X\setminus K)$ is a quasi-iso for each $K\subseteq U$. Then it follows by the fact that the inverse limit satisfies Mittag-Leffler. □

This in particular implies that we have maps $H_*^{\mathrm{lf}}(X)\to H_*^{\mathrm{lf}}(U)$ for $U\subseteq X$ open.

Lemma 2.13.4. *Let M be a compact manifold with a closed submanifold A (which can be the boundary). Then we have that the canonical map*

$$H_*(M,A)\to H_*^{\mathrm{lf}}(M,A)$$

is an isomorphism.

Example 2.13.5. We have $H_*^{\mathrm{lf}}(\mathbb{R}^n) = H_*^{\mathrm{lf}}(S^n,\mathrm{pt}) = H_*(S^n,pt) = \tilde{H}_*(S^n)$.

The canonical map $H_n^{\mathrm{lf}}(M;\Theta_M)\to\mathbb{Z}$ is an isomorphism as well as the maps

$$H^*(M,U)\to H_{n-*}^{\mathrm{lf}}(M,U) = H_{n-*}(M\setminus A)$$

2.14 Poincare Duality

Let us first sketch what Poincare duality is. Recall that for a topological manifold M we had the orientation bundle $\theta_M\to M$ which was a $\mathbb{Z}$-sheeted cover (its fibres are abstract groups isomorphic to $\mathbb{Z}$) and the orientation cover $\hat{M}\to M$ which was a 2-sheeted cover. Then orientations over M are in 1-1 correspondence to sections of $\hat{M}\to M$ which we could also interpret as sections of $\theta_M\to M$ which are fibrewise generators. We also proved that $H_n(M)\cong\Gamma_c(M;\Theta_M)$. For compact M we concluded that an orientation gives a class $[M]\in H_n(M)$ which was a generator (i.e. $H_n(M)\cong\mathbb{Z}$) and which was called the fundamental class. Now the cap product with the fundamental class induces a morphism

$$-\cap[M]: H^k(M)\to H_{n-k}(M)$$

where $n = \dim M$. The statement of Poincare duality is that these morphism are isomorphisms, i.e. $H^k(M) \cong H_{n-k}(M)$.

Since we want to prove such a result for arbitrary coefficients let us a call an R-orientation of M a section $\Gamma(M, \Theta_M \otimes R)$ which is fibrewise a generator. In the case $R = \mathbb{Z}$ this reduces to the ordinary notion of orientation, for $R = \mathbb{Z}/2$ every Manifold M has a canonical and unique R-orientation.

Definition 2.14.1 (Cohomology with compact support)**.** Let X be a locally compact topological space. Then we define the subcomplex

$$C_c^*(X; R) \subseteq C^*(X; R)$$

consisting of those morphismsm $f : C_n(X) \to R$ for which there exists a compact subset $K \subseteq X$ such that for all $\sigma : \Delta^n \to X \setminus K$ considered as element $\sigma \in C_n(X)$ we have $f(\sigma) = 0$. Then cohomology with compact support of M is defined as

$$H_c^*(M; R) := H^n(C_c^*(M; R))$$

There is a canonical morphism $H_c^*(X; R) \to H^*(X; R)$ given by the inclusion $C_c^*(X; R) \to C^*(X; R)$. Note that this morphism is not injective in general! There are also canonical morphisms $C^*(X, X \setminus K; R) \to C_c^*(X; R)$ and $H^*(X, X \setminus K; R) \to H_c^*(X; R)$ for $K \subseteq X$ compact.

Lemma 2.14.2. *1. The canonical morphism*

$$\varinjlim_{K \subseteq X\, compact} C^*(X, X \setminus K; R) \to C_c^*(X; R)$$

is an isomorphism.

2. The canonical morphism

$$\varinjlim_{K \subseteq X\, compact} C^*(X, \overline{X \setminus K}; R) \to C_c^*(X; R)$$

is an isomorphism.

3. The canonical morphism

$$\varinjlim_{K \subseteq X\, compact} H^*(X, X \setminus K; R) \to H_c^*(X; R)$$

is an isomorphism.

4. The canonical morphism

$$\varinjlim_{K \subseteq X\, compact} H^*(X, \overline{X \setminus K}; R) \to H_c^*(X; R)$$

is an isomorphism.

5. There is a canonical isomorphism

$$H_c^*(X; R) \xrightarrow{\sim} H^*(X^+, +; R)$$

where X^+ is the one-point compactification of X.

Remark 2.14.3. Here the limits are not indexed over the natural numbers $(\mathbb{N}, \leq)$ but over another partially ordered set, namely the set of all open compact subsets $K \subseteq X$ ordered by inclusion. Direct limits make sense with the exact same definition and formulas we had before (Definition **??**) for every poset $(P, \leq)$ as long as it has the property that is directed, i.e. for every pair of elements $p, p' \in P$ there is $q \in P$ with $p \leq q$ and $p' \leq q$. In our case this is clearly satisfied. Morally we should think of this direct limit as the limit when the compact subsets become larger and larger.

Example 2.14.4. For $\mathbb{R}^n$ we get

$$H_c^*(\mathbb{R}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } * = n \\ 0 & \text{else} \end{cases}$$

More generally for a compact manifold M we have

$$H_c^*(M \setminus \{m\}; \mathbb{Z}) = H^*(M; R)$$

for $* \geq 1$.

Corollary 2.14.5. *The graded group $H_c^*(X; R)$ is a non-unital ring with the cup product. In fact already $C_c^*(X; R)$ is a non-unital differential graded algebra. It is unital precisely if X is compact. The homology $H_*(X; R)$ is a graded module over $H_c^*(X; R)$, i.e. there is a morphism*

$$\cap : H_*(X; R) \otimes H_c^*(X; R) \to H_*(X; R)$$

induced by the family of morphisms

$$\cap : H_*(X; R) \otimes H^*(X, X \setminus K; R) \to H_*(X; R)$$

Definition 2.14.6. Let M be a topoloigcal manifold and $\theta \in \Gamma(M; \Theta_M \otimes R)$ be an R-orientation of M. The Poincare-Duality morphism is the morphism

$$- \cap [\theta] : H_c^k(M; R) \to H_{n-k}(M; R)$$

It is defined as the morphism induced by the family of morphisms

$$- \cap [\theta_K] : H^k(M, M \setminus K; R) \to H_{n-k}(M; R)$$

where $[\theta_K] \in H_k(M, M \setminus K; R) \cong \Gamma(K, \theta_M \otimes R)$ is the class corresponding to θ.

Remark 2.14.7. When M is compact then the Poincare-Duality morphism clearly restrict to the morphism we have discussed before, i.e. just capping with $[M]$.

Theorem 2.14.8. *(Poincare duality) Let M be an n-dimensional topological manifold and θ an R-orientation of M. Then the Poincare duality morphism*

$$- \cap \theta : H_c^k(M; R) \to H_{n-k}(M; R)$$

is an isomorphism.

Proof. This will be a special case of Alexander-Poincare duality that we will prove soon. □

Definition 2.14.9. Let M be an n-manifold, $L \subseteq M$ a closed subset. Define the group

$$\check{H}^*(L;R) := \underrightarrow{\operatorname{colim}}\{H^*(U;R) \mid U \text{neighborhood of } L \text{ in } M\}$$

here the limit is taken over all neighborhoods U ordered by the superset relation. Explicitly we can realize these groups as the cohomology groups of the chain complex

$$\check{C}^*(L;R) := C^*(M;R)/\sim$$

where $f \sim g$ if there exists $L \subseteq U \subseteq M$ such that $f|_U = g|_U$ ('germs of cocycles').

Remark 2.14.10. The group $\check{H}^*(L;R)$ seems to depend on the manifold M and the embedding of L into M. But in fact it does not. One can show that its isomorphic to another group called the Cech-cohomology which is independent of M. Moreover one can show that for a nice space L (e.g. CW-complexes or topological manifolds) the group $\check{H}^*(L;R)$ is naturally isomorphic to $H^*(L;R)$. We will not prove these facts here, but they can be found in most topology books. We will here freely use this fact.

Definition 2.14.11. Let M be a manifold and $L \subseteq M$ be a closed subset. We define the group

$$\check{H}^*_c(L;R) := \underrightarrow{\operatorname{colim}}\{H^*_c(U) \mid U \text{neighborhood of } L \text{ in } M\}$$

Here one has to be slightly careful since the group $H^*_c(-)$ is not functorial in all inclusions $U \subseteq V$ but only in proper maps. The best thing is to index the colimit only in closed neighborhoods. Again these groups are the cohomology groups of the cochain complex

$$\check{C}^*_c(L;R) = C^*_c(M;R)/\sim$$

where $\sim$ is the equivalence relation from before.

Lemma 2.14.12. *If $L = M$ then $\check{H}^*_c(L;R) \cong H^*_c(M;R)$ and $\check{H}^*(L;R) \cong H^*(M;R)$. If $L \subseteq M$ is compact, then $\check{H}^*_c(L;R) \cong \check{H}^*(L;R)$.*

Proof. The first two statements are immediately clear. Now assume $L \subseteq M$ is compact. We observe that for every neighboorhood $L \subseteq U$ there exists a compact neighborhood $V \subseteq U$. This shows that we can index our colimit also over compact neighborhoods and the result is isomorphic. Thus the result follows. $\square$

Lemma 2.14.13. *Let M be an R-oriented manifold $L \subseteq M$ closed and U, V neighborhoods of L with $U \subseteq V$. Then the diagram*

$$\begin{tikzcd}
H^k_c(V) \arrow[r, "-\cap[\theta|_V]"] \arrow[dd, "\text{res}"] & H_{n-k}(V) \arrow[dr] \arrow[dd] & \\
& & H_{n-k}(M, M\setminus L) \\
H^k_c(U) \arrow[r, "-\cap[\theta|_U]"] & H_{n-k}(U) \arrow[ur] &
\end{tikzcd}$$

commutes.

Proof. Follows from the definitions and the naturality $f : X \to Y$, $\alpha \in H^*(Y)$ and $\beta \in H_*(X)$ then $f_*(f^*(\alpha) \cap \beta) = \alpha \cap f_*(\beta)$ as stated in Proposition 2.8.23. $\square$

Corollary 2.14.14. *Let $\theta \in \Gamma(M, \Theta_M \otimes R)$ be an R-orientation of M and $L \subseteq M$ be a closed subset of M. Then there is a well-defined, natural map*

$$- \cap \theta : \check{H}^k_c(L; R) \to H_{n-k}(M, M \setminus L; R)$$

which we also call the Poincare-duality morphism.

Proof. Follows from the last proposition and the universal property of the colimit. $\square$

Lemma 2.14.15. *Let K and L be closed subsets of the n-manifold M with the R-orientation θ. Then there is a diagram*

$$\begin{tikzcd}
\dots \arrow[r] & \check{H}^k_c(K \cup L) \arrow[r] \arrow[d] & \check{H}^k_c(K) \oplus \check{H}^k_c(L) \arrow[r] \arrow[d] & \check{H}^k_c(K \cap L) \arrow[r] \arrow[d] & \check{H}^{k+1}_c(K \cup L) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & H_{n-k}(M, M \setminus K \cup L) \arrow[r] & H_{n-k}(M, M \setminus K) \oplus H_{n-k}(M, M \setminus L) \arrow[r] & H_{n-k}(M, M \setminus K \cap L) \arrow[r] & H_{n-k-1}(M, M \setminus K \cup L) \arrow[r] & \dots
\end{tikzcd}$$

which commutes, and has exact rows. The vertical morphisms are given by the Poincare duality morphisms and the lower sequence is the Mayer-Vietoris sequence.

Theorem 2.14.16 (Alexander-Poincare duality)**.** *Let $\theta \in \Gamma(M, \Theta_M \otimes R)$ be an R-orientation of M and $L \subseteq M$ be a closed subsets of M. Then the Poincare duality morphism*

$$- \cap \theta : \check{H}^p_c(L; R) \to H_{n-p}(M, M \setminus L; R)$$

is an isomorphism.

For the proof first recall the following Lemma, which we proved last term (Lemma 2.10.2):

Lemma 2.14.17. *(Bootstrap Lemma) Let $S(M; L)$ be a statement about pairs (L, M) where M is an n-manifold and L a closed subset (and which is invariant under isomorphism of pairs). If the following 5-points hold then $S(M, L)$ holds for all L and M:*

1. *L compact and convex in $\mathbb{R}^n$ then $S(\mathbb{R}^n; L)$ holds*
2. *If $S(M, L)$, $S(M, K)$ and $S(M, L \cap K)$ hold, then also $S(M, L \cup K)$ holds*
3. *For $L_1 \supset L_2 \supset L_3 \supset \dots$ all compact in M such that $S(M, L_i)$ holds, then it follows that $S(M; \cap L_i)$ holds*
4. *L_i compact in M with disjoint neighborhoods and $S(M, L_i)$ for all i, then $S(M, \cup L_i)$ holds*
5. *$S(W, L \cap W)$ for all open, relatively compact $W \subseteq M$, then $S(M, L)$*

Example 2.14.18. For every manifold we have isomorphisms $H^k_c(M, \mathbb{Z}/2) \cong H_{n-k}(M; \mathbb{Z}/2)$.

Corollary 2.14.19. *If $L \subseteq M$ is a proper, closed subset of an n-dimensional R-orientable, connected manifold M, then $\check{H}^n(L, R) = 0$.*

Proof. It is isomorphic to $H_0(M, M \setminus L; R)$ which is zero since $M \setminus L \neq \emptyset$. $\square$

Corollary 2.14.20 (Alexander duality)**.** *If $A \subseteq \mathbb{R}^n$ is compact, then*

$$\tilde{H}_q(\mathbb{R}^n \setminus A; R) \cong \check{H}^{n-q-1}(A; R)$$

Proof. $\check{H}^{n-q-1}(A; R) \cong H_{q+1}(\mathbb{R}^n, \mathbb{R}^n \setminus A; R) \cong \tilde{H}_q(\mathbb{R}^n \setminus A; R)$. □

Now we want to talk about manifolds with boundary.

Definition 2.14.21. A manifold with boundary M is a topological space such that for every point there is an open neighborhood that is homeomorphic to an open subset in

$$H^n := \{x \in \mathbb{R}^n \mid x_0 \geq 0\}$$

The boundary $\partial M \subseteq M$ is well defined and a manifold. We assume that every manifold with boundary has a *collar*, that is an open neighborhood $U \subseteq M$ of ∂M which is homeomorphic to $\partial M \times [0, 1)$. This is always true by a theorem of Brown.Easy to see for smooth but for topological ones it is much harder. One can also always achieve that by glueing an external collar. In examples it is usually easy to verify.

Definition 2.14.22. An R-orientation of a manifold with boundary is an R-orientation of its interior $\int M = M \setminus \partial M$.

Proposition 2.14.23. *For M compact, connected with boundary and R-orientation we have*

$$H_n(M, \partial M; R) \cong R$$

and we call the generator induced by the orientation a fundamental class $[M] \in H_n(M, \partial M; R)$. Capping with the fundamental class induces an isomorphism

$$H^k(M; R) \to H_{n-k}(M, \partial M; R)$$

Lemma 2.14.24. *If M is compact and R-orientable then ∂M is R-orientable and $[\partial M] = \delta_*[M]$ where δ_* is the connecting homomorphism*

$$H_n(M, \partial M; R) \to H_{n-1}(\partial M)$$

of the pair $(M, \partial M)$.

Proposition 2.14.25 (Lefschetz-Duality)**.** *For M compact and R-orientable the morphism*

$$- \cap [M] : H^k(M, \partial M; R) \to H_{n-k}(M; R)$$

is an isomorphism.

2.15 De Rham cohomology

Throughout this chapter we make the convention that all manifolds are smooth (possibly with boundary). Recall that a smooth structure on a topological manifold M is an atlas $\mathcal{A}$ such that the transition functions are smooth.

Example 2.15.1. The standard atlases of $S^n, \mathbb{R}P^n, \mathbb{C}P^n, L_{p,q}$ are all smooth. The connected sum of two smooth manifolds admits a canonical smooth structure, in paricular all orientable and non-orientable genus g-surfaces admit smooth structures. All Knot complements $S^3 \setminus K$ admit canonical smooth structures. The Disk D^n and all Products $M \times [0,1]$ are smooth manifolds with boundary.

In this section we want to understand the cohomology (with $\mathbb{R}$-coefficients) of a smooth manifold in more differential geometric terms. That often allows to make a bridge between these two worlds. To this end we have to repeat the concept of a differential form. (Recall the definition of smooth maps).

Definition 2.15.2. For a smooth manifold M we denote the smooth functions $M \to \mathbb{R}$ by $C^\infty(M;R)$. They form a commutative $\mathbb{R}$-algebra with pointwise multiplication. A derivation at x is a $\mathbb{R}$-linear morphism

$$D : C^\infty(M;R) \to \mathbb{R}$$

such that we have $D(fg) = D(f)\cdot g(x) + f(x)\cdot D(g)$. They form a vector space by pointwise addition and scalar multiplication. Then we let T_xM be the vector space of all derivations at x and call it the Tangent space at x.

Remark 2.15.3. The tangent space T_xM also admits the following more intuitive description: consider smooth curves $\gamma : \mathbb{R} \to M$. We define an equivalence relation of curves as follows: ... The equivalence class of the curve γ is written as $\gamma'(0)$. Then the set of all equivalence classes is denoted by $T_x^{curve}M$. With this description it is a priori not clear that this set admits a well-definied vector space structure. We now want to describe a bijection

$$T_x^{curve}M \to T_xM \qquad \gamma'(0) = [\gamma] \mapsto D_\gamma$$

where $D_\gamma(f) = (f \circ \gamma)'(0)$ where the latter is the derivative of a function $\mathbb{R} \to \mathbb{R}$. It is not so hard to check that this defines a well-defined bijection.

For every smooth map $f : M \to N$ we get an induced linear map $df_x : T_xM \to T_{f(x)}N$ as follows:

$$D \mapsto D \circ f^*$$

where $f^* : C^\infty(N) \to C^\infty(M)$ is precomposition. This assignment is functorial in the sense that $d\mathrm{id}_x = \mathrm{id}$ and $d(f \circ g)_x = df_x \circ dg_{f(x)}$. For a diffeomorphism $f : M \to N$ we get an isomorphism of vector space $T_xM \to T_{f(x)}N$. It is also not hard to check that for the inclusion of an open set $U \subseteq M$ we also get an isomorphism $T_xU \to T_xM$ for every $x \in U$.

Lemma 2.15.4. *The space $T_0\mathbb{R}^n$ is an n-dimensional vector space with basis given by $\frac{\partial}{\partial x_i}$ for $i = 1, ..., n$.*

Corollary 2.15.5. *For an n-dimensional manifold M the tangent space T_xM is an n-dimensional vector space.*

Definition 2.15.6. A vector field X on a smooth manifold M is a derivation $X : C^\infty(M) \to C^\infty(M)$, i.e. an $\mathbb{R}$-linear morphism such that $X(fg) = X(f)g + fX(g)$. We denote the set of all vector fields by $\mathfrak{X}(M)$. It is canonically a module over the ring $C^\infty(M)$ since for $f \in C^\infty(M)$ and $X \in \mathfrak{X}(M)$ the product fX is again a derivation:

Clearly every vector field X gives rise to a family of tangent vectors $X_x \in T_xM$ as follows: $X_x(f) = X(f)(x)$. That is why we can think of a vector field X as a family of tangent vectors for every point of M. Clearly the assignment $x \mapsto X_x$ uniquely determines X.

Remark 2.15.7. One can improve this analogy as follows: One can make the set $\bigsqcup_{x\in M} T_xM$ into a smooth manifold denoted TM such that the morphism $TM \to M$ is smooth (the dimension is 2n). This works very similar to the way we made the orientation bundle of a topological manifold. Then one can describe vector fields as smooth sections of $TM \to M$, i.e. smooth maps $M \to TM$ which take x to T_xM.

Example 2.15.8. For $M = \mathbb{R}^n$ we have that $\mathfrak{X}(M) \cong \bigoplus_{i=1}^n C^\infty(M)\frac{\partial}{\partial x_i}$, i.e. as a module it is free with basis $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}$. In particular every vector field X is described as a linear combination

$$X = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$$

where f_i are smooth functions $f_i : \mathbb{R}^n \to R$. We can even compute that $f_i = X(x_i)$. Such a vector field is equivalently described as a smooth map $\mathbb{R}^n \to \mathbb{R}^n$.

Proof. □

Definition 2.15.9. A differential k-form ω on a smooth manifold M is a $C^\infty(M)$-multilinear morphism

$$\omega : \mathfrak{X}(M) \times ... \times \mathfrak{X}(M) \to C^\infty(M)$$

which is alternating, i.e. we have $\omega(.., X, ..Y, ...,) = -\omega(..., Y, .., X, ...)$. We denote the set of all differential k-forms by $\Omega^k(M)$. This again forms a module over $C^\infty(M)$. We denote $\Omega^*(M) := \bigoplus_{k=0}^\infty \Omega^k(M)$. (Zero forms are smooth functions)

Again a differential form ω gives rise to local mutilinear maps

$$w_x : T_xM \times ... \times T_xM \to \mathbb{R}$$

by $\omega_x(x_1, ..., x_n) := \omega(X_1, ..., X_n)$ where X_i are vector fields extending x_i, i.e. $(X_i)_x = x_i$. This is well-defined and lets us interpret differential forms as assignments which assign to every point $x \in M$ a alternating multilinear morphism of the tangent space T_xM. In particular we see that $\Omega^k(M) = 0$ for $k > \dim M$. Again the assigment $x \mapsto \omega_x \in \mathrm{Alt}^k(T_xM)$ uniquely determines ω.

Example 2.15.10. 1-forms are given as $C^\infty(M)$-linear maps $\mathfrak{X}(M) \to C^\infty(M)$. For $M = \mathbb{R}^n$ we had a basis of $\mathfrak{X}(\mathbb{R}^n)$ (over $C^\infty(\mathbb{R}^n)$) by $\frac{\partial}{\partial x_i}$. We denote the dual basis by dx_i, i.e. $dx_i(\frac{\partial}{\partial x_j}) = \delta_{ij}$. Then 1-forms are given as

$$\Omega^1(\mathbb{R}^n) \cong \bigoplus_{k=1}^n C^\infty(\mathbb{R}^n) dx_i$$

in other words a 1-form is the same as a linear combination $\omega = \sum_{k=1}^n f_i dx_i$ where again we find that $f_i = \omega(\frac{\partial}{\partial x_i})$.

Definition 2.15.11. Let $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$. We define a new $(k+l)$-form for $w \wedge \eta$ called the wedge product as follows:

$$(\omega \wedge \eta)(X_1, ..., X_{k+l}) := \sum_{\sigma \in \Sigma_{k+l}} \mathrm{sign}(\sigma)\ \omega\big(X_{\sigma(1)}, ..., X_{\sigma(k)}\big) \cdot \eta\big(X_{\sigma(k+1)}, ..., X_{\sigma_{k+l}}\big)$$

For example for two 1-forms we find

$$(\omega \wedge \eta)(X, Y) = \omega(X)\eta(Y) - \omega(Y)\eta(X)$$

Lemma 2.15.12. *The product $\wedge$ makes $\Omega^*(M)$ into a graded commutative algebra.*

As a corollar we see that for odd dimensional forms ω we have $\omega \wedge \omega = 0$. In fact the formula for 1-forms as given above together with the requirement that $\wedge$ should be graded commutative and associative already determines it uniquely.

Example 2.15.13. For $M = \mathbb{R}^n$ we have the forms $dx_i \wedge dx_j \in \Omega^2(M)$ and the triple products $dx_i \wedge dx_j \wedge dx_k \in \Omega^3(M)$. More generally the $C^\infty(M)$ module $\Omega^k(M)$ has a basis over $C^\infty(M)$ consisting of forms

$$dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k} \qquad \text{with } 1 \leq i_1 \leq i_2 \leq ... \leq i_k \leq i_k$$

Thus it has rank $\binom{n}{k}$.

Definition 2.15.14. Given a smooth map $f : M \to N$ and a differential form $\omega \in \Omega^k(N)$ we get a k-form $f^*\omega \in \Omega^k(M)$ as follows:

$$f^*(\omega)_x(v) = \omega(df_x(v))$$

for $v \in T_xM$. This is again a well-defined differential form (which needs a proof that we will skip here).

Lemma 2.15.15. *The assignment $f^* : \Omega^*(N) \to \Omega^*(M)$ defines a morphism of graded commutative algebras, i.e. $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$. It is moreover functorial, i.e. $(fg)^* = g^*f^*$ and $\mathrm{id}^* = \mathrm{id}$.*

Now we finally want to define a derivative. For a function $f \in C^\infty(M)$ we let df be the 1-form defined as

$$df(X) := X(f)$$

and call it the derivative of f. This defined an $\mathbb{R}$-linear morphism $d : C^\infty(M) \to \Omega^1(M)$ with the property that $d(fg) = df \cdot g + g \cdot df$. Note that it is not $C^\infty(M)$ linear. For constant functions f we have $df = 0$. For $M = \mathbb{R}^n$ we find that

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$$

That also explains that dx_i is the derivative of the i-th coordinate function.

Theorem 2.15.16. *There is a unique degree 1 morphism $d : \Omega^*(M) \to \Omega^*(M)$ (i.e. it takes $\Omega^k(X)$ to $\Omega^{k+1}(X)$ which extends the derivative defined above and makes $\Omega^*(M)$ a CDGA. Explicitly this means*

1. *$df \in \Omega^1(M)$ is the derivative defined above for smooth functions $f \in C^\infty(M) = \Omega^0(M)$.*

2. *$d^2 = 0$ i.e. $(\Omega^*(X), d)$ is a cochain complex*

3. *$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{|\omega|}(\omega \wedge d\eta)$, i.e d it is a degree 1 differential.*

This derivative is natural in M, i.e. for a smooth map $f : M \to N$ the induced morphism $f^* : \Omega^*(N) \to \Omega^*(M)$ is a morphism of CDGAs.

Proof. □

Remark 2.15.17. One can even give an explicit formula for the exterior derivative which is sometimes useful. We will only remark that formula without proof. For $\omega \in \Omega^k(M)$ it is

$$d\omega(X_0, ..., X_k) = \sum_{i=0}^{n}(-1)^i X_i(\omega(X_0, ..., \hat{X}_i, ..., X_k) + \sum_{i<j}(-1)^{i+j}\omega([X_i, X_j], X_0, ..., \hat{X}_i, ..., \hat{X}_j, ...X_k)$$

where $[X_i, X_j] := X_i \circ X_j - X_j \circ X_i$ is the commutator of derivations and $\hat{X}$ denotes omission. For example for $k = 1$ we get

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y])$$

Now we can finally define the de Rham cohomology groups.

Definition 2.15.18. Let M be a smooth manifold. We define its de Rham cohomology groups as

$$H^k_{dR}(M) := H^k(\Omega^*(M))$$

For a smooth morphism $f : M \to N$ we denote the induced morphism by $f^* : H^k(N) \to H^k(M)$.

A form ω is called closed is $d\omega = 0$ and exact if it is of the form $\omega = d\eta$ for some η. Thus the closed forms are the cocycles in $\Omega^*(M)$ and the exact forms are the coboundaries. Thus the de Rham cohomology group is defined as closed forms modulo exact forms.

Corollary 2.15.19. *The de Rham cohomology groups $H^*_{dR}(M) := \bigoplus_{k=0}^\infty H^k_{dR}(M)$ form a graded commutative ring (even an $\mathbb{R}$-algebra) and the morphisms f^* are ring morphisms. We have that $H^k_{dR}(M) = 0$ for $k > \dim M$.*

Example 2.15.20. Let us explicitly compute the deRham cohomology. For $\mathbb{R}^0$ we have that $\Omega^*(\mathbb{R}^0) = \mathbb{R}$ thus $H^*(\mathbb{R}^0) = \mathbb{R}$ for $* = 0$ and 0 else. For $\mathbb{R}^n$ we have that a zero form $f : \mathbb{R}^n \to \mathbb{R}$ is closed if $df = 0$ which means that f has to be a constant function. Thus we get that $H^0_{dR}(\mathbb{R}^n) = \mathbb{R}$. The same argument works for all connected manifolds M.

Now let us consider $\mathbb{R}^1$ and try to compute its deRham cohomology. The only non-trivial case is $H^1(\mathbb{R}^1)$. Every 1-form is of the form $f dx$ for f smooth and is automatically closed. It is closed if it is of the form

$$dg = \frac{\partial g}{\partial x} dx$$

for some $g \in C^\infty(\mathbb{R})$. But this is always the case since for every smooth f we find g integrating it. Thus every closed form is exact which shows that $H^1(\mathbb{R}) = 0$.

Our main result will be that the de Rham cohomology $H^k_{dR}(M)$ is isomorphic to singular cohomology $H^k(M;\mathbb{R})$. In particular we have to show that the value on $\mathbb{R}^n$ is trivial. The computation made above works for all n as we will show first.

Theorem 2.15.21 (Poincare Lemma). *Let U be an open convex subset of $\mathbb{R}^n$. Then the complex*

$$\Omega^0(U) \to \Omega^1(U) \to \Omega^2(U) \to ...$$

is exact. Thus $H^n_{dR}(U) = 0$ for $n \geq 1$. Any function f with $df = 0$ is constant, thus $H^0(U) \simeq \mathbb{R}$.

Proof. WLOG $U \cong \mathbb{R}^n$ (in fact for the proof we give it suffices to assume $0 \in U$. We denote the coordinates in $\mathbb{R}^n$ as always with $x_1, ..., x_n$. We define

$$H : \Omega^{k+1}(\mathbb{R}^{n+1}) \to \Omega^k(\mathbb{R}^n)$$

as follows: for $\omega = f dx_{i_0} \wedge dx_{i_1} \wedge dx_{i_2} \wedge ... \wedge dx_{i_k}$ we set

$$H(\omega) = \left(\int_0^1 t^k f(tx)dt\right) \sum_{j=0}^{n} (-1)^j x_{i_j} dx_{i_0} \wedge ....\widehat{dx}_{i_j} ... \wedge dx_{i_k}$$

Now we compute that

$$dH(\omega) + Hd\omega = \omega$$

Thus if $d\omega = 0$ then ω is exact.

□

Example 2.15.22. Let us consider the case $n = 3$. Then the sequence

$$0 \to \mathbb{R} \to \Omega^0(\mathbb{R}^3) \to \Omega^1(\mathbb{R}^3) \to \Omega^2(\mathbb{R}^3) \to \Omega^3(\mathbb{R}^3) \to 0$$

is exact. Using the standard basis we get an identification

$$0 \to \mathbb{R} \to C^\infty(\mathbb{R}^3) \xrightarrow{\text{grad}} C^\infty(\mathbb{R}^3, \mathbb{R}^3) \xrightarrow{\text{curl}} C^\infty(\mathbb{R}^3, \mathbb{R}^3) \xrightarrow{\text{div}} \mathbb{C}^\infty(\mathbb{R}^3) \to 0$$

Proposition 2.15.23 (Mayer-Vietoris). *Let M be a smooth manifold M which is covered by two open sets $U, V \subseteq M$. Then we have a long exact sequence*

$$... \to H^*_{dR}(M) \to H^*_{dR}(U) \oplus H^*_{dR}(V) \to H^*_{dR}(U \cap V) \to H^{*+1}_{dR}(M) \to ...$$

Proof. We claim that the sequence

$$\Omega^*(M) \to \Omega^*(U) \oplus \Omega^*(V) \to \Omega^*(U \cap V)$$

is short exact. This is clear on the left two spots. Thus we only have to check surjectivitiy on the right. Let $\omega \in \Omega^*(U \cap V)$. We must show that on $U \cap V$ we have $\omega = \omega_U + \omega_V$ for some forms $\omega_U \in \Omega^*(U)$ and $\omega_V \in \Omega^*(V)$. We choose a function $f : M \to \mathbb{R}$ which is 0 in a neighborhood of $U \setminus U \cap V$ and 1 in a neighborhood of $V \setminus U \cap V$. Thus $f\omega$ is zero in a neighborhood of $U \setminus V$ and can be extended to U by zero. Similarly $(1 - f)\omega$ can be extended to V. Thus we find the decomposition $\omega = f\omega + (1 - f)\omega$. □

We assume knowledge of the notion of the Riemann (or Lebesgue) integral in several variables. Thus for a smooth function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ with compact support we get an integral

$$\int_{R^n} f\mu = \int_{\mathbb{R}^n} f dx_1 ... dx_m n$$

We say that an k-form $\omega \in \Omega^k(M)$ has compact support if there is a compact set $K \subseteq M$ such that for $y \in M \setminus K$ we have $\omega_y = 0$. We denote those forms by $\Omega^k_c(M)$. If M is a manifold and $U \subseteq M$ open then we can consider forms in $\Omega^k_c(U)$ as defined on the whole of M, i.e. we have an inclusion $\Omega^k_c(U) \subseteq \Omega^k_c(M)$ called extension by zero. We will freely identify that. If a map $f : M \to N$ is proper then the pullback of differential forms preserves compactly supported forms.

Remark 2.15.24. We are going to integrate differential n-forms on n-manifolds. In particular for $\mathbb{R}^n$ we will integrate forms and not functions as above. So the integral will be interpreted as the integral of the n-form $f dx_1 \wedge ... \wedge dx_n$. Let us indicate why we can not get a well-defined notion of how to integrate functions on manifolds. If this was possible it would have to be independent of the choice of chart. So for $\mathbb{R}^n$ we would be able to choose another chart, i.e. a diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ and express the integration in this new chart. But then the transformation formula gives us:

$$\int_{\mathbb{R}^n} f dx_1 ... dx_n = \int_{\mathbb{R}^n} f\phi |\det(J\phi)| dx_1 ... dx_n$$

Thus the new function we have to integrate is not $f\phi$ but $f\phi|\det(J\phi)|$. Fortunately it turns out that the pullback $\phi^*(\omega)$ of $\omega = f dx_1 \wedge ... \wedge dx_n$ is exactly given as

$$\phi^*(f)\phi^*(dx_1) \wedge ... \wedge \phi^*(dx_n) = f\phi \det(J\phi) dx_1 \wedge ... \wedge dx_n$$

Thus the integral of the n-form ω is independent of the choice of chart as long as the determinant of $J\phi$ is positive, i.e. $J\phi$ is orientation preserving.

Lemma 2.15.25. *Let M be a smooth, oriented n-manifold and $\omega \in \Omega^n_c(M)$. Choose an oriented smooth atlas $\mathcal{A} = \{h_i : U_i \to \mathbb{R}^n\}$. Then we can write ω as*

$$\omega = \sum_{i \in I} h_i^*(\eta_i)$$

for forms $\eta_i \in \Omega^n_c(\mathbb{R}^n)$ (recall that the forms are considered as forms on M by extension through zero. The sum is infinite but locally finite.

Proof. Let $(\epsilon_i)_{i \in I}$ be a smooth partition of unity subordinated to U_i. This means that the support of each ϵ_i lies in U_i and that we have $\sum_{i \in I} \epsilon_i = 1$ where the sum is locally finite. Then $\epsilon_i \omega|_{U_i} \in \Omega^*_c(U_i)$. Now let $\eta_i \in \Omega^n(\mathbb{R}^n)$ be the forms obtained as $(h_i^{-1})^*(\epsilon_i \omega|_{U_i})$. We have

$$\sum_{i \in I} h_i^*(\eta_i) = \sum_{i \in I} \epsilon_i \omega = \omega$$

□

Definition 2.15.26. We want to define the integral of an n-form $\omega \in \Omega^n_c(M)$. Choose an oriented smooth atlas $\mathcal{A} = \{h_i : U_i \to \mathbb{R}^n\}$ and write

$$\omega = \sum_{i \in I} h_i^*(\eta_i)$$

according to Lemma 2.15.25. Then define

$$\int_M \omega := \sum_{i \in I} \int_{\mathbb{R}^n} h_i$$

Proposition 2.15.27. *For a manifold M the integral is well-defined (i.e. independent of the choices) and has the following properties:*

1.
$$\int : \Omega^n_c(M) \to \mathbb{R}$$
is $\mathbb{R}$-linear, i.e
$$\int_M c\omega + d\mu = c \int_M \omega + d \int_M \mu$$

2. *If f is an orientation preserving diffeomorphism $M \to N$ then*
$$\int_N \omega = \int_M f^*\omega$$

3. *For M orientatied and $\overline{M}$ the opposite orientated manifold*
$$\int_M \omega = - \int_{\overline{M}} \omega$$

4. *If $U \subseteq M$ is open and contains the suppport of $\omega \in \Omega^*_c(M)$ then*
$$\int_M \omega = \int_U \omega|_U$$

Remark 2.15.28. It is very often still possible to integrate functions on a manifold M or define the volume of M. This however requires a choice of a fixed n-form ω on a manifold M (a so called volume form) and an orientation. Because then we can defined the integral over $f : M \to \mathbb{R}$ as the integral $\int_M f\omega$. For example if the manifold has a Riemannian metric and an orientation then there is a canonical choice of volume form. If the manifold M is embedded in $\mathbb{R}^n$ then we get a canonical choice of volume form as well. But it depends on the embedding.

For a smooth manifold M with boundary the boundary is also a smooth manifold. Recall that an orientation of M induces an orientation on ∂M (we have discussed that only for compact manifolds using fundamental classes but it also works for non-compact manifolds by looking at local orientations).

Theorem 2.15.29 (Stokes Theorem)**.** *Let $\omega \in \Omega^{n-1}(M)$ on an oriented manifold M of dimension n with boundary. Then*

$$\int_M d\omega = \int_{\partial M} \omega$$

We will not prove that here, but let us consider the special case for $M = [0,1]$ and $\omega = f$ (a zero form). Then we get

$$\int_{[0,1]} f' dx = \int_{\{0,1\}} f = f(1) - f(0)$$

This is the fundamental theorem of calculus.

Corollary 2.15.30. *For a closed, compact manifold M we get that*

$$\int_M d\omega = 0$$

Thus the integral defines a morphism

$$\int : H^n_{dR}(M) \to \mathbb{R}$$

It will be a consequence of the de Rham isomorphism and Poincare duality that this is an isomorphism.

Now we consider the simplex Δ^k as a subset of $\mathbb{R}^k$ by

$$\Delta^n \cong \{(x_1, ..., x_k) \mid x_i \geq 0, \sum x_i \leq 1\}$$

We say that a continuous map $\Delta^k \to M$ is smooth if it admits an extension to a smooth function on a small neighborhood of Δ^k in $\mathbb{R}^k$. Note that Δ^k is not a smooth manifold with boundary (the boundary is only piecewise smooth).

Definition 2.15.31. The smooth singular complex $C_*^{sm}(M)$ is the subcomplex of $C_*(M)$ defined in degree k to be the free abelian group generated by the smooth k-simplices $\Delta^k \to M$. Likewise the complex $C^*_{sm}(M; A)$ is defined to be the complex of cochains defined on smooth simplices with values in A. The associated homology and cohomology is called smooth singular homology resp. cohomology.

Since the differential clearly preserves smooth simplcies these form chain complexes and we have a canonical map

$$C^*(M; A) \to C^*_{sm}(M; A)$$

In particular we get induced morphisms

$$H^*(M; A) \to H^*_{sm}(M; A)$$

Lemma 2.15.32. *The smooth singular cohomology has the following properties: for a convex subset $U \in \mathbb{R}^n$ we have*

$$H^*_{sm}(U; A) = \begin{cases} A & \text{for } * = 0 \\ 0 & \text{else} \end{cases}$$

and the same for H_*^{sm}. For an open covering we obtain a Mayer-Vietoris Sequence in homology and cohomology which is compatible with the Mayer-Vietoris sequences in singular homology and cohomology.

The following Lemma is a variant of the bootstrap Lemma 2.10.2.

Lemma 2.15.33. *Let M be smooth manifold and $P(U)$ be a statement for all open subsets of $U \subseteq M$ such that*

1. *$P(U)$ is true for U diffeomorphic to a convex subset of $\mathbb{R}^n$*
2. *$P(U), P(V), P(U \cap V) \Rightarrow P(U \cup V)$*
3. *U_α disjoint and $P(U_\alpha)$ for all $\alpha \Rightarrow P(\bigcup U_\alpha)$.*

Then $P(M)$ is true.

Proof. Page 290 Bredon. □

Proposition 2.15.34. *The canonical morphisms*

$$H_*^{sm}(M; A) \to H_*(M; A) \qquad \text{and} \qquad H^*(M; A) \to H^*_{sm}(M; A)$$

are isomorphisms for all smooth manifolds M.

Proof. We do the case of cohomology. Homology follows with the same argument. We will use the bootstrap Lemma above. The statement $P(U)$ is that the morphism $H^*(M; A) \to H^*_{sm}(M; A)$ is an isomorphism. So lets start by considering convex open subsets of $\mathbb{R}^n$. We know by the Lemma above the all homology groups are abstractly isomorphic. It is also clear that in H^0 the induced morphism is an isomorphism (both are generated by any point $x : \Delta^0 \to M$ which is certainly smooth. The second assertion about $P(U)$ follows from the MV-Sequence and the third from the decomposition $H^*(\bigsqcup U_\alpha) \cong \prod H^*(U_\alpha)$. □

If we have a differential k-form $\omega \in \Omega^k(M)$ and a smooth map $f : \Delta^k \to M$ (which means that we have an extension $\tilde{f} : U \to M$ where $\Delta^k \subseteq U \subseteq \mathbb{R}^k$. We can define the integral even if Δ^k is not a smooth manifold

$$\int_{\Delta^k} f^*\omega \in \mathbb{R}$$

as the integral over Δ^k of $\tilde{f}^*\omega$. For that we use the canonical orientation of $\mathbb{R}^n$. Then we get using a variant of Stokes Theorem for piecwise smooth boundaries that

$$\int_{\Delta^k} f^* d\omega = \sum_{i=0}^{k} (-1)^i \int_{\Delta^{k-1}} (f^{(i)})^*\omega$$

where $f^i : \Delta^{k-1} \to M$ is the i-th face of the simplex. To see that we can either use a more general variant of Stokes theorem which essentially has the same proof. Or we use the following trick:approximate Δ^k by a sequence of compact subsets

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq A_3 \subseteq ....$$

where $\Delta^k = \cup A_i$ and all A_i are actually smooth manifolds $A_i \subseteq \mathbb{R}^n$ with boundary (diffeomorphic to disks). Then we get that

$$\int_{\Delta^k} f^* d\omega = \lim \int_{A_i} f^* d\omega|_{A_i} = \lim \int_{\partial A_i} f^*\omega = \int_{\partial \Delta_k} f^*\omega$$

Lemma 2.15.35. *Integration defined a well-defined morphism*

$$I : \Omega^*(M) \to C^*_{sm}(M; \mathbb{R})$$

which is natural in M and which is compatible with the associated MV-sequences.

Theorem 2.15.36 (de Rham's theorem)**.** *For every smooth manifold M the de Rham morphism induces an isomorphism*

$$H^*_{dR}(M) \to H^*_{sm}(M; \mathbb{R})$$

thus we get an induced equivalence $H^*_{dR}(M) \cong H^*(M; \mathbb{R})$.

There is also a complex of differential forms with compact support

$$\Omega^*_c(M)$$

and clearly it gives also a cohomology which we denote as $H^*_{dR,c}(M)$ and call it de Rham cohomology with compact support.

Example 2.15.37. On $\mathbb{R}^n$ we have a form $\omega = f dx_1 \wedge dx_2 ... dx_n$ where $f : \mathbb{R}^n \to \mathbb{R}$ is a positive bump function around the origin. Then $\omega \in \Omega^n_c(M)$. Clearly ω is closed. Thus there is a form $\mu \in \Omega^{n-1}(M)$ with $d\mu = \omega$. But μ can not be compactly supported since we have $\int_{R^n} \omega > 0$ (and it would have to be zero by Stokes otherwise). In other words $[\omega] = 0$ in $H^*_{dR}(M)$ but $[omega] \neq 0$ in $H^*_{dR,c}(M)$.

Similarly to compact forms with have $C^*_{sm,c}(M)$ and $C^*_c(M)$.

Lemma 2.15.38. *The de Rham morphism I induces a morphism of cochain complexex (not of graded algebras!!!!)*

$$\Omega^*_c(M) \to C^*_{sm,c}(M; R)$$

Theorem 2.15.39. *The induced morphisms $H^*_{dR,c}(M) \to H^*_{sm,c}(M; \mathbb{R})$ and $H^*_c(M, \mathbb{R})$ are isomorphisms.*

Proof. Consider $\Omega^*(X, U) := \ker(\Omega^*(X) \to \Omega^*(U))$ Then quasi iso

$$\Omega^*(X, U) \to \mathbb{C}^*_{sm}(X; U)$$

and then colimit. □

Corollary 2.15.40. *If M is oriented then we have an isomorphism*

$$H^*_{dR,c}(M) \cong H_*(M; \mathbb{R})$$

Corollary 2.15.41. *For M oriented the morphism*

$$\int : H^n_c(M; \mathbb{R}) \to \mathbb{R}$$

is an isomorphism.

We finall remark the following thing without proof:

Proposition 2.15.42 (without proof)**.** *The de Rham isomorphism*

$$I : H^n_{dR}(M) \to H^n_{sm}(M; \mathbb{R})$$

is compatible with products, i.e. its a morphism graded rings (this is not the case on chain level). Moreover for M oriented the Kronecker pairing between homology and cohomology corresponds under the de Rham isomorphism to the pairing

$$H^k_{dR}(M) \times H^{n-k}_c(M) \to \mathbb{R} \qquad (\omega, \nu) \mapsto \int_M \omega \wedge \nu$$

Chapter 3

Topology III, Winter term 2025/26

Organizational matters:

- Oral exams at the end of the semester. Allowance criteria:
    1. Participation in the Exercise classes
    2. Present a solution
- Exercices: One exercise sheet per week (starting this Wednesday). Turn in alone. Turn in solution Wednesday in class. Discussed in the excercises classes. The first sheet is uploaded this Wednesday.
- Date for the exercises: We ?, Maxime Ramzi
- Literature:
    - Bredon: Topology and Geometry
    - Hatcher: Algebraic Topology
    - tom Dieck: Algebraic Topology
- This is the follow-up lecture of Topology I and II. We will assume everything that has been covered in Topology I and from Topology II the basics of cohomology. If you do not satisify these requirements please come and talk to me.
- Time for lecture? Mondays and Thursday, 16:00-18:30 without break or 16:00-18:45 with break or 16:15 - 18:45 without break

3.1 Overview: Homotopy theory

Recall: two space X, Y are called homotopy equivalent if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $fg \simeq \mathrm{id}_Y$ and $gf \simeq \mathrm{id}_X$.

Definition 3.1.1 (Preliminary)**.** A homotopy type is an equivalence class of topological spaces under the equivalence relation 'homotopy equivalence'.

With this definition homotopy theory could be described as the study of homotopy types.

So far we have seen a bunch of invariants of homotopy types, for example the singular (co)homology groups. These were good invariants in the sense that they could distinguish a lot of homotopy types, were computable and also could answer 'geometric' questions.

The most important invariant that we will study in this course are the homotopy groups

$$\pi_n(X, x) := [S^n, X]_*$$

given as pointed homotopy classes of pointed maps from the n-sphere S^n to X. Similar to the case of π_1 we will often abusively drop the basepoint from the notation. It is a priori not clear that these are even groups, but we will define a group structure and see that $\pi_n(X)$ for $n \geq 2$ is even *abelian*.

It will turn out that these homotopy groups are *much* harder to compute than homology and cohomology groups, for example until today the homotopy groups of the 2-sphere S^2 are *not* completely known. We will establish some tools to compute homotopy groups, which we will review later.

While they are hard to compute, the homotopy groups are much better than the homology groups in distinguishing spaces. Let us make this precise now:

Definition 3.1.2. A map $f : X \to Y$ of topological spaces is called *weak homotopy equivalence* if it induces an isomorphism on all homotopy groups, i.e. $\pi_n(X, x) \to \pi_n(Y, f(x))$ for all $n \geq 0$ and all choices of basepoint $x \in X$ (of course for $n = 0$ no choice of basepoint is needed).

We will see that singular homology and cohomology are actually invariant under weak homotopy equivalence, that is if $f : X \to Y$ is a weak homotopy equivalence then $f_* : H_*(X) \to H_*(Y)$ and $f^* : H^*(Y) \to H^*(X)$ are isomorphisms.

Why is this called weak homotopy equivalence? That is because of the following result:

Theorem 3.1.3 (Whitehead)**.** *If X and Y are CW complexes then a map $f : X \to Y$ is a weak homotopy equivalence iff it is a homotopy equivalence. Moreover for every space X there is a CW complex X' which is weakly homotopy equivalent to X.*

Warning 3.1.4. This does not mean that two CW complexes X and Y with isomorphic homotopy groups are homotopy equivalent!

Whitehead's theorem in particular implies that the canonical map

$$\frac{\{\text{CW complexes}\}}{\text{homotopy equivalence}} \xrightarrow{\cong} \frac{\{\text{topological Spaces}\}}{\text{weak homotopy equivalence}}$$

is a bijection.

Definition 3.1.5 (Improved)**.** A homotopy type is a topological space up to weak homotopy equivalence.

Thus homotopy theory is the study of these (weak) homotopy types. Of course we even study maps between those, thus the homotopy category Ho(CW) of CW complexes, aka the weak homotopy category of spaces. One of the goals of this lecture is to give a purely combinatorial model for this category using the theory of simplicial sets.

After this formal discussions we will return to the study of homotopy groups (which are invariants of a homotopy type) and establish the following properties:

1. For a covering $\widetilde{X} \to X$ the induced map $\pi_n(\widetilde{X}) \to \pi_n(X)$ is an isomorphism for $n \geq 2$. This will in particular let us compute all homotopy groups of surfaces except for S^2 and $\mathbb{R}P^2$.

2. Every map $f : X \to Y$ between CW complexes is homotopic to a cellular map (Cellular approximation theorem). This implies that $\pi_k(S^n) = 0$ for $k < n$.

3. There is a homomorphism $\pi_n(X) \to H_n(X)$ induced by pulling back the fundamental class $[S^n] \in H_n(S^n)$. If $\pi_k(X) = 0$ for $k < n$ and $n \geq 2$ then this map is an isomorphism (Hurewicz theorem). This implies that $\pi_n(S^n) = \mathbb{Z}$.

4. For a *fibre sequence* $F \to E \to B$ of homotopy types (a notion that we will define in the lecture) there is a long exact sequence in homotopy groups
$$... \to \pi_{n+1}B \to \pi_n F \to \pi_n E \to \pi_n B \to \pi_{n-1}F \to ...$$
In particular for a covering $p : E \to B$ we have a fibre sequence $F \to E \to B$ where $F := p^{-1}(b)$ is the fibre, which is a set considered as a discrete space.

5. Suspension induces a homomorphism of groups
$$\pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1})$$
which is an isomorphism for $n > k + 1$. In particular for large n the group $\pi_{n+k}(S^n)$ is independent of n and called the k-th stable homotopy group (of spheres). These stable homotopy groups are in a sense the 'primes numbers of homotopy theory' and a lot of current homotopy theory is devoted to studying those.

So far we have talked about homotopy types like spheres and more generally finite CW complexes, whose homotopy groups are extremely complicated while the homology tends to be quite computable. But there are also spaces which show a different behaviour.

Definition 3.1.6. A homotopy type X is called Eilenberg–MacLane space if there exists a positive integer n such that $\pi_k(X) = 0$ for $k \neq n$.

The foundational result that we will show is that for every n and every group A (abelian if $n \geq 2$) there exists an Eilenberg Mac–Lane space $X = K(A, n)$ with $\pi_n X = A$. Moreover we will see that $K(A, n)$ is uniquely determined by this requirement. These Eilenberg–MacLane spaces are important for two reasons:

1. We will show that every homotopy type X can be 'decomposed' into Eilenberg–MacLane spaces in a certain sense: There is a 'filtration' of X (called the *Whitehead tower*)
$$\cdots \to \tau_{\geq 3}X \to \tau_{\geq 2}X \to \tau_{\geq 1}X \to \tau_0 X = X$$
such that there are fibre sequences
$$K(\pi_n X, n-1) \to \tau_{\geq n+1}X \to \tau_{\geq n}X \ .$$
Here $\tau_{\geq 1}X \to X$ is the inclusion of the connected component of the basepoint of X and $\tau_{\geq 2}X \to \tau_{\geq 1}X$ is the universal cover.

2. We will show that the Eilenberg–MacLane spaces $K(A, n)$ represent cohomology groups in the sense that for every abelian group and every n there is a bijection

$$H^n(X; A) = [X, K(A, n)] .$$

This will be very useful to study cohomology.

The second fact might seem very suprising, that cohomology groups are 'just' homotopy groups of maps into some carefully defined space. But it turns out that this holds in much greater generality: for every generalized cohomology theory E^* (which is invariant under weak homotopy equivalence) and every $n \in \mathbb{Z}$ there are homotopy types $E(n)$ such that for every homotopy type X we have

$$E^n(X) = [X, E(n)] .$$

Moreover these homotopy types $E(n)$ are uniquely determined by the cohomology theory E and there are homotopy equivalences

$$E(n) \simeq \Omega E(n+1) .$$

The data of the homotopy types $E(n)$ and these homotopy equivalences is called a *spectrum*. In particular we have seen that every cohomology theory determines a spectrum.

Spectra are the fundamental objects of *stable homotopy theory* and are closely related to that stable homotopy groups discussed above. If time allows, then we will discuss spectra and stable homotopy theory a bit.

3.2 Homotopy groups

Recall that at the very beginning we defined the homotopy 'groups' of a pointed space (X, x) to the pointed homotopy classes of maps

$$\pi_n(X, x) = [S^n, X]_* = [S^n, 1; X, x]$$

Sometimes it will be convenient to give a slightly different but equivalent definition as

$$\pi_n(X, x) \cong [I^n, \partial I^n; X, x]$$

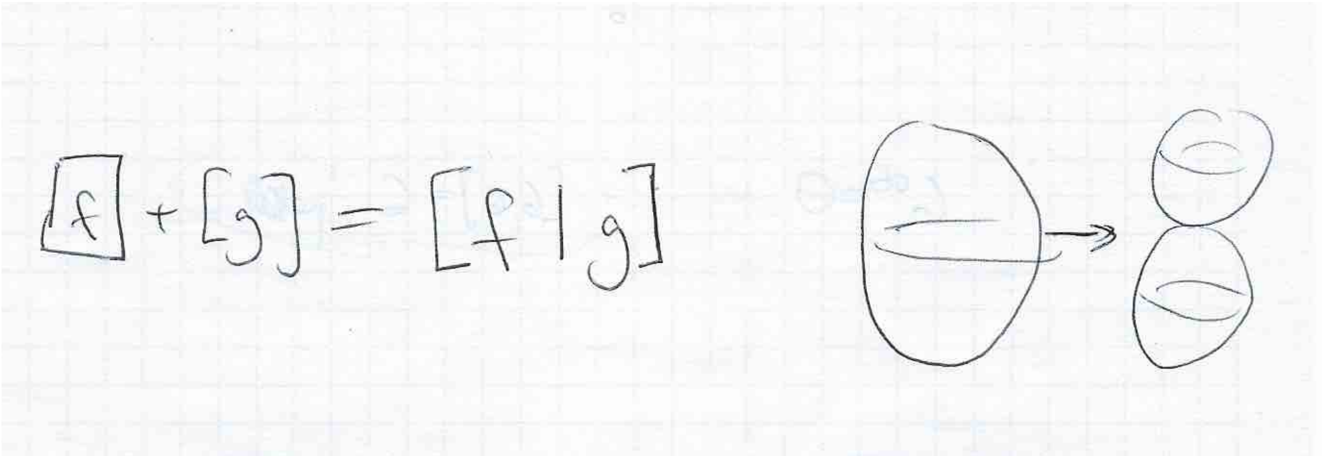
where $I^n = [0, 1]^n$ is the cube. This even makes sense for $n = 0$ where $I^0 = *$ and $\partial I^0 = \emptyset$. We will freely switch between the two perspectives and hope that this does not lead to confusion.

Definition 3.2.1. The sum operation for $[f], [g] \in \pi_n(X, x)$ (with $f, g : I^n \to X$) is defined as follows:

$$(f+g)(s_1, ..., x_n) = \begin{cases} f(2s_1, s_2, ..., s_n) & s_1 \in [0, 1/2] \\ g(2s_1 - 1, s_2, ..., s_n) & s_1 \in [1/2, 1] \end{cases}$$

Alternatively we can define it using the pinch map $S^n \to S^n \vee S^n$ that collapses $S^{n-1} \subseteq S^n$ to a point as the composite

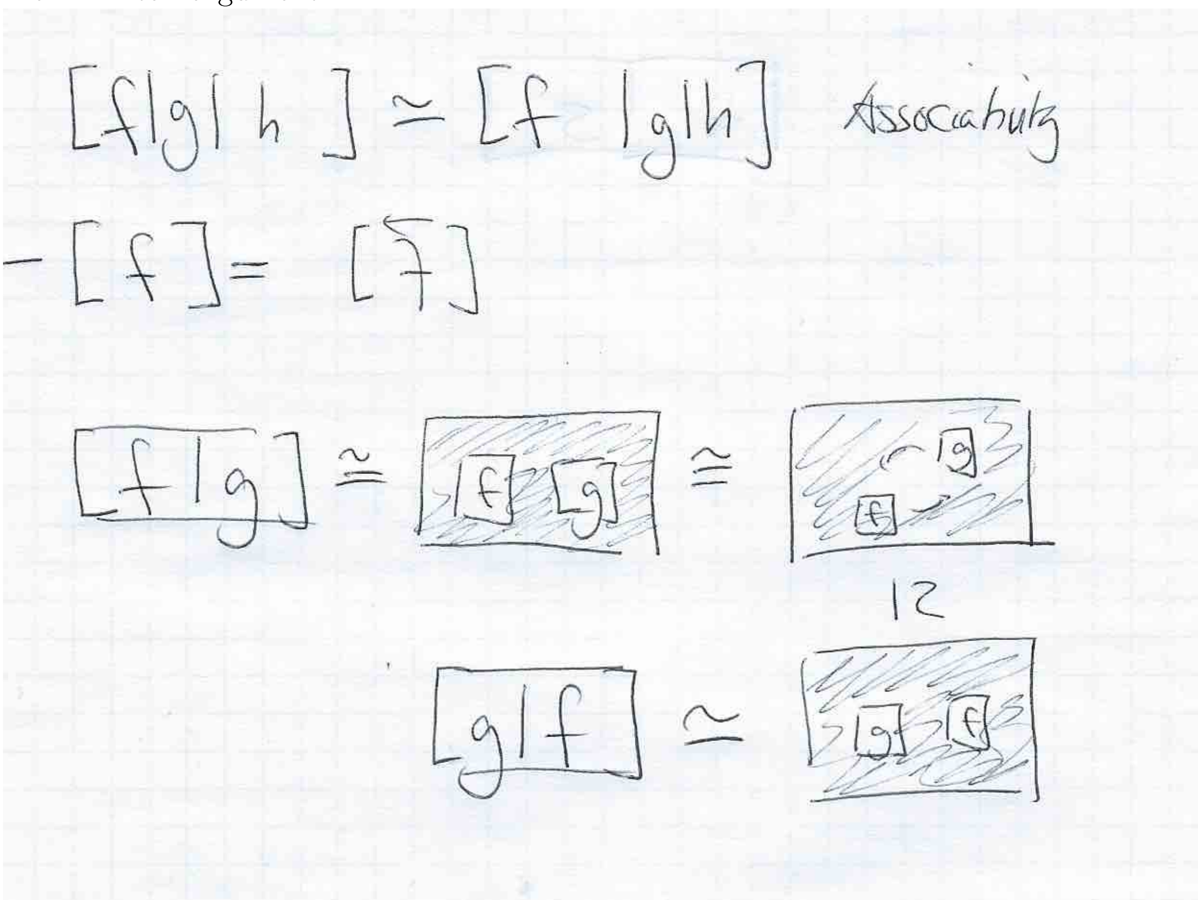
$$S^n \to S^n \vee S^n \xrightarrow{f \vee g} X$$



Note that the same formula also works for the group structure in the case $n = 1$, but we don't want to use additive notation for that since its not abelian. However:

Proposition 3.2.2. *The sum operation equips $\pi_n(X, x)$ with a well-defined, abelian group structure.*

Proof. The proof that $\pi_n(X, x)$ is a group is the same as for $\pi_1(X, x)$ and is left as an exercise. Some pictures are below. To see that the group is abelian we invoke the following Eckmann-Hilton argument:



There is also a slighly less graphical way of making this argument precise as follows which is also very instructive, so that we want to make it precise here. We have defined an addition operation $+ : \pi_n(X) \times \pi_n(X) \to \pi_n(X)$ using the first coordinate. But we can also use the second variable to define another group structure:

$$(f * g)(s_1, ..., x_n) = \begin{cases} f(s_1, 2s_2, ..., s_n) & s_2 \in [0, 1/2] \\ g(s_1, 2s_2 - 1, ..., s_n) & s_2 \in [1/2, 1] \end{cases}$$

Since the coordinates do not speak to each other we get that $* : \pi_n(X, x) \times \pi_n(X, x) \to \pi_n(X, x)$ is a group homomorphism with respect to $+$, i.e. we have

$$(a + b) * (c + d) = (a * c) + (c * d) \ .$$

Then the following important argument given in Lemma 3.2.3 finishes the proof. □

Lemma 3.2.3 (Eckmann-Hilton)**.** *Assume a set M has two different unital monoid structures $\cdot$ and $*$ such that one is a homomorphism for the other, i.e.*

$$(a \cdot b) * (c \cdot d) = (a * c) \cdot (b * d) \ .$$

Then the two monoid structures agree and are commutative.

Proof. Let us denote the units of these monoid structures by 1 and 1_*. Then we have

$$1_* = 1_* * 1_* = (1 \cdot 1_*) * (1_* \cdot 1) = (1 * 1_*) \cdot (1_* * 1) = 1 \cdot 1 = 1$$

and thus

$$a * b = (a \cdot 1) * (1 \cdot b) = (a * 1) \cdot (1 * b) = a \cdot b$$

and

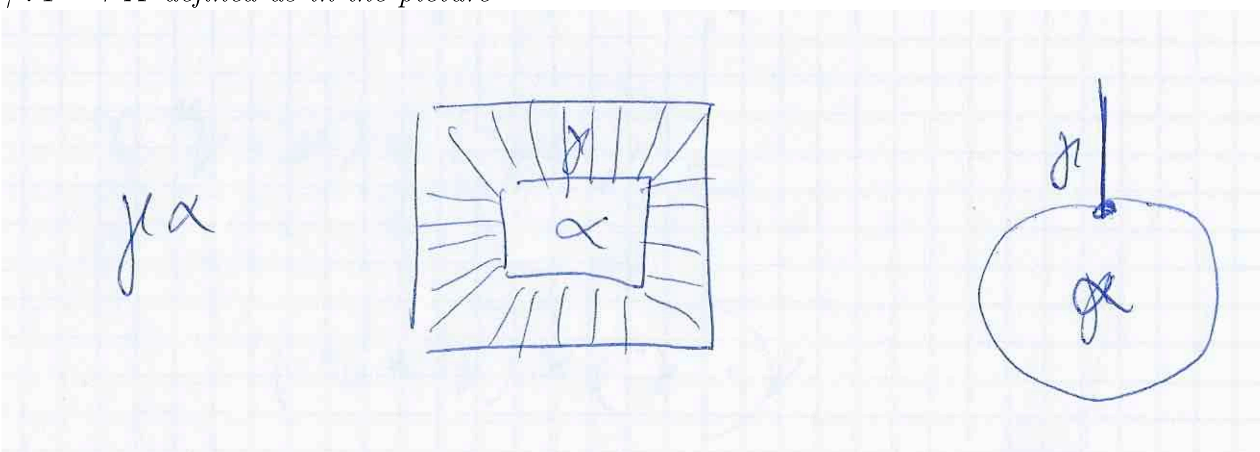
$$a \cdot b = (1 * a) \cdot (b * 1) = (1 \cdot b) * (a \cdot 1) = b * a \ .$$

□

Lemma 3.2.4. *For a path $\gamma : [0, 1] \to X$ we get a canonical isomorphism*

$$- \cdot \gamma : \quad \pi_n(X, \gamma_0) \to \pi_n(X, \gamma_1)$$

defined as follows: for a representative $\alpha : I^n \to X$ the class $\gamma \cdot [\alpha]$ is represented by the map $\alpha \cdot \gamma : I^n \to X$ defined as in the picture



Proof. Works as for π_1. Details left for the reader. □

Lemma 3.2.5. *For paths γ, γ' from x_0 to x_1 which are homotopic relative basepoints we find that $[f\gamma] = [f\gamma']$ for each $[f] \in \pi_n(X, x_0)$. Moreover we have that*

- $(f+g)\gamma = f\gamma + g\gamma$
- $f(\gamma * \eta)f = (f\gamma)\eta$
- $f1 = f$

where here by f we denote a class in $\pi_n(X, x_0)$ as opposed to the representative.

Proof. Clear. □

Corollary 3.2.6. *For a pointed space (X, x) the above defined multiplication makes $\pi_n(X, x)$ into a right module over $\pi_1(X, x)$, i.e. a right module over the group ring $\mathbb{Z}[\pi_1(X, x)]$.*

Note that the formula also works for $n = 1$ but in that case all the actions are just given by conjugation, hence by inner automorphism of $\pi_1(X, x_0)$. Clearly for every continuous map of pointed spaces $(X, x) \to (Y, y)$ we get an induced map on homotopy groups $\pi_n(X, x) \to \pi_n(Y, y)$ which is compatible with the group structure. Thus π_n is a covariant functor:

$$\mathrm{Top}_* \to \mathrm{Ab} \qquad (X, x) \to \pi_n(X, x)$$

Moreover this functor is clearly pointed homotopy invariant. For a product of pointed spaces $\prod X_i$ we find

$$\pi_n\left(\prod_{i\in I} X_i, (x_i)_{i\in I}\right) = \prod_{i\in I} \pi_n(X_i, x_i)$$

Proposition 3.2.7. *Let $(E, e) \to (X, x)$ be a covering space. Then the induced morphisms*

$$\pi_n(E, e) \to \pi_n(X, x)$$

for $n \geq 2$ are isomorphisms.

Proof. For surjectivity we have to lift a map $S^2 \to X$ through E which is possible since S^2 is simply connected. For injectivity we have to lift a homotopy between maps $S^2 \to X$ which also works by covering theory. □

Note that we also completely understand the effect of a covering $p : E \to X$ on π_1 in terms of covering theory, namely the map $\pi_1(E, e) \to \pi_1(X, x)$ is injective and the quotient $\pi_1(X, x)/\pi_1(E, e)$ is given as a $\pi_1(X, x)$-set by the fibre $F = p^{-1}(x)$.

Example 3.2.8.

1. The higher homotopy groups $\pi_n(S^1, 1)$ for $n \geq 2$ are all zero (look at $\mathbb{R} \to S^1$).
2. The torus $T = S^1 \times S^1$ has trivial higher homotopy groups.
3. The higher homotopy groups of $\mathbb{R}P^\infty$ are also all trivial. This follows by looking at the universal covering $S^\infty \to \mathbb{R}P^\infty$.
4. The higher homotopy groups of surfaces Σ_g for $g \geq 1$ are trivial since the universal covers are contractible in all cases.

Definition 3.2.9. A space X for which $\pi_n(X, x_0) = 0$ for all $n \geq 2$ and all choices of basepoints is called aspherical.

Example 3.2.10. What about non-orientable surfaces? Σ_g^-? Idea: Consider the orientation cover, which happens to be $\Sigma_{g-1} \to \Sigma_g^-$ (Euler characteristic). Then we get that all non-orientable surfaces except for $\mathbb{R}P^2$ are aspherical. But $\mathbb{R}P^2$ can not be aspherical as it is covered by S^2 which has a non-trivial π_2 and π_3 as we have already seen in the last term.

Warning 3.2.11.

1. It is not true that the homotopy groups $\pi_n(S^k)$ for $n \geq k$ are zero, as the example of the Hopf-invariant $[S^3 \to S^2] \in \pi_3(S^2)$ has shown (make a choice of basepoint). This also shows that there can not be a formula for $\pi_n(X \wedge Y)$ in terms of $\pi_n(X)$ and $\pi_n(Y)$ or some easy suspension isomorphism.

2. It is true however that for $n < k$ we have $\pi_n(S^k) = 0$ and that $\pi_n(S^n) = \mathbb{Z}$ as we will see later

3. For a wedge sum of spaces (X, x) and (Y, y) in general we have $\pi_n(X \vee Y, (x, y)) \neq \pi_n(X, x) \oplus \pi_n(y, Y)$ as the following example shows: Let $X = S^1$ and $Y = S^2$ with the canonical basepoints. We have

$$\begin{aligned} \pi_2(S^1 \vee S^2) &\cong \pi_2\left(\mathbb{R} \amalg_{\mathbb{Z}} (\mathbb{Z} \times S^2)\right) \\ &\cong \pi_2\left(\bigvee S^2\right) \end{aligned}$$

   The group $\pi_2\left(\bigvee S^2\right)$ surjects onto $H_2\left(\bigvee S^2\right) = \bigoplus_{\mathbb{Z}} \mathbb{Z}$ (in fact it is isomorphic but we do not yet know this). This implies that it cannot be isomorphic to $\pi_2(S^1) \oplus \pi_2(S^2) = \mathbb{Z}$. This is in contrast to the case of fundamental groups where we have $\pi_1(X \vee Y) \cong \pi_1(X) * \pi_1(Y)$ given that both have NDR basepoints by Seifert-van-Kampen.

   Note also that $\pi_2(S^1 \vee S^2)$ is also not finitely generated, although $S^2 \vee S^1$ is a finite CW complex!

Definition 3.2.12. A space X is said to be n-connected for $n \geq 0$ if $\pi_0(X) = *$ and for all $k \leq n$ we have $\pi_k(X) \cong 0$ (here the basepoint doesn't matter). It is called n-truncated if $\pi_k(X, x) = 0$ for all $k > n$ and all choices of basepoints.

Example 3.2.13. 0-connected is (path)-connected (note that the empty space is not 0-connected). 1-connected means simply connected. 1-truncated means aspherical,

3.3 Whiteheads theorem

Our first big goal is Whitehead's theorem: a morphism $f : X \to Y$ where X and Y are CW complexes is a homotopy equivalence if and only if it is a weak homotopy equivalence, that is if all the induced morphisms $f_* : \pi_0(X) \to \pi_0(Y)$ and all $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ for all $x \in \pi_0(X)$ and $n \geq 1$ are isomorphisms! But first we need to define relative homotopy groups similar to relative Homology groups. In many respects homotopy groups behave like homology.

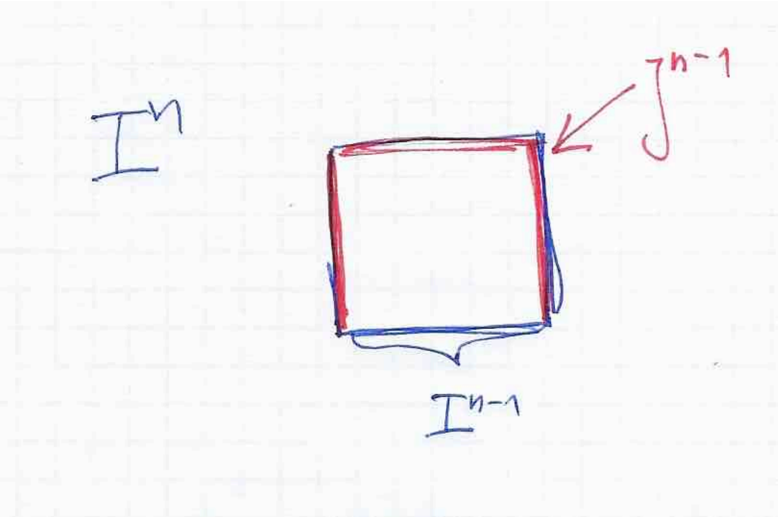
Definition 3.3.1. Let (X, A) be a pair of spaces and $x \in A$ a basepoint. Then define $\pi_n(X, A, x)$ for $n \geq 1$ to be the set of pointed homotopy classes of pairs

$$\pi_n(X, A, x) = [(D^n, S^{n-1}, 1), (X, A, x)]$$

Another formula is given by

$$\pi_n(X, A, x) \cong [(I^n, \partial I^n, J^{n-1}), (X, A, x)]$$

where J^{n-1} is the closure of $\partial I^n \setminus I^{n-1}$ with I^{n-1} the front face of I^n, i.e. the face with last coordinate $s_n = 0$.



Note that $\pi_0(X, A, x)$ is not defined! For $A = \{x_0\}$ we have that $\pi_n(X, \{x_0\}, x_0) = \pi_n(X, x_0)$. Thus relative homotopy groups generalize homotopy groups. For $n = 1$ we find that $\pi_1(X, A, x_0) = [([0, 1], \{0, 1\}, 1), (X, A, x_0)]$ Thus its the set of homotopy classes of paths from a some point in A to the fixed base point x_0. This has no group (or monoid structure) in a natural way but it is still a pointed set, where the basepoint (zero) is given by the constant path.

Definition 3.3.2. The sum operation for $[f], [g] \in \pi_n(X, A, x_0)$ with $f, g : (I^n, \partial I^n, J^n) \to (X, A, x_0)$ and $n \geq 2$ is defined as before:

$$(f + g)(s_1, ..., x_n) = \begin{cases} f(2s_1, s_2, ..., s_n) & s_1 \in [0, 1/2] \\ g(2s_1 - 1, s_2, ..., s_n) & s_1 \in [1/2, 1] \end{cases}$$

Alternatively consider the pinch map

$$D^n \to D^n \vee D^n$$

which collapses $D^{n-1} \subseteq D^n$ to a point. It sends $S^{n-1} \to S^{n-1} \vee S^{n-1}$ and the base point to the basepoint. Thus it induces morphisms

$$+ : \pi_n(X, A, x_0) \times \pi_n(X, A, x_0) \to \pi_n(X, A, x_0) \qquad (f, g) \mapsto (D^n \to D^n \vee D^n \xrightarrow{f \vee g} X) \ .$$

Proposition 3.3.3. *The set $\pi_n(X, A, x_0)$ equipped with the above structure is a group for $n = 2$ and an abelian group for $n \geq 3$.*

Proof. Same proof as before, just that we do not have the last coordinate s_n available and thus everything shifts by 1. $\square$

Lemma 3.3.4. *A map $f : (D^n, S^{n-1}, 1) \to (X, A, x_0)$ represents zero in $\pi_n(X, A, x_0)$ precisely if it is homotopic relative S^{n-1} to a map $g : (D^n, S^{n-1}, 1) \to (A, A, x_0) \subseteq (X, A, x_0)$ (image contained in A).*[1]

[1]Relative S^{n-1} here means that the map $f|_{S^{n-1}}$ is pointwise fixed throughout the homotopy.

Proof. If f and g are homotopic relative S^{n-1} then $[f] = [g]$. But $[g] = 0$ via the homotopy obtained by composing g with a retraction of D^n to 1.

Conversely if $[f] = 0$ via a homotopy $H : D^n \times I \to X$. We restrict H along a morphism $F : D^n \times I \to D^n \times I$ with $F(x, 0) = (x, 0)$ and $\mathrm{Im}(F(-, 1)) = D^n \times \{1\} \cup S^{n-1} \times I$ all of which have the same boundary. Basically this is the map to pulls the right hand disk over the cylinder. Then the restricted homotopy is stationary on S^{n-1}. □

Every map $\phi : (X, A, x_0) \to (Y, B, y_0)$ induces maps $\phi_* : \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$ which are homomorphisms for $n \geq 2$ and are functorial in the morphisms:

$$(\phi\psi)_* = \phi_* \circ \phi_* \qquad (\mathrm{id})_* = \mathrm{id}$$

and $\phi_* = \psi_*$ if $\phi \simeq \psi$ homotopic through maps $(X, A, x_0) \to (Y, B, y_0)$. There is also a homomorphism

$$\partial_* : \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0)$$

given by restricting $(D^n, S^{n-1}, x_0) \to (X, A, x_0)$ to the induced map $(S^{n-1}, x_0) \to (A, x_0)$. This is by definition of the group structures a group homomorphism for $n \leq 2$. For easier notation we shall also suppress basepoints in the relative homotopy groups if convenient.

Theorem 3.3.5. *For a pointed pair (X, A, x_0) the sequence*

$$\begin{aligned} ... \to \pi_{n+1}(X, A) \to &\pi_n(A) \to \pi_n(X) \to \pi_n(X, A) \to \pi_{n-1}(A) \to ... \\ &... \\ &... \to \pi_1(A) \to \pi_1(X) \to \pi_1(X, A) \to \pi_0(A) \to \pi_0(X) \end{aligned}$$

is exact. At the last three steps its exact as a sequence of pointed sets (i.e. the image of a morphism is the preimage of the basepoint under the next).

Proof. We will check exactness at various places. We suppress basepoints in the notation. We also observe that the composition of any two adjacent maps in the diagram is clearly zero.

Exactness at $\pi_n(A)$: Given $[f] \in \pi_n(A)$ represented by a map $S^n \to A$ which is zero in $\pi_n(X)$. Then using a nullhomotopy of f in X we find a map $F : D^{n+1} \to X$ with $F|S^n = f$. Then $F : (D^{n+1}, S^n) \to (X, A)$ represents a class $[F] \in \pi_{n+1}(X, A)$ mapping to $[f]$.

Exactness at $\pi_n(X)$ for $n \geq 1$: Assume given $[f] \in \pi_n(X)$ represented by $f : D^n \to X$ with $f(S^{n-1}) = \{x_0\}$. If is is zero in $\pi_n(X, A)$ then by Lemma 3.3.4 it is homotopic relative S^{n-1} to a map g with image contained in A, thus lies in the image of $\pi_n(A)$.

Exactness at $\pi_n(X, A)$ for $n \geq 1$: Assume that $F : (D^n, S^{n-1}) \to (X, A)$ represents a class in $\pi_n(X, A)$ which is mapped to zero in $\pi_{n-1}(A)$. Then we find a nullhomotopy of $f = F|_{S^{n-1}} : S^{n-1} \to A$ within A. We can picture this nullhomotopy as a map on the annulus $S^{n-1} \times [1, 1+\epsilon]$ which takes the inner circle to f and the outer circle to the basepoint. Then we by pasting this circle to the original map F we get a map $F_{+\epsilon} : D^n_{+\epsilon} \to X$ that takes the boundary to the basepoint and which represents the same element in $\pi_n(X, A)$, by the homotopy gradually removing the outer glued circle. This clearly lies in the image of $\pi_n(X) \to \pi_n(X, A)$. □

Example 3.3.6. Let X be path connected and CX be the cone on X (i.e. $CX = X \times [0, 1]/X \times \{0\}$). There is an inclusion $X \to CX$ and we get

$$0 = \pi_n(CX) \to \pi_n(CX, X) \cong \pi_{n-1}(X) \to \pi_{n-1}(CX) = 0$$

thus $\pi_n(CX, X) \cong \pi_{n-1}(X)$ for $n \geq 1$.

Definition 3.3.7. We say that a pair (A, X) has the Homotopy extension property (HEP) if given a map $f : X \to Y$ and a homotopy

$$H : A \times I \to Y$$

with $H_0 = f|_A$ then there always exists and extension of H to a homotopy

$$H' : X \times I \to Y$$

with $H'_0 = f$.

In other words, we can always find a lift in the following diagram

$$\begin{tikzcd} A \times I \cup_{A\times\{0\}} X \times \{0\} \arrow[r] \arrow[d] & Y \\ X \times I \arrow[ur, dashed] & \end{tikzcd}$$

Since we can let Y be $A \times I \cup_{A\times\{0\}} X \times \{0\}$ we see that the HEP implies that inclusion $A \times I \cup_{A\times\{0\}} X \times \{0\} \to X \times I$ admits a retraction. Conversely its easy to see that the existence of such a retraction implies the HEP for (X, A).

Lemma 3.3.8. *If (X, A) is a CW pair, then (X, A) has the HEP*

Proof. Erercises. ☐

Lemma 3.3.9 (Compression Lemma)**.** *Let (X, A) be a CW pair and let (Y, B) be any pair with B nonempty. For each n such that $X \setminus A$ has cells of dimension n assume that $\pi_n(Y, B, y_0) = 0$ for all $y_0 \in B$. For $n = 0$ the vanishing of the relative homotopy group condition should be replaced by the condition that $\pi_0(B) \to \pi_0(Y)$ is surjective. Then every map*

$$f : (X, A) \to (Y, B)$$

is homotopic relative A to a map $X \to B$ (i.e. A is fixed pointwise through the homotopy).

Proof. We induct over the skeleta $X^n \subseteq X$. Assume inductively that f has already been homotoped to take X^{n-1} to B. Now for any cell with a characteristic map

$$\chi : D^n \to X$$

with $\chi|_{S^{n-1}} : S^{n-1} \to X^{n-1}$ we have that the composite

$$f\chi : D^n \to Y$$

takes the boundary S^{n-1} to Y thus representing a class in $\pi_n(Y, B, y_0)$ with $y_0 = f\chi(1)$. By assumption this class is zero, i.e. we can find a homotopy to a map with image in B. Doing this for all n-cells simultaneously we find a homotopy of $f|_{X^n} : X^n \to Y$ to a map f' with image in B. By the homotopy extension property (Lemma 3.3.8) applied to the pair (X, X^n) we can extend this homotopy to a homotopy of $f : X \to Y$ to a map which takes X^n to B.

Finally we compose the homotopies transfinitely by performing them in shorter and shorter time intervalls (converging to the interval) which does not cause problems since the constructed homotopy leaves the previous skeleta fixed. ☐

Example 3.3.10. If $A = \emptyset$ and $B = \{y\}$. Then the Lemma states that every map $X \to Y$ is homotopic to the trivial maps provided that $\pi_n(Y, y)$ vanishes for all n such that X has cells of dimension n. In particular if X has only cells through dimension n and Y is n-connected, then every map $X \to Y$ is homotopic to a constant map.

Theorem 3.3.11 (Whiteheads theorem)**.** *Let $f : X \to Y$ be a map between CW-complexes X and Y. Assume that f is a weak homotopy equivalence, then f is a homotopy equivalence.*

Proof. First assume that $f : X \to Y$ is the inclusion of a subcomplex which is a weak equivalence. Then we find that the relative homotopy groups $\pi_n(Y, X)$ all vanish by the long exact sequence (for $n = 0$ and $n = 1$ one should make separate arguments). Thus we can apply the compression Lemma to the identity map

$$(Y, X) \to (Y, X)$$

to get that it is homotopic to a map $Y \to X$ which is then the homotopy inverse to f. In fact we have shown that $X \to Y$ is a deformation retract.

In general we want to consider the mapping cylinder

$$M_f = \frac{(X \times I) \amalg Y}{(x, 1) \sim f(x)}$$

which is homotopy equivalent to Y and contains $X = X \times \{0\}$ as a subspace, i.e. it gives a factorization

$$X \hookrightarrow M_f \xrightarrow{\simeq} Y$$

Thus it suffices to show that $X \hookrightarrow M_f$ is a homotopy equivalence. If f was a cellular map then $X \to M_f$ is a CW inclusion and we are done by the first part. We could now use the fact that every map f is homotopic to a cellular map (as we will prove soon) to finish the proof. But let us give an argument that does not use this fact: first the relative homotopy groups (M_f, X) all vanish since f is a weak equivalence. Now consider the inclusion

$$(Y \amalg X, X) \to (M_f, X)$$

and apply the compression lemma to homotop this to a map with image in X. Since $(M_f, X \amalg Y)$ satisfies the HEP (clearly?) we can extend this homotopy to a homotopy of the identity $\mathrm{id} : M_f \to M_f$ to a map $g : M_f \to M_f$ that takes $X \amalg Y$ to X. Then we apply the compression lemma again to the composition

$$\big((X \times I) \amalg Y, (X \times \partial I) \amalg Y\big) \to (M_f, X \amalg Y) \xrightarrow{g} (M_f, X)$$

to produce a deformation retract of M_f onto X which finishes the proof. $\square$

Warning 3.3.12. Whiteheads theorem does NOT say that CW complexes X and Y with isomorphic homotopy groups are homotopy equivalent! Consider $\mathbb{R}P^2$ and $S^2 \times \mathbb{R}P^\infty$. We have that

$$\pi_1(\mathbb{R}P^2) \cong \mathbb{Z}/2 \cong \pi_1(S^2 \times \mathbb{R}P^\infty)$$

and

$$\pi_n(\mathbb{R}P^2) \cong \pi_n(S^2) \cong \pi_1(S^2 \times \mathbb{R}P^\infty)$$

But

$$H_*(\mathbb{R}P^2; \mathbb{F}_2) = \mathbb{F}_2[\alpha]/\alpha^3 \qquad |\alpha| = 1$$

and

$$H_*(S^2 \times \mathbb{R}P^\infty; \mathbb{F}_2) \cong \mathbb{F}_2[x, y]/x^2 \qquad |x| = 2, |y| = 1$$

Thus they are not homotopy equivalent!

Corollary 3.3.13. *If a CW complex X has $\pi_n(X) = 0$ for all n. Then X is contractible.*

In particular this implies that a CW complex X is ashpherical if and only if the universal cover $\tilde{X}$ (which is also a CW complex) is contractible.

Definition 3.3.14. We say that a space X has the homotopy type of a CW complexes if it is homotopy equivalent to a CW complex.

Corollary 3.3.15. *Let X and Y have to homotopy type of CW complexes. Then $f : X \to Y$ is a weak homotopy equivalence if and only if it is a homotopy equivalence.*

Proof. We choose CW complexes $X' \simeq X$ and $Y' \simeq Y$. Then the composition

$$X' \xrightarrow{\simeq} X \xrightarrow{f} Y \xrightarrow{\simeq} Y'$$

is a weak homotopy equivalence since f is a weak equivalence and the outer homotopy equivalences $X' \xrightarrow{\simeq} X$ and $Y \xrightarrow{\simeq} Y''$ also clearly are weak equivalences. Thus by Whitehead's theorem this composition is a homotopy equivalence with inverse $g : Y' \to X'$. But then also f is a homotopy equivalence since we can compose g with the inverse homotopy equivalences $Y \xrightarrow{\simeq} Y'$ and $X' \xrightarrow{\simeq} X$ to obtain a homotopy inverse to f. □

3.4 Simplicial sets

The goal of this section is to introduce a purely combinatorial version of homotopy types and CW complexes.

Definition 3.4.1. The category Δ is the category of finite, non-empty, linearly ordered sets and order preserving maps.

Every object in Δ is canonically isomorphic to a linearly ordered set of the form

$$[n] = \{0 \leq 1 \leq 2 \leq ... \leq n\}$$

for a natural number $n \in \mathbb{N}$. Thus there is no harm in replacing Δ by an equivalent category which has only the objects $[n]$ and we will sometimes implicitly do this. We will need some special morphisms in Δ that we will introduce now. For every $n \geq 0$ and every $0 \leq k \leq n-1$ there is a *degeneracy map*

$$\sigma^k : [n] \to [n-1]$$

defined as the unique surjective morphism that hits $k \in [n-1]$ twice. For every $n \in \mathbb{N}$ and every $0 \leq k \leq n+1$ there is a *face map*

$$\partial^k : [n] \to [n+1]$$

defined as the unique injective map that misses $k \in [n+1]$.

Lemma 3.4.2. *We have the following simplicial identities:*

$$\partial^j \partial^i = \partial^i \partial^{j-1} \qquad \textit{for } i < j$$
$$\sigma^j \sigma^i = \sigma^i \sigma^{j+1} \qquad \textit{for } i \leq j$$
$$\sigma^j \partial^i = \begin{cases} \mathrm{id} & \textit{for } i = j, j+1 \\ \partial^i \sigma^{j-1} & \textit{for } i < j \\ \partial^{i-1} \sigma^j & \textit{for } i > j+1 \ . \end{cases}$$

Every morphism $f : [n] \to [m]$ in Δ can be written as a composition

$$f = \partial^{k_1} \cdots \partial^{k_n} \sigma^{k_{n+1}} \cdots \sigma^{k_m} \ .$$

of face and degeneracy maps and such a decomposition is unique up to the first two relations above .

Proof. The relation are a straighforward verification. For the second part we factor an arbitrary morphism f into an surjective map followed by an injective map and then get the result by writing both as compsitions of faces resp. degeneracies. □

Remark 3.4.3. Lemma 3.4.2 gives a generators and relations description of the category Δ and one can in fact introduce Δ in this way. This is often done in the literature.

Definition 3.4.4. A simplicial set is a functor $X : \Delta^{\mathrm{op}} \to \mathrm{Set}$. The category sSet of simplicial sets is the functor category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{Set})$.

Notation 3.4.5. Let X be a simplicial set. If S is an object of Δ, then we write X_S instead of $X(S)$. In particular, we will write X_n instead of $X([n])$. We will refer to X_n as the set of n-simplices of X. We will write $X(\partial^i) = \partial_i : X_n \to X_{n-1}$ for the function induced by the i-th coface map and refer to ∂_i as the i-th face map. Similarly, we will write $X(\sigma^i) = \sigma_i : X_n \to X_{n+1}$ for the function induced by the i-th codegeneracy map and refer to σ_i as the i-th degeneracy map.

Remark 3.4.6. It follows from Remark 3.4.3 that a simplicial set can be identified up to isomorphism with a sequence of sets $\{X_n\}_{n \geq 0}$ and functions $\partial_i : X_n \to X_{n-1}$, $\sigma_i : X_n \to X_{n+1}$ which satisfy the dual of the simplicial identities. This is the classical perspective on simplicial sets.

Remark 3.4.7. The category of simplicial sets admits small limits and colimits and these are computed levelwise. More precisely, if $F : I \to \mathrm{sSet}$ is a small I-shaped diagram in sSet, then for every integer $n \geq 0$, there are canonical bijections

$$(\lim_{i \in I} F)_n \simeq \lim_{i \in I}(F(i)_n) \qquad (\mathrm{colim}_{i \in I} F)_n \simeq \mathrm{colim}_{i \in I}(F(i)_n)$$

For example if X and Y are simplicial sets, then the product $X \times Y$ is given by

$$(X \times Y)_n = X_n \times Y_n$$

for every integer $n \geq 0$, and the simplicial operators are defined by $\partial_i(x, y) = (\partial_i x, \partial_i y)$.

We now present some important examples.

Example 3.4.8 (The n-simplex)**.** Let $y : \Delta \to \mathrm{sSet}$ denote the Yoneda embedding. Recall that if S is an object of Δ, then $y(S) = \mathrm{Hom}_\Delta(-, S)$ is the representable functor associated to S. For every integer $n \geq 0$, we define a simplicial set $\Delta^n : \Delta^{\mathrm{op}} \to \mathrm{Set}$ by $\Delta^n = y([n])$ which we refer to as the n-simplex. Concretely, the simplicial set Δ^n is given by

$$(\Delta^n)_S = \mathrm{Hom}_\Delta(S, [n]).$$

If $S \to S'$ is a morphism in Δ, then the function $(\Delta^n)_{S'} \to (\Delta^n)_S$ is given by precomposition with $S \to S'$. Furthermore, if X is a simplicial set, then there exists a natural bijection

$$\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, X) \xrightarrow{\simeq} X_n$$

by virtue of the Yoneda lemma.

Example 3.4.9 (The topological n-simplex)**.** Let $n \geq 0$ be an integer. Recall that the topological n-simplex $|\Delta^n|$ is defined as the following subspace of $\mathbb{R}^{n+1}$

$$|\Delta^n| = \{(x_0, x_1, \dots, x_n) \mid x_0 + \dots + x_n = 1, x_i \geq 0\} \subseteq \mathbb{R}^{n+1}.$$

The topological n-simplex $|\Delta^n|$ was denoted by Δ^n in Definiton 1.2.1 but we will now write $|\Delta^n|$ to distinguish it from the simplicial set Δ^n. Later we will see that the topological n-simplex arises as the geometric realization of the simplicial set Δ^n which explains the choice of notation. More generally, if S is an object of Δ, then we define

$$|\Delta^S| = \{x \in \bigoplus_S \mathbb{R} \mid \sum_{s \in S} x_s = 1, x_s \geq 0\} \subseteq \bigoplus_S \mathbb{R},$$

where $\bigoplus_S \mathbb{R}$ denotes the free $\mathbb{R}$-vector space on S. If $S \to S'$ is a morphism in Δ, we obtain a map of topological spaces $|\Delta^S| \to |\Delta^{S'}|$. Consequently, we obtain a functor $\Delta \to \mathrm{Top}$ determined by the construction $S \mapsto |\Delta^S|$.

Example 3.4.10 (The singular simplicial set)**.** In this example we explain how to associate a simplicial set to a topological space. Let X be a topological space. Let $\mathrm{Sing}(X)$ denote the simplicial set defined by

$$\mathrm{Sing}(X)_S = \mathrm{Hom}_{\mathrm{Top}}(|\Delta^S|, X).$$

If $S \to S'$ is a map in Δ, then the function $\mathrm{Sing}(X)_{S'} \to \mathrm{Sing}(X)_S$ is given by precomposition with the continuous map $|\Delta^S| \to |\Delta^{S'}|$ from Example 3.4.9. If $X \to Y$ is a map of topological spaces, then we obtain a function $\mathrm{Sing}(X)_S \to \mathrm{Sing}(Y)_S$ given by postcomposition with $X \to Y$ for every object S of Δ. It follows that the construction $X \mapsto \mathrm{Sing}(X)$ defines a functor

$$\mathrm{Sing} : \mathrm{Top} \to \mathrm{sSet}.$$

One of the major goals of our discussion of simplicial sets is to show that the functor Sing is a sort of equivalence of homotopy types, i.e. that, up to an appropriate notion of equivalence, every simplicial set arises in this way from a unique homotopy type X and that $\mathrm{Sing}(X)$ also restores all the information about the homotopy type X. This is the way in which simplicial sets form a combinatorial model for homotopy types.

Example 3.4.11 (Singular homology)**.** In this example we explain that the singular homology groups of a topological space only depend on the underlying singular simplicial set. For every integer $i \geq 0$, the singular homology $H_i(X, \mathbb{Z})$ of a topological space X is defined as the following composite of functors

$$\mathrm{Top} \xrightarrow{C_*(-)} \mathrm{Ch}(\mathbb{Z}) \xrightarrow{H_i} \mathrm{Ab},$$

where $\mathrm{Ch}(\mathbb{Z})$ denotes the category of chain complexes of abelian groups and the functor $C_*(-)$ assigns to every topological space the singular chain complex on X. We show that C_* factors over the category of simplicial sets. Let X be a simplicial set. Define a chain complex $C_*(X)$ by $C_n(X) = \mathbb{Z}[X_n]$, where $\mathbb{Z}[X_n]$ denotes the free abelian group on the set of n-simplices of X, and whose differential $d : C_{n+1}(X) \to C_n(X)$ is defined by

$$d = \sum_{i=0}^{n+1} (-1)^i \partial_i.$$

Using the simplicial identities one checks that d is a differential. The singular homology functor can now be obtained as the following composite of functors

$$\mathrm{Top} \xrightarrow{\mathrm{Sing}} \mathrm{sSet} \xrightarrow{C_*(-)} \mathrm{Ch}(\mathbb{Z}) \xrightarrow{H_i} \mathrm{Ab}$$

which shows that the singular homology of a topological space depends only on the underlying singular simplicial set.

Example 3.4.12 (The nerve)**.** In this example we explain how to extract a simplicial set from a small category. Let Cat denote the category of small categories and functors between them. If $\mathcal{C}$ is a small category, then we define a simplicial set $N\mathcal{C} : \Delta^n \to \mathrm{Set}$ by $(N\mathcal{C})_n = \mathrm{Hom}_{\mathcal{C}\mathrm{at}}([n], \mathcal{C})$, where $[n]$ is considered as a poset category. If $[m] \to [n]$ is a morphism in Δ, then $(N\mathcal{C})_n \to (N\mathcal{C})_m$ is given by precomposition with $[m] \to [n]$. Note that the set of 0-simplices of $N\mathcal{C}$ can be identified with the set of objects of the category $\mathcal{C}$. Similarly, the set of 1-simplices of $N\mathcal{C}$ can be identified with the set of morphisms of $\mathcal{C}$. In general, an n-simplices of $N\mathcal{C}$ is given by a string

$$c_0 \to c_1 \to \cdots \to c_n$$

of n-composable morphisms in $\mathcal{C}$. The i-th face map $\partial_i : (N\mathcal{C})_n \to (N\mathcal{C})_{n-1}$ is given by

$$(c_0 \to \cdots \to c_{i-1} \to c_i \to c_{i+1} \to \cdots \to c_n) \mapsto (c_0 \to \cdots \to c_{i-1} \to c_{i+1} \to \cdots \to c_n)$$

The i-th degeneracy map $\sigma_i : (N\mathcal{C})_{n-1} \to (N\mathcal{C})_n$ is given by

$$(c_0 \to \cdots \to c_i \to \cdots \to c_{n-1}) \mapsto (c_0 \to \cdots \to c_i \xrightarrow{\mathrm{id}} c_i \to \cdots \to c_{n-1})$$

The nerve construction defines a functor $N : \mathcal{C}\mathrm{at} \to \mathrm{sSet}$. If $\mathcal{C} \to \mathcal{D}$ is a functor of small categories, then the function $(N\mathcal{C})_n \to (N\mathcal{D})_n$ is given by postcomposition with $\mathcal{C} \to \mathcal{D}$.

Example 3.4.13 (The classifying space of a group)**.** Let G be a group considered as a category with a single element $\star$ and $\mathrm{Hom}_G(\star, \star) = G$. In this situation we define the classifying space BG of G by $BG = NG$, where N denotes the nerve construction from Example 3.4.12.

The topological n-simplex $|\Delta^n|$ has some important subspaces which in some sense arise from simplicial subsets of the n-simplex Δ^n.

Definition 3.4.14. A simplicial subset of a simplicial set X is a monomorphism $U \hookrightarrow X$ in the category of simplicial sets.

Definition 3.4.15 (The boundary of Δ^n)**.** Let $n \geq 0$ be an integer. Define a simplicial set $\partial\Delta^n : \Delta^{\mathrm{op}} \to \mathrm{Set}$ by

$$(\partial\Delta^n)_S = \{\alpha \in \mathrm{Hom}_\Delta(S, [n]) \mid \alpha \text{ is not surjective}\}.$$

If $S \to S'$ is a morphism in Δ, then $(\partial\Delta^n)_{S'} \to (\partial\Delta^n)_S$ is given by precomposition with $S \to S'$. The inclusion $(\partial\Delta^n)_S \hookrightarrow (\Delta^n)_S$ defines a monomorphism of simplicial sets which exhibit $\partial\Delta^n$ as a simplicial subset of Δ^n. We refer to $\partial\Delta^n$ as the boundary of Δ^n.

Definition 3.4.16 (The horn Λ^n_i)**.** Let $n \geq 0$ be an integer and let $0 \leq i \leq n$. Define a simplicial set $\Lambda^n_i : \Delta^{\mathrm{op}} \to \mathrm{Set}$ by

$$(\Lambda^n_i)_S = \{\alpha \in \mathrm{Hom}_\Delta(S, [n]) \mid [n] \notin \alpha(S) \cup [i]\}.$$

If $S \to S'$ is a morphism in Δ, then $(\Lambda^n_i)_{S'} \to (\Lambda^n_i)_S$ is given by precomposition with $S \to S'$. The inclusion $(\Lambda^n_i)_S \hookrightarrow (\partial\Delta^n)_S$ defines a monomorphism of simplicial sets which exhibit Λ^n_i as a simplicial subset of $\partial\Delta^n$. We refer to Λ^n_i as the i-th horn of Δ^n.

We end this section by giving a coequalizer formula for the boundary $\partial\Delta^n$ and the horns Λ^n_i. For each $0 \leq i \leq n$, the i-th face map $[n-1] \to [n]$ induces a monomorphism $\Delta^{n-1} \to \Delta^n$ which we consider as a subobject of Δ^n. We will denote this subobject by $\partial_i\Delta^n$. Similarly, for every pair $0 \leq i, j \leq n$, the injective map $[n-2] \to [n]$ whose image does not contain i and j induces a monomorphism $\Delta^{n-2} \to \Delta^n$ which we identify with a subobject of Δ^n which we denote by $\partial_{i,j}\Delta^n$. There are inclusion maps $\partial_{i,j}\Delta^n \to \partial_i\Delta^n$ and $\partial_{i,j}\Delta^n \to \partial_j\Delta^n$.

Lemma 3.4.17. *The simplicial set $\partial\Delta^n$ is a coequalizer of the following diagram*

$$\textstyle\coprod_{i,j} \partial_{i,j}\Delta^n \rightrightarrows \coprod_i \partial_i\Delta^n.$$

Similarly, the k-th horn Λ^n_k is the coequalizer of the following diagram

$$\textstyle\coprod_{i,j\neq k} \partial_{i,j}\Delta^n \rightrightarrows \coprod_{i\neq k} \partial_i\Delta^n.$$

Proof. Follows from Remark 3.4.7 since colimits are computed levelwise. □

3.5 Geometric realization

In this section we describe a way to associate a topological space to a simplicial set. In some sense, this assignment is inverse to the construction $X \mapsto \mathrm{Sing}(X)$. We will prove the following result.

Theorem 3.5.1. *The functor* Sing : Top $\to$ sSet *admits a left adjoint*

$$|-| : \mathrm{sSet} \to \mathrm{Top}$$

called geometric realization. The functor $|-| : \mathrm{sSet} \to \mathrm{Top}$ *is determined up to unique natural isomorphism by the requirement that* $|-| : \mathrm{sSet} \to \mathrm{Top}$ *preserves colimits and that the composite*

$$\Delta \xrightarrow{y} \mathrm{sSet} \xrightarrow{|-|} \mathrm{Top}$$

is equivalent to the functor $\Delta \to \mathrm{Top}$ *determined by the construction* $[n] \mapsto |\Delta^n|$.

We will give a proof of this result later. Let us first discuss some consequences.

Remark 3.5.2. Let us spell out the meaning of the uniqueness part of Theorem 3.5.1. Suppose that $F : \mathrm{sSet} \to \mathrm{Top}$ is a functor which preserves colimits and satisfy that $F(\Delta^n) \simeq |\Delta^n|$ for every integer $n \geq 0$, then there exists a unique natural isomorphism of functor $|-| \to F$.

Example 3.5.3 (Simplicial sets as colimits of representables)**.** In a presheaf category, every presheaf is a colimit of representables. We briefly indicate how this works in the presheaf category of simplicial sets. Let $X : \Delta^{\mathrm{op}} \to \mathrm{Set}$ be a simplicial set. Let I denote the small category whose objects are morphisms of simplicial sets $\Delta^n \to X$ for every integer $n \geq 0$, and whose morphisms are commutative diagrams

$$\begin{tikzcd} \Delta^n \arrow[r] \arrow[d] & X \\ \Delta^m \arrow[ur] & \end{tikzcd}$$

Then X is equivalent to the colimit of the following composite $I \to \Delta \xrightarrow{y} \mathrm{sSet}$, where $I \to \Delta$ is the functor determined by $(\Delta^n \to X) \mapsto [n]$. Concisely, we will write this as follows

$$X \simeq \mathrm{colim}_{\Delta^n \to X} \Delta^n.$$

Let us now compute the geometric realization of X. Using that geometric realization $|-|$ preserves colimits we find that

$$|X| \simeq |\mathrm{colim}_{\Delta^n \to X} \Delta^n| \simeq \mathrm{colim}_{\Delta^n \to X} |\Delta^n|.$$

This explains why geometric realization is uniquely characterized by preserving colimits and sending the n-simplex Δ^n to the topological n-simplex $|\Delta^n|$.

Lemma 3.5.4. *Let* $n \geq 1$ *be an integer. The inclusion of simplicial sets* $\partial\Delta^n \hookrightarrow \Delta^n$ *induces a homeomorphism from* $|\partial\Delta^n|$ *to the boundary of the topological* n*-simplex* $|\Delta^n|$*, which is defined by*

$$\{(x_0, \ldots, x_n) \in |\Delta^n| : x_j = 0 \textit{ for some } j\}.$$

Note that $|\partial\Delta^n|$ *is homotopy equivalent to* S^{n-1}.

Proof. Since geometric realization preserves with colimits it follows from Lemma 3.4.17 that the geometric realization of $\partial\Delta^n$ is given by the coequalizer of the following diagram of topological spaces

$$\coprod_{i,j} |\Delta^{n-2}| \rightrightarrows \coprod_i |\Delta^{n-1}|,$$

where we have used that $\partial_{i,j}\Delta^n$ is abstractly isomorphic to Δ^{n-2} and that $\partial_i\Delta^n$ is abstractly isomorphic to Δ^{n-1}. This coequalizer is homeomorphic to the boundary of $|\Delta^n|$ as wanted. □

We have the following result whose proof is similar to that of Lemma 3.5.4.

Lemma 3.5.5. *Let $n \geq 0$ be an integer and let $0 \leq i \leq n$. The inclusion of simplicial sets $\Lambda^n_i \hookrightarrow \Delta^n$ induces a homeomorphism from $|\Lambda^n_i|$ with the i-th horn of $|\Delta^n|$, which is defined by*

$$\{(x_0, \ldots, x_n) \in |\Delta^n| : x_j = 0 \text{ for some } j \neq i\}.$$

Example 3.5.6. Now we can draw some pictures which might aid your intuition.

Definition 3.5.7. A simplicial set X is a Kan complex if it satisfies the following condition: For $n > 0$ and $0 \leq i \leq n$, any map of simplicial sets $f_0 : \Lambda^n_i \to X$ admits an extension $f : \Delta^n \to X$ such that the following diagram commutes

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "f_0"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dotted, "f"'] & \end{tikzcd}$$

Example 3.5.8. If $n > 0$, then the n-simplex Δ^n is not a Kan complex.

Example 3.5.9. Let $\mathcal{C}$ be a small category. In general, the nerve $N\mathcal{C}$ is not a Kan complex. If $\mathcal{C}$ is a groupoid, then $N\mathcal{C}$ is a Kan complex. For example, if G is a group considered as a category with a single element, then the classifying space BG of G is a Kan complex.

Proposition 3.5.10. *If X is a topological space, then* $\mathrm{Sing}(X)$ *is a Kan complex.*

Proof. Given a pair of integers $0 \leq i \leq n$ and a map of simplicial sets $f_0 : \Lambda^n_i \to \mathrm{Sing}(X)$, we want to show that there exists an extension $f : \Delta^n \to \mathrm{Sing}(X)$ of f_0. Since geometric realization is left adjoint to Sing it suffices to show that there exists an extension in the following diagram

$$\begin{tikzcd} |\Lambda^n_i| \arrow[r] \arrow[d, hook] & X \\ |\Delta^n| \arrow[ur, dotted] & \end{tikzcd}$$

We may choose a continuous retraction $r : |\Delta^n| \to |\Lambda^n_i|$ of $|\Delta^n|$ onto $|\Lambda^n_i|$ and define $|\Delta^n| \to X$ by the composite $|\Delta^n| \xrightarrow{r} |\Lambda^n_i| \to X$. This provides the desired extension which ends the proof. □

Definition 3.5.11. An n-simplex $\Delta^n \to X$ is called degenerate, if it factors through some degeneracy map $\Delta^n \to \Delta^{n-1}$. Otherwise it is called non-degenerate.

For a degenerate n-simplex $\Delta^n \to X$ the geometric realization $|\Delta^n| \to |X|$ factors over the projection to some side (which is the degeneracy map). So they are in some sense determined by a lower dimensional datum. Every 0-simplex is automatically non-degenerate.

Example 3.5.12. Consider $X = \Delta^k$. Then n-simplices are given by maps $[n] \to [k]$ in Δ. Such a map is non-degenerate precisely if it is injective. In particular all n-simplices for $n > k$ are degenrated. There is precisely one non-degenerated k-simplex given by the identity. For every vertex $i = 0, 1, ..., n$ there is a degenerate 1-simplex given by the non-injective map $[1] \to [k]$ sending everything to i. The non-degenerate 1-simplices are given by subsets of $\{0, ..., n\}$ of cardinality 2. There are thus $\binom{k}{2}$ such 1-simplices. Similarly the non-degenerare n-simplices are given by subsets of $\{0, ..., k\}$ and there are $\binom{k}{n}$ of them.

Example 3.5.13. The simplicial set $\partial\Delta^k$ has all the same non-degenerate n-simplices except the top one of Δ^k.

Example 3.5.14. The simplicial set $\Delta^k/\partial\Delta^k$ has precisely one non-degenerate k-simplex and one non-degenerate 0-simplex. All other simplices are degenerate. In particular we see that faces of non-degenerate simplices can easily be degenerate.

Lemma 3.5.15. *For every n-simplex $\Delta^n \to X$ there exists a unique (up to unique isomorphism) factorization*

$$\Delta^n \to \Delta^{n'} \to X$$

where $\Delta^{n'} \to X$ is non-degenerate and $\Delta^n \to \Delta^{n'}$ is the composition of degeneracy maps.

Proof. The existence of such a factorization is clear since if $\Delta^n \to X$ is degenrate, there exists by definition such a factorization and iterating this one has eventually to arrive at a non-degenerated one after finitely many steps. It remains to show uniqueness. That is assume we have factorizations

$$\Delta^n \to \Delta^{n'} \to X \qquad and \Delta^n \to \Delta^{n''} \to X$$

Then □

3.6 Categorical Interlude: Kan Extensions

Our goal is now to prove the Sing-realization (Theorem 3.5.1) adjunction between topological spaces and simplicial sets, and to introduce the skeletal filtration on simplicial sets. In order to do so, we will use some categorical machinery.

Recall we want to "extend" a functor $\Delta\colon$ sSet $\to$ Top to a functor defined on presheaves on Δ. More generally, we might ask the following question: Given functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{C} \to \mathcal{E}$, when can we find a functor $K\colon \mathcal{D} \to \mathcal{E}$ such that $K \circ F \cong G$?

It turns out that in order to investigate the former question, is better to split it in two asymmetric notions, with two dual universal properties, one "initial" and one "terminal".

Given a functor $F\colon \mathcal{C} \to \mathcal{D}$ and a category $\mathcal{E}$, in most of the concrete cases, one is interested in finding left and right adjoints to precomposition with F, considered as a functor

$$\mathrm{Fun}(\mathcal{D}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E});$$

as we'll see, sometimes one can get such functors, some other times this is not possible, but it's still possible to determine functors that have the correct behavior for a fixed functor $G\colon \mathcal{C} \to \mathcal{E}$.

Definition 3.6.1. Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{C} \to \mathcal{E}$ be functors. A *left Kan extension of G along F* consist of a functor $\operatorname{Lan}_F G\colon \mathcal{D} \to \mathcal{E}$ together with a natural transformation $\eta\colon G \Rightarrow \operatorname{Lan}_F G \circ F$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "G"] \arrow[d, "F"'] \arrow[dr, phantom, "\Searrow\eta" near start] & \mathcal{E} \\
\mathcal{D} \arrow[ur, "\operatorname{Lan}_F G"'] &
\end{tikzcd}$$

which makes it initial among natural transformations from G to functors of the form $F \circ -$; that is, such that, given any other functor $H\colon \mathcal{D} \to \mathcal{E}$ together with a natural transformation. $\alpha\colon G \Rightarrow H \circ F$ there exists a unique natural transformation $\chi\colon \operatorname{Lan}_F G \Rightarrow H$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "G"] \arrow[d, "F"'] \arrow[dr, phantom, "\Searrow\eta" near start] & \mathcal{E} \\
\mathcal{D} \arrow[ur, ""{name=L}] \arrow[ur, bend right=60, "\forall H"'{name=H}] & \arrow[phantom, from=L, to=H, "\exists\Rightarrow\chi"]
\end{tikzcd}$$

such that $\alpha = F\chi \circ \eta$

$$\begin{tikzcd}
G \arrow[r, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "\eta"'] & H \circ F \\
\operatorname{Lan}_F G \circ F. \arrow[ur, dotted, "F\chi"'] &
\end{tikzcd}$$

Remark 3.6.2. Notice that $\operatorname{Nat}(\operatorname{Lan}_F G, H) \cong \operatorname{Nat}(G, H \circ F)$. A fancier way to phrase this (and to make clear that left Kan extensions are uniquely determined up to unique isomorphism) is to say that $\operatorname{Lan}_F G$ is an object corepresenting the functor

$$\begin{aligned}
\operatorname{Fun}(\mathcal{D}, \mathcal{E}) &\to \operatorname{Set} \\
H &\mapsto \operatorname{Nat}(G, - \circ F).
\end{aligned}$$

This also means, that if $\operatorname{Lan}_F G$ exists for each functor $G : \mathcal{C} \to \mathcal{E}$ (for fixed $\mathcal{E}$) then we get that the left Kan extension assembles into functor

$$\operatorname{Lan}_F : \operatorname{Fun}(\mathcal{C}, \mathcal{E}) \to \operatorname{Fun}(\mathcal{D}, \mathcal{E})$$

which is left adjoint to precomposition with F, considered as a functor $\operatorname{Fun}(\mathcal{D}, \mathcal{E}) \to \operatorname{Fun}(\mathcal{C}, \mathcal{E})$.

We will just analyze the case of left Kan extensions, as everything can be easily adapted to right Kan extensions. For the sake of clarity, let's just spell out once what a right Kan extension is.

Definition 3.6.3. Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{C} \to \mathcal{E}$ be functors. A *right Kan extension of G along F* consist of a functor $\operatorname{Ran}_F G\colon \mathcal{D} \to \mathcal{E}$ together with a natural transformation $\varepsilon\colon \operatorname{Ran}_F G \circ F \Rightarrow G$ such that, given any other functor $H\colon \mathcal{D} \to \mathcal{E}$ together with a natural transformation. $\beta\colon H \circ F \Rightarrow G$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "G"] \arrow[d, "F"'] \arrow[dr, phantom, "\Nwarrow\varepsilon" near start] & \mathcal{E} \\
\mathcal{D} \arrow[ur, ""{name=R}] \arrow[ur, bend right=60, "\forall H"'{name=H}] & \arrow[phantom, from=R, to=H, "\exists\Leftarrow\xi"]
\end{tikzcd}$$

there exists a unique natural transformation $\xi\colon H \Rightarrow \operatorname{Ran}_F G$ such that $\beta = \varepsilon \circ \xi_F$

$$
\begin{tikzcd}
 & \operatorname{Ran}_F G \circ F \arrow[d, Rightarrow, "\varepsilon"] \\
H \circ F \arrow[ur, dotted, Rightarrow, "\xi_F"] \arrow[r, Rightarrow, "\beta"'] & G
\end{tikzcd}
$$

Example 3.6.4. Let $\mathcal{D} = [0]$, then $\operatorname{Lan}_F G \cong \operatorname{colim} G$.

Remark 3.6.5. In general, η is just a natural transformation, not necessarily a natural *equivalence*. However, we will see in the Exercises that it is an isomorphism if $F : \mathcal{C} \to \mathcal{D}$ is fully faithful. This is a very important statement, which says that Kan extensions along fully faithful functors are extensions (up to natural isomorphism)! In general Kan extensions are only extensions in a rather weak sense.

Example 3.6.6. Let $\mathcal{C} = B\mathbb{N}$, $\mathcal{D} = B\mathbb{Z}$, let F be the inclusion, and $\mathcal{E} = \mathrm{CMon}$; then a functor $(A, f)\colon B\mathbb{N} \to \mathrm{CMon}$ is just a Abelian monoid together with an endomorphism of A (recall Cayley's Theorem, aka Yoneda's Lemma: this is just an element of A). Then $Lan_F(A, f) \cong A[1/f]$ and the natural transformation is just the localization map $A \to A[1/f]$.

The following theorem can be stated with tighter hypoteses, but for us this generality will suffice.

Theorem 3.6.7. *Let $\mathcal{C}$ be a small category, and let $\mathcal{E}$ be a cocomplete category. Then given any pair of functors $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{C} \to \mathcal{E}$, the left Kan extension $\operatorname{Lan}_F G$ exists, and is given pointwise by the colimit*

$$\operatorname{colim}(F_{/d}) \to \mathcal{C} \to \mathcal{E}.$$

Definition 3.6.8. Here for $d \in \mathcal{D}$ and $F : \mathcal{C} \to \mathcal{D}$ the category $F_{/d}$ has objects as pairs consisting of an object $c \in \mathcal{C}$ and a morphism $f : F(c) \to d$. A morphism from (c, f) to (c', f') in $F_{/d}$ is given by a morphism $c \to c'$ in $\mathcal{C}$ such that the diagram

$$
\begin{tikzcd}
F(c) \arrow[rr] \arrow[dr] & & F(c') \arrow[dl] \\
 & d &
\end{tikzcd}
$$

commutes. Composition is defined in the (hopefully) obvious way.

There is a clear forgetful functor $F_{/d} \to \mathcal{C}$. If F is fully faithful, which is the situation that we are mostly into, then $F_{/d}$ is a full subcategory of the slice $\mathcal{D}_{/d}$.

Proof. (**Sketch**) Functoriality follows from the universality of all the objects (spell out a bit). Let us temporarily denote this functor by L. Let $c \in \mathcal{C}$, then the natural transformation at c is given by the canonical map into the colimit

$$Gc = G \circ i(Fc) \to \operatorname{colim}_{F_{/Fc}} G \circ i$$

and naturality is a straightforward check. Let H be another functor, and let $\alpha\colon G \Rightarrow H \circ F$ be a natural transformation. We construct $\chi\colon L \Rightarrow H$ on $d \in \mathcal{D}$ as follows

$$\begin{tikzcd}
Gc \arrow[r] \arrow[d] \arrow[dd, bend right] & Gc' \arrow[d] \arrow[ddl] \\
HFc \arrow[r] \arrow[d] \arrow[dr] & HFc' \arrow[d] \\
Ld \arrow[r, dotted] & Hd.
\end{tikzcd}$$

Naturality and universality are a (tedious) exercise. $\square$

Remark 3.6.9. In the hypotheses of the theorem ($\mathcal{C}$ small and $\mathcal{E}$ cocomplete), one can make left Kan extensions functorial on the functor categories, i.e. one can define a functor

$$\mathrm{Lan}_F\colon \mathrm{Fun}(\mathcal{C}, \mathcal{E}) \to \mathrm{Fun}(\mathcal{D}, \mathcal{E})$$

left adjoint to precomposition with F.

We are now ready to prove the Sing-realization adjunction!

Definition 3.6.10. We define $|-|\colon \mathrm{sSet} \to \mathrm{Top}$ to be the left Kan extension of $|-|$ (as defined last time on Δ) along the Yoneda embedding.

We just have to prove that its right adjoint is given by Sing and in particular $|-|$ preserves colimits (like any left adjoint functor).

Proposition 3.6.11. $|-| \dashv \mathrm{Sing}$.

Proof.

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{sSet}}(X, \mathrm{Sing}\, T) &= \mathrm{Hom}_{\mathrm{sSet}}(\mathrm{colim}_{\Delta^n \to X} \Delta^n, \mathrm{Sing}\, T) \\
&= \lim_{\Delta^n \to X} \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, \mathrm{Sing}\, T) \\
&= \lim_{\Delta^n \to X} \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, T) \\
&= \mathrm{Hom}_{\mathrm{Top}}(\mathrm{colim}_{\Delta^n \to X} |\Delta^n|, T) \\
&= \mathrm{Hom}_{\mathrm{Top}}(|X|, T).
\end{aligned}$$

$\square$

Remark 3.6.12. The definition as a left Kan extension, together with the fact that it is a left adjoint functor, make clear that $|-|$ is uniquely determined by its restriction to Δ and the fact that commutes with colimits.

3.7 Skeleta and the skeletal filtration

Definition 3.7.1. Let $\Delta_{\leq n}$ be the full subcategory of Δ spanned by nonempty finite sets of cardinality $\leq n+1$. That is, up to equivalence, $\Delta_{\leq n}$ has object $[0], [1], ..., [n]$.

For all $n \in \mathbb{N}$, we have a fully faithful inclusion functor $i_n \colon \Delta_{\leq n} \to \Delta$. In particular, we have the following adjunction

$$\begin{tikzcd} \mathrm{Fun}(\Delta^{op}_{\leq n}, \mathrm{Set}) \arrow[r, shift left=1.5, "\mathrm{Lan}_{i_n}"] \arrow[r, phantom, "\perp"] & \mathrm{sSet} \arrow[l, shift left=1.5, "i_n^*"] \end{tikzcd}$$

the counit of the adjunction induces a natural transformation

$$\mathrm{sk}_n := \mathrm{Lan}_{i_n} \circ i_n^* \Rightarrow \mathrm{id}$$

the composite $\mathrm{sk}_n := \mathrm{Lan}_{i_n} \circ i_n^*$ is called the *n-th skeleton functor*.

We note that by the fact that i_n is fully faithful, we find that also Lan_{i_n} is fully faithful, equivalently the natural transformation $\mathrm{sk}_n(X) \to X$ for any simplicial sets induces an isomorphism when restriction along i_n or even simpler: this map induces a bijection on all k simplices for $k \leq n$.

Definition 3.7.2. Notice that we also have functors

$$i_{n,n+1} \colon \Delta_{\leq n} \to \Delta_{\leq n+1}$$

inducing a precomposition-Lan adjunction. We also have natural transformations

$$\mathrm{Lan}_{i_{n,n+1}} \circ i_n^* \Rightarrow i_{n+1}^*$$

induced by

$$\begin{tikzcd} \Delta^{op}_{\leq n} \arrow[r, "i_n^* X"] \arrow[d, "i_{n,n+1}"'] & Set \\ \Delta^{op}_{\leq n+1} \arrow[ur, "\mathrm{Lan}"] \arrow[ur, bend right, "i^*_{n+1}X"'] & \end{tikzcd}$$

hence we get a chain of natural transformations

$$\begin{tikzcd} & & \mathrm{sSet} \arrow[dll, "i_0^*"'] \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr, "i_4^*"] & & \\ \mathrm{Fun}(\Delta^{op}_{\leq 0}) \arrow[r, "\mathrm{Lan}_{i_{01}}"'] \arrow[drr, "\mathrm{Lan}_{i_0}"'] & \mathrm{Fun}(\Delta^{op}_{\leq 1}) \arrow[r] \arrow[dr] & \mathrm{Fun}(\Delta^{op}_{\leq 2}) \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^{op}_{\leq 3}) \arrow[r] \arrow[dl] & \cdots \arrow[dll, "\mathrm{Lan}_{i_4}"] \\ & & \mathrm{sSet} & & \end{tikzcd}$$

in the upper triangles, whereas the lower ones just commute. This in turn gives a chain of natural transformations

$$\mathrm{sk}_0 \Rightarrow \mathrm{sk}_1 \Rightarrow \mathrm{sk}_2 \Rightarrow \mathrm{sk}_3 \Rightarrow \mathrm{sk}_4 \Rightarrow \cdots$$

The evaluation at X of the above chain is called the *skeletal filtration of* X, i.e. for every simplicial set we get a diagram

$$\mathrm{sk}_0 X \Rightarrow \mathrm{sk}_1 X \Rightarrow \mathrm{sk}_2 X \Rightarrow \mathrm{sk}_3 X \Rightarrow \mathrm{sk}_4 X \Rightarrow \cdots \Rightarrow X$$

Here we have used that the composition of left Kan extensions along functors F, F' is given by the left Kan extension along the composite. Another way of constructing the map

$$\mathrm{sk}_n X \to \mathrm{sk}_{n+1} X$$

would be the use the universal propoerty of the left Kan extension to see that such a map is equivalently described as a map $i_n^* X \to i_n^* \mathrm{sk}_{n+1} X$ and noting that the target is naturally isomorphic to $i_n^* X$, thus we can take the identity.

Lemma 3.7.3. *The map $\mathrm{sk}_n X \to X$ is injective and exhibits sk_n as the smallest simplicial subset that contains all the n-simplices of X (hence also all the k-simplices for $k < n$).*

Proof. The colimit formula for Kan extensions immediately yields that the set of m-simplices of $\mathrm{sk}_n(X)$ is given by pairs (p, x)

$$\Delta^m \xrightarrow{p} \Delta^k \xrightarrow{x} X$$

for $k \leq n$ modulo the relation that $(p, x) \sim (p', x')$ if there exists a diagram

$$\begin{tikzcd} & \Delta^k \arrow[dd] \arrow[dr, "x"] & \\ \Delta^m \arrow[ur, "p"] \arrow[dr, "p'"'] & & X \\ & \Delta^{k'} \arrow[ur, "x'"'] & \end{tikzcd}$$

Now it is easy to see that this quotient set does in fact not change if we assume that the map p is surjective (as a map $[m] \to [k]$ by replacing $[k]$ by the image. But then we can even assume that the simplex x is non-degenerate by the factorization through a non-degenerate simplex. Thus we see that this set is exactly equivalent to the set of m-simplices of X whose non-degenerate factorization $\Delta^m \to \Delta^k \to X$ satisfies $k \leq n$.

For $m \leq n$ these are all n-simplices. For $m = n+1$ these are the degenerate m-simplices. For larger m these are the degenerate simplices with nin-degenerate simplex of dimension $k \leq n$.

The last statement is then immediate: an m-simplex is in the set precisely if it is obtained from a k-simplex by degeneration. But the smallest subsimplicial set contains all degenrations. $\square$

Example 3.7.4. We have $\mathrm{sk}_{n-1}\Delta^n = \partial\Delta^n$.

Proposition 3.7.5. *We have that all the maps $\mathrm{sk}_n X \to \mathrm{sk}_{n+1}$ are monomorphisms and sit in a pushout*

$$\begin{tikzcd} \coprod_{nondeg\ X_{n+1}} \partial\Delta^{n+1} \arrow[r] \arrow[d] & \mathrm{sk}_n X \arrow[d] \\ \coprod_{nondeg\ X_{n+1}} \Delta^{n+1} \arrow[r] & \mathrm{sk}_{n+1} X \ . \end{tikzcd}$$

The canonical maps $\mathrm{sk}_n X \to X$ exhibt an equivalence $X = \mathrm{colim}_n(\mathrm{sk}_n X)$.

Proof. The colimit statement is clear, since the colimit consists of all the simplices of X whose degeneration has dimension $\geq n$ for some n. For the first statement , since colimits (and limits) are pointwise in simplicial sets it suiffices to check that the square of m-simplices is a pushout for every m. For $m \leq n$ both vertical maps are in fact isomorphism so that this is clear.

For $m > n$ the idea is that since both vertical maps are inclusions, we need to show that the lower horizontal map induces a bijection on complements, i.e. on those simplices whose degeneration is $n+1$-dimensional on either side. Injectivity follows from the uniqueness of the degeneration and surjcktivity is clear. □

Corollary 3.7.6. *The geometric realization of a simplicial set X has a canonical CW-structure with $X_n = |\mathrm{sk}_n|$ and*

Proof. Since $|-|$ preserves colimits, in particular coproducts and since $\mathrm{sk}_0 X$ is a disjoint union of Δ^0 we find that $|\mathrm{sk}_0|$ is a disjoint union of points. Then $\mathrm{sk}_1 X$ is obtained as a pushout along $|\partial\Delta^1| \to |\Delta^1|$ which is a closed inclusion. Thus also $|\mathrm{sk}_0| \to |\mathrm{sk}_1|$ is a closed inclusion, in particular injective (and $|\mathrm{sk}_0 X|$ carries the subspace topology). We conclude that

$$|\mathrm{sk}_0 X| \to |\mathrm{sk}_1 X| \to ...$$

defines a filration of $|X$ whose colimit (= union) is $|X|$ and that it is a CW structure. □

Definition 3.7.7. We say that a simplicial set X is n-dimensional, if the map $\mathrm{sk}_n X \to X$ is an isomorphism.

Corollary 3.7.8. *Being n-dimensional is equivalent to fact that all non-degenerate simplices are in dimension $\leq n$.*

Warning 3.7.9. For an n-dimensional CW complex X the singular complex $\mathrm{Sing}\, X$ is not necessarily n-dimensional. But the 'converse' is true: if X is n-dimensional, then so is $|X|$.

3.8 Simplicial homotopy theory

Recall that we had the notion of a Kan complex and that $\mathrm{Sing}(X)$ for a topological space X is always a Kan complex. A Kan complex was defined to admits 'horn-fillers'. A more fancy way of saying that is that it admits the lifting property against the maps $\Lambda^n_k \to \Delta^n$. But it turns out that for formal reasons it then also admits the lifting property against a much larger set of maps.

Definition 3.8.1. A map $X \to Y$ of simplicial sets is called anodyne if it lies in the smallest class of maps of simplicial sets that contains the maps $\Lambda^n_k \to \Delta^n$ and which is closed under the following operations:

1. <u>Pushout</u>: If a map $X \to Y$ is contained in the class, then also every pushout of this class. That means for every maps $X \to X'$ the induced map $X' \to X' \cup_X Y$ is in the class.

2. Retract: If a map $j : X \to Y$ is contained in the class, then also every retract j' of j is contained in this class. That is if we have a diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "j'"] & X \arrow[r] \arrow[d, "j"] & X' \arrow[d, "j'"] \\ Y' \arrow[r] & Y \arrow[r] & X \end{tikzcd}$$

where the upper and lower horizontal compositions are the identity.

3. Transfinite composition The class is closed under transfinite composition. That means first of all, that it is closed under composition of maps. A transfinite composition is the following: given a non-zero ordinal α assume that we are given a simplicial set X and a diagram $X : \alpha \to \mathrm{sSet}_{X/}$ with the property that for every $\beta < \alpha$ the canonical morphism
$$\mathrm{colim}_{i<\beta} X_i \to X_\beta$$
is in our class, then also the map $X \to \mathrm{colim}_{i<\alpha} X_i$ is in the class.

Lemma 3.8.2. *A Kan complex K admits lifts against anodyne maps. That is if we have a map $X \to K$ and an anodyne map $j : X \to Y$ then we find a lift f as in the diagram*

$$\begin{tikzcd} X \arrow[r, "f_0"] \arrow[d, "j"] & K \\ Y \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

Every anodyne map is injective (that is a monomorphism of simplicial sets).

Proof. We consider the class of maps that have the lifting property against Kan complexes. This class contains the horn inclusions by definition and the proof will be finished by showing that this is class is closed under pushouts, retracts and transfinite compositions. But this is clear (spell it out).

To see that anodyne maps are injective we note that the class of injective maps is closed under these constructions and contains the horn inclusions. □

Example 3.8.3. 1. Every face map $\Delta^1 \to \Delta^2$ is anodyne. For the last face this can be seen by writing it as a composition $\Delta^1 \to \Lambda^2_0 \to \Delta^2$. The first inclusion is the pushout of Δ^1 along $\Delta^0 \to \Delta^1$ and thus anodyne.

It follows that each vertex $\Delta^0 \to \Delta^2$ is also anodyne.

2. More generally we claim that every map
$$\bigcup_{i \in I} \partial_i \Delta^n \to \Delta^n$$
for some non-empty, non-trivial subset $I \subseteq [n]$ is anodyne. To see this we use induction over n.

We set
$$B_I \Delta^n = \bigcup_{i \in I} \partial_i \Delta^n$$

and find a sequence $I = I_0 \subseteq I_1 \subseteq I_2 \subseteq ... \subseteq I_k \subseteq I_{k+1} = [n]$ in which each I_i has one more element than I_{i-1}. We consider the chain of inclusions

$$B_{I_0}\Delta^n \subseteq B_{I_1}\Delta^n \subseteq ... \subseteq B_{I_k}\Delta^n \subseteq \Delta^n$$

The last map is a horn inclusion. The map $B_{I_0}\Delta^n \subseteq B_{I_1}\Delta^n$ is obtained by attaching (i.e. pushout) of an $(n-1)$-simplex along non-full part of its boundary. Thus is anodyne by inductive assumption. Thus the composition $B_{I_0}\Delta^n \to \Delta^n$ is also anodyne.

3. In particular any face map is anodyne. More generally every map $\Delta^k \to \Delta^n$ induced from an injective map in Δ is anodyne as a composition of face maps.

Lemma 3.8.4. *For every injective map $i : A \to B$ and every anodyne map $j : X \to Y$ the pushout-product map*

$$i \boxtimes j : (A \times Y) \cup_{(A\times X)} (B \times X) \to (B \times Y)$$

induced from the commutative square

$$\begin{tikzcd} A \times X \arrow[r] \arrow[d] & B \times X \arrow[d] \\ A \times Y \arrow[r] & B \times Y \end{tikzcd}$$

is anodyne.

This Lemma is the technical key to everything that will follows. Before we prove it, we first want to discuss consequences. Note that for $A = \emptyset$ the lemma implies that $B \times X \to B \times Y$ is anodyne.

Definition 3.8.5. Let K be a Kan complex and X be an arbitrary simplicial set. A homotopy between maps $f, g : X \to K$ is a map

$$X \times \Delta^1 \to K$$

restricting to f resp. g along the two maps $X \to X \times \Delta^1$ induced from the two maps $\Delta^0 \to \Delta^1$. If $A \subseteq X$ and $f |_A = g |_A$ then homotopy relative to A is a map $X \times \Delta^1 \to K$ such that the restriction to $A \times \Delta^1 \subseteq X \times \Delta^1$ is constant in Δ^1, i.e. given by the composition

$$A \times \Delta^1 \xrightarrow{\text{pr}} A \xrightarrow{f|_A = g|_A} X \ .$$

The latter case in particular covers pointed homotopies.

Proposition 3.8.6. *1. The notion of (relative) homotopy between maps $X \to K$ is an equivalence relation, in particular we have well-defined sets $[X, K]$ of homotopy classes of maps and $[X, K]_*$ of pointed maps.*

2. For Kan complexes K and L composition induces a well-defined maps

$$[X, K] \times [K, L] \to [X, L] \qquad [X, K]_* \times [K, L]_* \to [X, L]_*$$

3. For every anodyne map $j : X \to Y$, every Kan complex K and every diagram

$$\begin{tikzcd} X \arrow[r, "f_0"] \arrow[d, "j"] & K \\ Y \end{tikzcd}$$

any two lifts $Y \to K$ of f_0 are homotopic relative X. That is lifts are unique up to homotopy.

Proof. ad 1: We first treat the case of absolute homotopy: The reflexivity of the relation $\simeq$ of being homotopic is clear as we can just consider the homotopy

$$X \times \Delta^1 \xrightarrow{\mathrm{pr}} X \xrightarrow{f} K$$

as a homotopy $f \simeq f$. We will refer to this as the identity homotopy on f. To see that it is symmetric assume that we have a homotopy $h : X \times \Delta^1 \to K$ from f to g. Then we form the map

$$X \times \Lambda^2_0 = (X \times \partial_1\Delta^2) \cup_{(X \times \partial_{1,2}\Delta^2)} (X \times \partial_2\Delta^2) \to K$$

which is given by h on $X \times \partial_2\Delta^2$ and by the identity homotopy on f on $X \times \partial_1\Delta^2$. By Lemma 3.8.4 the map $X \times \Lambda^2_0 \to X \times \Delta^2$ is anodyne so that we can find an extension to a map $X \times \Delta^2$. The restriction of this map to $\partial_0\Delta^2$ is then a homtopy from g to f. For transitivity we proceed similarly. We combined homotopies $h : f \simeq g$ and $h' : g \simeq k$ to a map

$$X \times \Lambda^2_1 \to K$$

and extend to a map $X \times \Delta^2 \to K$ whose restriction to $\partial_1\Delta^2$ is the 'composition'.

For the notion of relative homotopy we proceed similarly: for example for transitivity we combine homotopies $h : f \simeq g$ and $h' : g \simeq k$ relative to A into a map

$$(A \times \Delta^2) \cup_{A \times \Lambda^2_1} (X \times \Lambda^2_1)$$

which is in the first summand given by the 'identity' 2-simplex on the map $f|_A = g|_A = k|_A$. Then we can extend this to a map $X \times \Delta^2 \to K$ and the restriction to $\partial_1\Delta^2$ is a relative homotopy from f to k.

ad 2: We have to show that for $g \simeq g'$ we have $fg \simeq fg'$ and $gk \simeq gk'$. Both of these cases are trivial as we simplt post/precomose the homotopy $h : g \to g'$ with f resp $g \times \mathrm{id}_{\Delta^1}$.

ad 3: For two extensions $f, f' : Y \to K$ of f_0 we get a map

$$(X \times \Delta^1) \cup_{X \times \partial\Delta^1} (Y \times \partial\Delta^1) \to K$$

given on the first summand by $X \times \Delta^1 \xrightarrow{pr} X \xrightarrow{f_0} K$ and by $Y \times \partial\Delta^1 = Y \amalg Y \xrightarrow{(f,g)} K$ on the second summand. Using Lemma 3.8.4 (note that $\boxtimes$ is obviously symmetric) we extend to a map $Y \times \Delta^1 \to K$ which is the desired relative homotopy. $\square$

Definition 3.8.7. We define the homotopy category Ho(Kan) of Kan complexes to have objects Kan complexes and morphism homotopy classes of maps between Kan complexes. A map $K \to L$ is called homotopy equivalence if it is an isomorphism in Ho(Kan). Similar we define Ho(Kan_*) and the notion of pointed homotopy equivalence.

Clearly for every simplicial set X the set $[X, K]$ is invariant under homotopy equivalence in K. We claim that the notion of homotopy is compatible with geometric realization. This follows from the following Lemma:

Lemma 3.8.8. *For every simplicial set X and every finite simplical set Y the canonical map $|X \times Y| \to |X| \times |Y|$ is a homeomorphism.*

Proof Sketch. The functor $- \times |Y|$ commutes with all colimits since it has a right adjoint given by $\mathrm{Map}(|Y|, -)$. This is in fact true for every locally compact Hausdorff space in place of Y and uses crucially that Y is finite so that $|Y|$ i a finite CW complex and thus locally compact. Thus since the map in question is natural we can write X as a colimit of simplicies and reduce the general question to the case of the map $|\Delta^m \times Y| \to |\Delta^m| \times |Y|$. Now we perform the same argument with interchanged roles and reduce to the case of the map

$$|\Delta^m \times \Delta^n| \to |\Delta^n| \times |\Delta^m|$$

Now one has to explicitly analyse the simplices in $\Delta^m \times \Delta^n$ and see that this is the union of non-degenerated simplices Δ^{n+m} indexed by shuffles of n and m. We have a similar decomposition of $|\Delta^n| \times |\Delta^m|$ and so one sees that the two spaces agree under the canonical map. □

Warning 3.8.9. It is not in general true that the map $|X \times Y| \to |X| \times |Y|$ is a homeomorphism. This fails for point-set pathological reasons as the functor $\mathrm{Map}(Y, -)$ (with the compact open topology) is not for every space Y right adjoint to $- \times Y$. One way to deal with that problem is to work in the category of compactly generated topological spaces instead and perform all the constructions there. We will not need this and prefer to work simplicially anyhow to avoid all sorts of point set problems.

In particular we see that $|X \times \Delta^1| = |X| \times |\Delta^1|$. Thus for any simplicial set X and every topological space Y we get a bijection

$$\Big[X, \mathrm{Sing}\, Y\Big] \cong \Big[|X|, Y\Big]$$

induced from the adjunction between Sing and $|...|$. We in particular immediately deduce

Proposition 3.8.10. *The functors* Sing *and* $|...|$ *induce adjunctions*

$$\begin{aligned} |...| : \mathrm{Kan} &\leftrightarrow \mathrm{CW} : \mathrm{Sing} \\ |...| : \mathrm{Kan}_* &\leftrightarrow \mathrm{CW}_* : \mathrm{Sing} \\ |...| : \mathrm{Ho}(\mathrm{Kan}) &\leftrightarrow \mathrm{Ho}(\mathrm{CW}) : \mathrm{Sing} \\ |...| : \mathrm{Ho}(\mathrm{Kan}_*) &\leftrightarrow \mathrm{Ho}(\mathrm{CW}_*) : \mathrm{Sing} \end{aligned}$$

Definition 3.8.11. We define $\pi_0(K)$ for any Kan complex as $[\Delta^0, K]$. For a pointed Kan complex K we define

$$\pi_n(K, x) := [\Delta^n/\partial\Delta^n, K]_*$$

where $\Delta^n/\partial\Delta^n$ carries the obvious basepoint.

Note that we can equivalently describe $\pi_n(K)$ as the set of maps $\Delta^n \to K$ that sends the boundary $\partial\Delta^n \to \Delta^n$ to the basepoint module homotopy relative to the boundary. In particular the set $\pi_n(K, x)$ is a quotient of the subset of K_n consisting of those n-simplicies all of whose faces are given by the basepoint. Note that a pointed simplicial set is the same as choosing a basepoint in all K_n which is preserved by all face and degeneracy maps.

Proposition 3.8.12. *For a (pointed) topological space X we have canonical bijections $\pi_n(X) \cong \pi_n(\operatorname{Sing} X)$. In particular $\pi_n(X)$ also (like homology and cohomology) only depends on $\operatorname{Sing} X$.*

We now want to equip the sets $\pi_n(K)$ for $n \geq 1$ with the structure of groups, similar to the topological case. To this end assume that we have two pointed maps

$$f, g : \Delta^n / \partial\Delta^n \to K$$

We can combine f and g into a map

$$\Lambda_1^{n+1} \to K$$

which is given by f on $\partial_2\Delta^{n+1}$, by g on $\partial_0\Delta^{n+1}$ and by the constant map on all the other faces of Λ_1^{n+1}. Then we find a lift

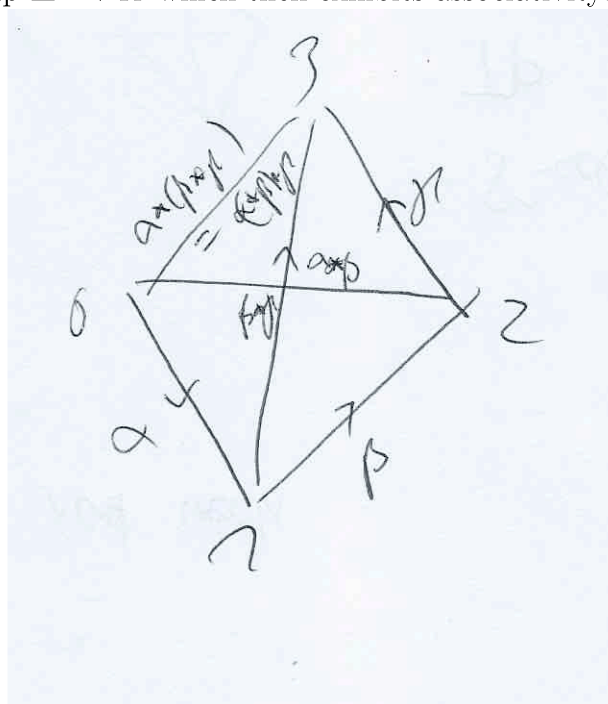
$$\Delta^{n+1} \to K$$

and we define the composition to be the restriction of this map to $\partial_1\Delta^{n+1}$. PICTURE!

Proposition 3.8.13. *This operation is well-defined and equips $\pi_n(K)$ with the structure of a group for $n \geq 1$.*

Proof. For well-definedness we note that two different lifts of $\Delta^{n+1} \to K$ are homotopic relative to the horn by Proposition 3.8.6. But then the restrictions to $\partial_1\Delta^{n+1} = \Delta^n$ are homotopic relative boundary.

For associativity we consider the case $n = 1$ for simplicity first. We first claim that the map $\operatorname{Spine}(2) = \Delta^1 \cup_{\Delta^0} \Delta^1 \cup_{\Delta^0} \Delta^1 \to \Delta^2$ given by the spine inclusion is anodyne. Then for three given classes $\alpha, \beta, \gamma \in \pi_1(K)$ we get an induced map $\operatorname{Spine}(2) \to K$ and lift it to a map $\Delta^2 \to K$ which then exhibits associativity:



For the case of higher n we form the following...

The case of inverses follows similarly by lifting against the correct horn. □

We now want to show that the $\pi_n(K)$ are abelian for $n \geq 2$. This can be done completely elementarily by clever applications of the horn filling condition, but we want to give a slightly more conceptual argument similar to the Eckmann-Hilton argument we gave in the topological setting.

For this we introduce

Definition 3.8.14. We define simplicial sets $\square^n := (\Delta^1)^{\times n}$, and $\partial\square^n \subseteq \square^n$ as the union of all sub-simplicial sets $\Delta^1 \times \ldots \times \partial\Delta^1 \times \ldots \times \Delta^1$. PICTURE

For a Kan complex K, we denote by $\pi_n^\square(K)$ the pointed homotopy classes

$$[\square^n/\partial\square^n, K].$$

Of course, these agree with the simplicial homotopy groups we defined previously. To see this, we need a lemma:

Lemma 3.8.15. *If $A \subseteq B \subseteq X$, and $A \to B$ is anodyne, then the map*

$$[X/B, K]_* \to [X/A, K]_*$$

is an equivalence.

Proof. For surjectivity, we need to show that any map $f : X \to K$ which is constant on A is homotopic rel A to a map which is constant on B. Now observe that $f|_B : B \to K$ is an extension of the constant map $A \to K$, but since extensions along anodyne maps are unique up to homotopy (Proposition 3.8.6) and the constant map $B \to K$ is another such extensions, we get a homotopy $B \times \Delta^1 \to K$ rel A between $f|_B$ and the constant map.

Using that $X \times \{0\} \cup B \times \Delta^1 \to X \times \Delta^1$ is anodyne (since it is the pushout product $(X, B) \boxtimes (\Delta^1, \{0\})$), we can now extend the homotopy of $f|_B$ to a homotopy $X \times \Delta^1 \to K$, rel A, between f and a map constant on B, as desired. (This is the general principle of HEP, which we've just given the argument for again.)

For injectivity, we need to show that if two maps $f, g : X \to K$ which are constant on B, are homotopic rel A, they are also homotopic rel B. So we choose a homotopy $h : X \times \Delta^1 \to K$ between f and g rel A, and the strategy will be to deform this map h rel $X \times \partial\Delta^1 \cup A \times \Delta^1$ to one that is constant on $B \times \Delta^1$, by a very similar trick like the one that gave us surjectivity above.

Consider the restriction $h|_{B\times\Delta^1}$. This is constant on $B \times \partial\Delta^1 \cup A \times \Delta^1$, and since

$$B \times \partial\Delta^1 \cup A \times \Delta^1 \subseteq B \times \Delta^1$$

is anodyne (pushout product $(B, A)\boxtimes(\Delta^1, \partial\Delta^1)$), we obtain a homotopy rel $B\times\partial\Delta^1\cup A\times\Delta^1$ between $h|_{B\times\Delta^1}$ and the constant map. By gluing this with the constant homotopy of $h|_{X\times\partial\Delta^1}$, we get a homotopy rel $X \times \partial\Delta^1 \cup A \times \Delta^1$ between $h|_{X\times\partial\Delta^1\cup B\times\Delta^1}$ and a map which is constant on $B \times \Delta^1$. Now, we extend this homotopy to all of $X \times \Delta^1$, again just using HEP (which here boils down to the anodyneness of the pushout product $(X \times \Delta^1, X \times \partial\Delta^1 \cup B \times \Delta^1) \boxtimes (\Delta^1, \{0\})$). We thus obtain a homotopy, rel $X \times \partial\Delta^1 \cup A \times \Delta^1$, between $h : X \times \Delta^1 \to K$ and a map h' which is constant on $B \times \Delta^1$. But this h' now is a homotopy between f and g rel B, as desired. $\square$

In $\square^n$, there is a specific n-simplex $\Delta^n \to \square^n$, given on posets by the map $[n] \to [1]^{\times n}$ sending $i \in [n]$ to the sequence in $[1]^{\times n}$ given by 1 at indices $\leq i$ and by 0 at indices $> i$.

We define $C \subseteq \square^n$ as the minimal sub-simplicial set containing all $(n-1)$-simplices, and all n-simplices except for the one described above.

We also have a map $r : \square^n \to \Delta^n$, sending a sequence $x \in [1]^{\times n}$ to $i \in [n]$, where i is minimal such that $x_i = 0$. This is a retract to the inclusion $\Delta^n \to \square^n$, and it induces a map

$$r^* : \pi_n(K) = [\Delta^n/\partial\Delta^n, K]_* \to [\square^n/\partial\square^n, K]_* = \pi_n^{\square}(K)$$

Proposition 3.8.16. *The above map $r^* : \pi_n(K) \to \pi_n^{\square}(K)$ is a bijection.*

Proof. It factors as the composite

$$[\Delta^n/\partial\Delta^n, K]_* \to [\square^n/C, K]_* \to [\square^n/\partial\square^n, K]_*.$$

The first map comes actually from precomposition with an isomorphism of simplicial sets $\square^n/C \to \Delta^n/\partial\Delta^n$. One checks this for example by establishing that $\square^n$ is obtained from C by gluing in the missing n-simplex.

The second map is a bijection because of Lemma 3.8.15, using that $\partial\square^n \to C$ is anodyne. This one can see explicitly by attaching the n-simplices of C in the right order. $\square$

We now can endow $\pi_n^{\square}(K)$ with n a priori different group structures, which we momentarily denote by $\cdot_i$ for $1 \leq i \leq n$. We can form for two $\alpha, \beta : \square^n/\partial\square^n \to K$ the combined map

$$(\beta, \alpha) : \Delta^1 \times \ldots \times \Lambda_1^2 \times \ldots \times \Delta^1 \to K.$$

with Λ_1^2 in the i-th factor. We can then extend to the product $\Delta^1 \times \ldots \times \Delta^2 \times \ldots \times \Delta^1$ and restrict along the face $\partial_1\Delta^2$ to obtain a new map $\alpha \cdot_i \beta : \Delta^1 \times \ldots \times \Delta^1 \times \ldots \times \Delta^1 \to K$. This defines an associative group structure precisely by the same arguments as for $\pi_1(K)$ (and in fact, we can identify this group $\pi_n^{\square}(K)$ as π_1 of some Kan complex obtained as suitable internal Hom).

Lemma 3.8.17. *For $n \geq 2$, the above group structures $\cdot_i$ on $\pi_n^{\square}(K)$ agree, and they are abelian.*

Proof. We do this for the first two to simplify notation, the other pairs work completely analogously. For $\alpha, \beta, \gamma, \delta : \square^n/\partial\square^n \to K$, we consider

$$(\delta, \gamma, \beta, \alpha) : \Lambda_1^2 \times \Lambda_1^2 \times \Delta^1 \times \ldots \times \Delta^1 \to K$$

putting the four maps on the four different copies of $\square^n$ the domain consists of. Now if we first extend to $\Delta^2 \times \Lambda_1^2 \times \ldots$ and then to $\Delta^2 \times \Delta^2 \times \ldots$, we obtain a representative of $(\alpha \cdot_1 \beta) \cdot_2 (\gamma \cdot_1 \delta)$ when restricting to $\Delta^1 \times \Delta^1 \times \ldots$. If we do it the other way, we get a representative of $(\alpha \cdot_2 \gamma) \cdot_1 (\beta \cdot_2 \delta)$. But since extensions along anodynes are unique up to homotopy, we learn that these two classes agree, i.e. $\cdot_1$ and $\cdot_2$ satisfy the assumptions of Eckmann-Hilton. Thus, the claim follows. $\square$

Proposition 3.8.18. *The bijection $r^* : \pi_n(K) \to \pi_n^{\square}(K)$ is a group isomorphism. In particular, $\pi_n(K)$ is abelian for $n \geq 2$.*

Proof. We show that the map is a homomorphism with respect to $\cdot_1$, the composition in $\pi_n^{\square}(K)$ using the first coordinate. Given $\alpha, \beta : \Delta^n \to K$, we find an $(n+1)$-simplex σ :

$\Delta^{n+1} \to K$, whose faces are given by $(\beta, \alpha \cdot \beta, \alpha, 0, \ldots)$. We now embed Δ^{n+1} into $\Delta^2 \times (\Delta^1)^{n-1}$ by the map $[n+1] \to [2] \times [1]^{\times n-1}$ given by

$$(0,0,0,\ldots) \to (1,0,0,\ldots) \to (2,0,0,\ldots) \to (2,1,0,\ldots) \to (2,1,1,\ldots) \to \ldots$$

There is a retraction $\Delta^2 \times (\Delta^1)^{n-1} \to \Delta^{n+1}$ onto this $n+1$-simplex, which on posets sends a sequence $x \in [2] \times [1]^{\times n-1}$ to

$$x \mapsto \begin{cases} n+1 \text{ if all } x_i \neq 0, \\ i+1 \text{ if the smallest index } i \text{ at which } x_i = 0 \text{ satisfies } i \geq 2, \\ 0 \text{ if } x \text{ starts with } (0,0,\ldots), \\ 1 \text{ if } x \text{ starts with } (1,0,\ldots), \\ 2 \text{ if } x \text{ starts with } (2,0,\ldots). \end{cases}$$

Now we precompose the map $\Delta^{n+1} \to K$ with faces $(\beta, \alpha \cdot \beta, \alpha, 0, \ldots)$ with this retraction. We obtain a map $\Delta^2 \times (\Delta^1)^{\times n-1} \to K$, which on the faces $\partial_i \Delta^2 \times (\Delta^1)^{\times n-1}$ is given by $r^*\beta$ for $i = 0$, $r^*(\alpha\beta)$ for $i = 1$, $r^*\alpha$ for $i = 2$, as we will check in a moment. This implies the result, since this extension to $\Delta^2 \times (\Delta^1)^{\times n-1}$ witnesses $r^*(\alpha\beta) = r^*(\alpha) \cdot_1 r^*(\beta)$.

For the remaining claim, it suffices to check that on all three faces, the composite is constant on the respective $C \subseteq \square^n$, and is given by β, $\alpha\beta$ resp. α on the last n-simplex. This one can work out directly from the description of the map on posets. For example, on the face $\partial_0 \Delta^2 \times (\Delta^1)^{\times n-1}$, the map never hits 0, i.e. factors through $\partial_0 \Delta^{n+1}$, and sends all simplices to degenerate simplices or boundary simplices except for the one given by

$$(1,0,\ldots) \to (2,0,\ldots) \to (2,1,0,\ldots) \to \ldots$$

One similarly proceeds for the other faces. $\square$

Now we finally want to prove the key Lemma 3.8.4. The proof will crucially rely on the following claim:

Lemma 3.8.19. *The class of anodyne morphisms is the smallest class closed under pushouts, retracts and transfinite compositions containing the morphisms $k \boxtimes l$ for $k : A \to B$ any monic and l any map $\Delta^0 \to \Delta^1$.*[2]

Before we verify Lemma 3.8.19 let us first show that it implies Lemma 3.8.4. We want to prove that for a given monomorphism i and anodyne morphism j the morphism $i \boxtimes j$ is anodyne. For fixed i the constrtuction $j \mapsto j \boxtimes i$ commutes with pushouts, retracts and transfinite compositions as one easily checks. Thus using Lemma 3.8.19 it suffices to treat the case $j = k \boxtimes l$. But then we have that

$$i \boxtimes (k \boxtimes l) \cong (i \boxtimes k) \boxtimes l$$

and the claim will follow from the fact that for monics i and k the induced morphism $i \boxtimes k$ is monic as well together with another application of Lemma 3.8.19. The statement about monics is clear since the corresponding statement obviously holds in the category of sets.

[2] There are of couse precisely two maps

Proofsketch of Lemma 3.8.19. We first claim that any morphism $k \boxtimes l$ is anodyne. We write the map $k : A \to B$ as an iterated pushout along horn inclusions, i.e. $A = A_0 \to A_1 \to \ldots \to B$ where A_n is the subsimplicial set of B generated by the non-degenerate n-simplices in $B_n \setminus A_n$. This is a relative version of the skeletal filtration. It follows that the map $A \to B$ is transfinite composition of pushouts along $\partial\Delta^n \to \Delta^n$. Since $k \boxtimes l$ preserves pushouts and transfinite compositions in k we can thus without loss of generality assume that $k = (\partial\Delta^n \to \Delta^n)$.

Thus we have to show that

$$(\partial\Delta^n \times \Delta^1) \amalg_{\partial\Delta^n} \Delta^n \to \Delta^n \times \Delta^1$$

is anodyne where Δ^n is attached to the upper or lower part of the cylinder (picture). In order to see this we observe that $\Delta^n \times \Delta^1$ has $(n+1)$-nondegenerate $(n+1)$-simplices corresponding to the maps

$$h_j : \Delta^{n+1} \to \Delta^n \times \Delta^1$$

given by shuffles:

$$(0,0) \to (0,1) \to \ldots \to (0,j) \to (1,j) \to (1,j+1) \to \ldots \to (1,n)\ .$$

We have to attach them one after the other in the order $h_n, h_{n-1}, \ldots, h_0$ along horn inlcusions, that the morphism in question is

$$X_0 \to X_1 \to \ldots \to X_n$$

such that each morphism fits into a pushout square

$$\begin{tikzcd} \Lambda^{n+1}_{n-i} \arrow[r] \arrow[d] & X_i \arrow[d] \\ \Delta^{n+1} \arrow[r] & X_{i+1} \end{tikzcd}$$

This shows the first direction.

Conversely for $k < n$ we consider the retract $\Delta^n \to \Delta^n \times \Delta^1 \to \Delta^n$ in which the first map is given by $[n] \to [n] \times [1]$ with $j \mapsto (1,j)$ and the map $[n] \times [1] \to [n]$ by the diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 1 \arrow[r] \arrow[d] & \ldots \arrow[r] & k \arrow[r] \arrow[d] & k \arrow[r] \arrow[d] & \ldots \arrow[r] & k \arrow[d] \\ 0 \arrow[r] & 1 \arrow[r] & \ldots \arrow[r] & k \arrow[r] & k+1 \arrow[r] & \ldots \arrow[r] & n \end{tikzcd}\ .$$

Then by construction we have a retract diagram

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & (\Delta^n \times \{0\}) \amalg_{(\Lambda^n_k \times \{0\})} (\Lambda^n_k \times \Delta^1) \arrow[r] \arrow[d] & \Lambda^n_k \arrow[d] \\ \Delta^n \arrow[r] & \Delta^n \times \Delta^1 \arrow[r] & \Delta^n \end{tikzcd}$$

which shows that the horn inclusions are retracts of maps in the class generated by the maps in question if $k < n$. For the case $k = n$ we either make a symmetrical argument or pass to opposite simplicial sets. $\square$

Definition 3.8.20. A map of simplicial sets $X \to Y$ is called weak homotopy equivalence if for every Kan complex K the induced map $[Y, K] \to [X, K]$ is a bijection.

Example 3.8.21. Every anodyne morphism is a weak homotopy equivalence. To see this we can construct and inverse by lifting, which is unique by Proposition 3.8.6.

Remark 3.8.22. One can show that also the converse holds true: a monomorphism of simplicial sets is anodyne precisely if it is a weak homotopy equivalence. This is a deep result which we will not prove here and also not need it.

Proposition 3.8.23. *A map $K \to K'$ between Kan complexes is a weak homotopy equivalence precisely if it is a homotopy equivalence. For every weak equivalence $X \to Y$ of simplicial sets the inuduced map $|X| \to |Y|$ is a homotopy equivalence of spaces.*

Proof. The first part is Yoneda and the second follows by adjunction since we get that $[|Y|, M] \to [|X|, M]$ is a bijection. □

Proposition 3.8.24. *For every simplicial set X there exists a weak homotopy equivalence $X \xrightarrow{\simeq} X_K$ where X_K is a Kan complex. Any two such Kan replacements are homotopy equivalent. In fact, the map $X \to X_K$ can even be chosen to be anodyne and the construction $X \mapsto (X \to X_K)$ functorial in X.*

Proof. Small object argument. □

Definition 3.8.25. For any pointed simplcial set X we define

$$\pi_n(X, x) := \pi_n(X_K, x)$$

These homotopy groups are clearly invariant under weak equivalence in X using that for if $X \simeq_w Y$ then $X_K \simeq Y_K$.

3.9 Comparison of homotopy theories

The main purpose of this section is to show that the homotopy theories of simplicial sets and the one of topological spaces are equivalent. Recall that we had the adjunction

$$|...| : \mathrm{sSet} \leftrightarrow \mathrm{Top} : \mathrm{Sing}$$

and this in particular comes with a unit transformation

$$K \to \mathrm{Sing}\,|K|$$

for every simplicial set K.

Theorem 3.9.1. *For every Kan complex K the map $K \to \mathrm{Sing}\,|K|$ is a homotopy equivalence.*

We will prove this theorem (maybe) later but now want to take it on faith and use it to develop the theory. We note that this result is in fact equivalent to the seemingly stronger result that for every simplicial set X the induced map

$$X \to \mathrm{Sing}\,|X|$$

is a weak equivalence. To see the non-obvious implication we simply note that for X we consider the map $X \to X_K$ and get that $|X| \to |X_k$ is a homotopy equivalence, thus $\operatorname{Sing}|X| \to \operatorname{Sing}|X_K|$ is a homtotopy equivalence. But then in the square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X_K \arrow[d] \\ \operatorname{Sing}|X| \arrow[r] & \operatorname{Sing}|X_K| \end{tikzcd}$$

all morphisms except the left vertical are weak equivalences, thus also the left vertical one.

Corollary 3.9.2. *1. For any simplicial set X we have an isomorphism of groups $\pi_n(X,x) = \pi_n(|X|,x)$.*

2. A map $X \to Y$ of simplicial sets if a weak equivalence iff $|X| \to |Y|$ is a homotopy equivalence iff it is a π_* iso (simplicial Whitehead) [3]

3. A map $f : X \to Y$ of spaces is a weak equivalence iff $\operatorname{Sing} X \to \operatorname{Sing} Y$ is a homotopy equivalence of Kan complexes.

4. For any topological space X the map $|\operatorname{Sing} X| \to X$ is a weak homotopy equivalence. In particular each space is weakly homotopy equivalent to a CW complex.

5. The functor Sing and $|...|$ induce inverse equivalences

$$\mathrm{Ho}(\mathrm{CW}) \simeq \mathrm{Ho}(\mathrm{Kan}) \ .$$

6. The full inclusion

$$\mathrm{Ho}(\mathrm{CW}) \subseteq \mathrm{Ho}(\mathrm{Top})$$

admits a right adjoint given by $X \mapsto |\operatorname{Sing} X|$ or any other CW approximation.[4]

Proof. 1) We note that the map $X \to \operatorname{Sing}|X|$ qualifies as a Kan replacement, thus can be used to compute π_n for X. But $\pi_n(\operatorname{Sing}|X|) = \pi_n(|X|)$.
2) By the first and the classical Whitehead we find that the second two conditions are equivalent. If $|X| \to |Y|$ is a homotopy equivalence, then so is $\operatorname{Sing}|X| \to \operatorname{Sing}|Y|$ and thus also $X \to Y$.
3) Since $\pi_n(\operatorname{Sing} X) = \pi_n(X)$ we get this immediately from (2).
4) We apply the criterion of (3) and use the commutative triangle

$$\begin{tikzcd} \operatorname{Sing} X \arrow[r] \arrow[dr, "\mathrm{id}"'] & \operatorname{Sing}|\operatorname{Sing} X| \arrow[d] \\ & \operatorname{Sing} X \end{tikzcd}$$

in which the upper horizontal map is a weak equivalence by Theorem 3.9.1.
(5) Clear, since unit and counit are equivalences. (6) We have to show that for weak homotopy equivalence $\hat{X} \to X$ of spaces such that $\hat{X}$ is a CW complexes and every CW complex Y the induced map

$$[Y,\hat{X}] = \mathrm{Hom}_{\mathrm{Ho(CW)}}(Y,\hat{X}) \to \mathrm{Hom}_{\mathrm{Ho(Top)}}(Y,X) = [Y,X]$$

[3] In reality one has to prove this theorem first in order to deduce 3.9.1. Thus the argument here is a bit circular.

[4] That is a CW complex X' with a weak homotopy equivalence $X' \to X$.

is a bijektion. To see this we can without loss of generality assume that $Y = |K|$ for a Kan complex and then we get that this map is given by

$$[K, \operatorname{Sing} \hat{X}] \to [K, \operatorname{Sing} X]$$

which is a bijection since the map $\operatorname{Sing} \hat{X} \to \operatorname{Sing} X$ is a homotopy equivalence. □

Now we want to see some immediate consequences.

Proposition 3.9.3. *Any weak homotopy equivalence of spaces $X \to Y$ induces isomorphisms $f_* : H_n(X; M) \to H_n(Y; M)$ and $H^*(Y; M) \to H^*(X; M)$ for all coefficients M.*

Proof. Follows immediately from the fact that for a weak equivalence of spaces $X \to Y$ the induced map of simplicial sets is a (weak) homotopy equivalence and the fact that homology for simplicial sets is invariant. □

Remark 3.9.4. Recall the Eilenberg-Steenrod axioms: a (co)homology theory was a pair consisting of a functor $E_* : \mathrm{Top}^2 \to \mathrm{grAb}$ or $E^* : (\mathrm{Top}^2)^{op} \to \mathrm{grAb}$ together with natural morphisms $\delta : E_n(X, A) \to E_{n-1}(A)$ such that the following axioms are satisfied: Homotopy invariance, long exact sequence, additivity, excision and dimension (without the last it was a generalized cohomology theory). Homotopy invariance was that a map

> (Homotopy invariance) If $f \simeq g$ then $f_* = g_*$ (resp. $f^* = g^*$.

This is equivalent to say that for f a homotopy equivalence the induced maps f_* is an isomorphism. Only using the axioms we could show that the value of a cohomology theory is uniquely determined for CW complexes . But now we observe that singular homology has a much stronger invariance property, namely that we get isos for weak homotopy equivalences between spaces. Thus it satisifes the following variant:

> (Weak Homotopy invariance) If $f : X \to Y$ is a weak homotopy equivalence, then f_* (resp. f^*) induces and isomorphism $E_*(X) \to E_*(Y)$ (resp. $E^*(Y) \to E^*(X)$).

In that case we say that the homology (or cohomology theory) satisifies weak homotopy invariance. We now get that an ordinary homology theory which satisfies weak homotopy invariance and the dimension axiom is uniquely determined by its value on the point (and agrees with singular homology).

Lemma 3.9.5. *If a simplicial set X already has fillers for all horns of simplices of dimension $i \leq n$ then we can find a Kan replacement $X \to X_K$ that is an isomorphism on i-cells for $i < n$.*

Proof. We use a version of the small object argument to construct X_K where we attach in each step only those horns that do not already admit fillers. Since the inclusion of a horn into Δ^j for $j > n$ is an isomorphism on i-cells for $i < n$ this does not change the simplices in dimension $< i$ at each step. □

Theorem 3.9.6. *We have that $\pi_k(S^n) = 0$ for $k < n$.*

Proof. We observe that $|\Delta^n/\partial\Delta^n| = S^n$ so that it suffices to show that $\pi_k(\Delta^n/\partial\Delta^n) = 0$ for $k < n$. The simplicial set $\Delta^n/\partial\Delta^n$) is a point in each simpicial degree $< n$. As a result it has fillers for all horns of Δ^i with $i \leq n$ as all the maps $\Lambda^i_j \to \Delta^n/\partial\Delta^n$ are constant. Thus by the precious lemma we can construct a Kan replacement which also does not have any non-trivial cells in dimension $< n$ which immediately shows the claim. □

Definition 3.9.7. Let $\mathcal{C}$ be a category and W be a class of morphisms in $\mathcal{C}$. We say that a functor $\kappa : \mathcal{C} \to \mathcal{C}'$ exhibits $\mathcal{C}'$ as the localization of $\mathcal{C}$ with respect to W if it satisfies:

1. κ sends the morphisms in W to isomorphisms in $\mathcal{C}'$.

2. For every category $\mathcal{D}$ the induced functor

$$\kappa^* : \mathrm{Fun}(\mathcal{C}', \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

is fully faithful with essential image consisting of those functors $\mathcal{C} \to \mathcal{D}$ that sends W to isomorphisms.

In this case we shall also write $\mathcal{C}' = \mathcal{C}[W^{-1}]$.

Example 3.9.8. Consider the category Top and let H be the class of homotopy equivalences. Then the functor $\kappa : \mathrm{Top} \to \mathrm{Ho(Top)}$ exhibits Ho(Top) as the localization at H.

To see note that for some category $\mathcal{D}$ we have that κ^* embeds $\mathrm{Fun}(\mathrm{Ho(Top)}, \mathcal{D})$ as the full subcategory of $\mathrm{Fun}(\mathrm{Top}, \mathcal{D})$ consisting of those functors $\mathrm{Top} \to \mathcal{D}$ which send homotopic morphisms in Top to the same morphism in $\mathcal{D}$. Thus the claim follows from the assertion that a functor $F : \mathrm{Top} \to \mathcal{D}$ sends H to isomorphisms in $\mathcal{D}$ precisely if it sends homotopic maps to the same map in $\mathcal{D}$. Lets prove this:

If F sends H to isos, then it sends $pr : X \times I \to X$ to an isomorphism. But then it sends $i_0, i_1 : X \to X \times I$ to the same map as they are both one sided inverses to pr. Thus homotopic maps $f, g : X \to Y$ are also sent to the same map. The converse is clear.

Example 3.9.9. Similar to the last example we see that $\mathrm{Kan} \to \mathrm{Ho(Kan)}$ and $\mathrm{CW} \to \mathrm{Ho(CW)}$ are localizations at the respective homotopy equivalences.

Proposition 3.9.10. *Any two localizations of $\mathcal{C}$ at W are canonically equivalent. More precisely for $\kappa : \mathcal{C} \to \mathcal{C}'$ and $\kappa' : \mathcal{C} \to \mathcal{C}''$ localizations there is an equivalence $f : \mathcal{C}' \xrightarrow{\simeq} \mathcal{C}''$ of categories together with a natural isomorphism $f\kappa \simeq \kappa'$ and this f is unique up to unique natural isomorphism.*

Proof. Clear by the universal property. □

Proposition 3.9.11. *For any large[5] category $\mathcal{C}$ and any class W there exists a localization $\kappa : \mathcal{C} \to \mathcal{C}[W^{-1}]$ with $\mathcal{C}[W^{-1}]$ large as well and the functor κ is essentially surjective.*

Proof. If one closed the class W under the 3 for 2 property and adds all identity morphisms then the localization does not change (by the universal property). Thus we can without loss of generality assume that W is closed under 3-for-2 and contains all identities.

Now we define a candidate category $\mathcal{C}[W^{-1}]$ as follows: the objects are the same as the objects of $\mathcal{C}$. For any pair of objects $c, c' \in \mathcal{C}$ a morphism in $\mathcal{C}[W^{-1}]$ is represented by a ZigZag

$$d_0 \xleftarrow{\sim} d_1 \to d_2 \xleftarrow{\sim} d_3 \to \dots \to d_n \xleftarrow{\sim} c'.$$

where the arrows pointing to the left are in W. Two such ZigZags are identified under the following operations:

[5] Meaning that the objects as well as the morphisms form a *large* set.

1. Skipping an identity and composing adjacent morphisms:
$$...d_k \xleftarrow{f} d_{k+1} \xrightarrow{\mathrm{id}} d_{k+2} \xleftarrow{g} d_{k+3}... \qquad \text{is equivalent to} \qquad d_k \xleftarrow{fg} d_{k+3}$$
and similar for left pointing identities.

2. Hammocks: two morphisms are equivalent if there is a diagram of the form
$$\begin{tikzcd}
 & d_0 \arrow[d, "\sim"] & d_1 \arrow[l, "\sim"'] \arrow[r] \arrow[d, "\sim"] & d_2 \arrow[d, "\sim"] & d_3 \arrow[l, "\sim"'] \arrow[r] \arrow[d, "\sim"] & \dots \arrow[r] & d_n \arrow[d, "\sim"] & \\
c \arrow[ur] \arrow[dr] & & & & & & & f \arrow[ul, "\sim"'] \arrow[dl, "\sim"] \\
 & d_0 & d_1 \arrow[l, "\sim"'] \arrow[r] & d_2 & d_3 \arrow[l, "\sim"'] \arrow[r] & \dots \arrow[r] & d_n &
\end{tikzcd}$$

Then we define composition of morphisms by concatenation of ZigZags. The identity on c is defined to be the ZigZag of length 0 (or equivalently the ZigZag $c \xrightarrow{\mathrm{id}} c \xleftarrow{\mathrm{id}} c$. Then it is clear that this defines a category as the composition if clearly well-defined on equivalence classes.

There is also a functor $\mathcal{C} \to \mathcal{C}[W^{-1}]$ which sends the object c to c and a morphism $c \to c'$ to the ZigZag
$$c \to c' \xleftarrow{\mathrm{id}} c'$$
and it is again clear by definition that this is a functor. Moreover this functor has the property that it sends W to isomorphisms in $\mathcal{D}$. To see this we claim that the morphism $c' \xrightarrow{\mathrm{id}} c' \xleftarrow{w} c$ is the inverse to w.

We now want to verify that this functor satisfies the universal property. To this end we claim that a functor
$$F : \mathcal{C} \to \mathcal{D}$$
that sends W to isomorphisms extends uniquely to $\mathcal{C}[W^{-1}]$. Indeed the extended functor has to send the ZigZag ... to ... A similar argument shows that natural transformations extend uniquely. This shows that the category of functors $\mathrm{Fun}(\mathcal{C}[W^{-1}], \mathcal{D})$ is in fact isomorphic to the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ consisting of those functors that send W to isomorphisms. In particular it is equivalent and this finishes the proof. □

Warning 3.9.12. Even if $\mathcal{C}$ is locally small then $\mathcal{C}[W^{-1}]$ does not need to be locally small in general! This is why sometimes it is claimed that the localization might not exist in general. The reason is that the morphisms in our construction of the localization above are quantified over all objects $d_0, .., d_n$ which are not a small set of $\mathcal{C}$ has a large set of objects.

Now we consider the category sSet of simplicial sets and want to study the localization of sSet at the class of weak homotopy equivalences $\mathrm{sSet}[W^{-1}]$.

Proposition 3.9.13. *The functors* $\mathrm{CW} \to \mathrm{Top} \xrightarrow{\mathrm{Sing}} \mathrm{Kan} \to \mathrm{sSet}$ *induce equivalences*
$$\mathrm{CW}[W^{-1}] \to \mathrm{Top}[W^{-1}] \xrightarrow{\mathrm{Sing}} \mathrm{Kan}[W^{-1}] \to \mathrm{sSet}[W^{-1}]$$
where W denotes the set of respective weak homotopy equivalences.

Proof. We have inverse functors given by

1. Kan replacement (which we do not need to chose functorially here, it is automatically functorial on the homotopy category)
2. Geometric realization
3. CW approximation

These descend to the localized categories by the universal property and then it is easy to see that they are inverse. It is easy to see that these are inverse. □

Remark 3.9.14. The homotopy categories in question capture the essence of being homotopic and homotopy classes of maps. But they are from a categorical point of view not very well behaved, e.g. they do not admit colimits and limits in general. The reason is that the homotopy category Ho Kan or any of the equivalent categories is only a 'truncation' of the real object, namely the ∞-categorical localization of Kan at the homotopy equivalences. For our purposes the homotopy category will be good enough, but we will get a glimpse at some of the ∞-categorical aspects when we discuss homotopy limits.

3.10 Cellular approximation

Recall that a map f between CW complexes X and Y is called cellular if $f(X^n) \subseteq Y^n$, i.e. cells of dimension n are sent to cells of dimension less or equal to n. The result that we want to show in this section is the following:

Theorem 3.10.1 (Cellular approximation)**.** *Every map $f : X \to Y$ of CW complexes is homotopic to a cellular map. If f is already cellular on a subcomplex $A \subseteq X$ then the homotopy can be taken constant on that subcomplex.*

Corollary 3.10.2. *If a CW complex X has one 0-cell and all other cells have dimension $> n$ then it is n-connected.*

Proof. We get $\pi_k(X) = 0$ for $k \leq n$ using that maps from spheres can be made cellular. □

Note that this argument again implies that the lower homotopy groups of spheres are trivial (and this is the standard proof). It is basically a topological version of the simplicial proof that we gave.

Definition 3.10.3. A homotopy $X \times I \to Y$ is called cellular if it is cellular for the product cell structure on $X \times I$ induced from the cell structure on the intervall $I = |\Delta^1|$, i.e. it has two 0-cells and one 1-cell. We say that two cellular maps are cellular homotopic if there exists a cellular homotopy between the two maps. We define Ho(Cell) to be the cellular homotopy category of CW complexes, i.e. maps are given by

$$[X, Y]_{cell} := \frac{\{\text{Cellular maps } X \to Y\}}{\text{cellular homotopy}}$$

A cellular homotopy equivalence is an isomorphism in this category.

Remark 3.10.4. The description of the category depends on the fact that cellular homotopy is an equivalence relation compatible with composition. This is not completely obvious but can be checked. We will not do this here since it also directly follows from the next description.

Corollary 3.10.5. *Two cellular maps are cellular homotopic precisely if they are homotpic. The functor* $\mathrm{Ho}(\mathrm{Cell}) \to \mathrm{Ho}(\mathrm{CW})$ *is an equivalence of categories.*

Proof. We replace a given homotopy $h : X \times I \to Y$ (not necessarily cellular) by a homotopic cellular map $h' : X \times I \to Y$ that agrees with h on the subcomplex $X \times \partial I \subseteq X \times I$. This shows the first part and the fact that the functor is faithful. The fact that it is full immediately follows from the cellular approximation Theorem. □

Warning 3.10.6. Assume that our spaces X and Y are already cellular homeomorphic to realizations of Kan complexes. Then by the equivalence of homotopy categories between Kan complexes and CW complexes (and since realization sends maps to cellular maps) we immediately see that every map $f : X \to Y$ is homotopic to a cellular map. This give a proof (of the non-relative version) of the cellular approximation theorem in this case. However, not every CW complex X is homeomorphic to the geometric realization of a Kan complex. This is only true up to homotopy equivalence as the map $|\operatorname{Sing} X| \to X$ is a homotopy equivalence. The problem is that this map is not only not a homeomorphism but not even cellular, thus the cellular approximation theorem can not be deduced along these lines (at least not without some effort). However, it follows from the cellular approximation Theorem that this map $|\operatorname{Sing} X| \to X$ can be replaced by a cellular homotopy equivalence which one would otherwise have to prove by other means.

Proof of Theorem 3.10.1. We want to use induction over a cell structure of the pair (X, A) to make the map gradually cellular on a larger subspace. Using the using the usual argument of composing infinite strings of homotopies it thus suffices to make a map $f : X \to Y$ that is cellular on A into a homotopic map $f' : X \to Y$ that is cellular on $A \cup e \subseteq X$. We will first homotop the restriction $f|_{A \cup e}$ to a map that is cellular and the adhere to homotopy extension. Thus we can without loss of generality assume that (X, A) consists of a single cell. Furthermore using that X is the pushout of A along a map $\partial D^n \to D^n$ we can even assume that $(X, A) = (D^n, \partial D^n)$.

We thus consider a map $f : D^n \to Y$ that sends ∂D^n to the (n-1)-skeleton of Y. It will hit finitely many cells of Y. Let $e^m \subseteq Y$ be one of these cells of maximal dimension. We can assume $m > n$, otherwise the map f is already cellular. We consider the subcomplex $Y' \subseteq Y$ which consists of the $(m-1)$-skeleton together with all other m-cells of Y. Thus we have that the m-skeleton of Y is given by

$$Y_m = Y' \cup e^m = Y' \amalg_{\partial D^m} D^m$$

We claim that we can homotop our map f relative to ∂D^n to a map that avoids the cell e^m. This then inductively finishes the proof as we can then inductively avoid all the cells. To see the claim it suffices to show that the relative homotopy group $\pi_n(Y' \cup_{\partial D^m} D^m, Y')$ vanishes (by the definition of relative homotopy groups this is exactly the obstruction). By the long exact sequence in homotopy groups

$$\to \pi_n(Y') \to \pi_n(Y' \amalg_{\partial D^m} D^m) \to \pi_n(Y' \amalg_{\partial D^m} D^m, Y') \to \pi_{n-1}(Y') \to \pi_{n-1}(Y' \amalg_{\partial D^m} D^m) \to \ldots$$

it suffices to show that the maps

$$\pi_n(Y') \to \pi_n(Y' \amalg_{\partial D^m} D^m) \quad \text{and} \quad \pi_{n-1}(Y') \to \pi_{n-1}(Y' \amalg_{\partial D^m} D^m)$$

are isomorphisms. We show the first for arbitrary $n < m$ and arbitrary Y' so that the second also follows.

To show this we can replace the space Y' by any homotopy equivalent space since we already know that the homotopy type of the pushout only depends on the homotopy class of the attaching map. Thus we replace Y' by the geometric realization of a Kan complex K. We can morever assume that the attaching map $\partial D^m \to |K|$ is given by a map of simplicial sets $\partial\Delta^m \to K$ since we know every map is homotopic to one of those. Thus we consider the question whether for any Kan complex K and any such map the induced map

$$K \to K \amalg_{\partial\Delta^m} \Delta^m$$

induces an isomorphism on π_n. But by Lemma 3.9.5 we can find a Kan replacemement K' for $K \amalg_{\partial\Delta^m} \Delta^m$ so that the induced map $K \to K'$ is an isomorphism on the $(m-1)$-Skeleton which immediately implies the claim since $n \leq m-1$. □

3.11 Simplicial Whitehead

In this and the next section we want to develop tools to sketch a proof of Theorem 3.9.1. For that we will need a simplicial Whitehead theorem. We first prove a simplicial version of the compression Lemma.

Lemma 3.11.1. *Let $A \hookrightarrow K$ be an inclusion of Kan complexes which induces an isomorphism on all homotopy groups. Then for a simplicial set X the map $[X, A] \to [X, K]$ is bijective.*

Proof. We claim: every map $\Delta^n \to K$, whose boundary maps to A, is homotopic relative boundary to one which lands entirely in A. Believing that claim, we can proceed to show inductively that for every pair (X, Z), a map $X \to K$ sending Z to A is homotopic rel Z to one that sends X into A. To see this, we can induct over skeleta, homotoping the map rel Z to map $\mathrm{sk}^n X \cup Z$ into A. In the inductive step, we need to homotope maps $\Delta^{n+1} \to K$ rel boundary into A, which we can by the claim.

Using this statement about pairs, surjectivity of $[X, A] \to [X, K]$ is done by applying to the pair $(X, \emptyset)$, while injectivity uses the pair $(X \times \Delta^1, X \times \partial\Delta^1)$.

To check the claim, first observe that it is enough to check that any map of pairs $(\Delta^n, \partial\Delta^n) \to (K, A)$ is homotopic to one with image in A through maps of pairs (not necessarily relative boundary). Namely, given such a homotopy

$$\Delta^n \times \Delta^1 \to K,$$

we can homotope it on $\partial\Delta^n \times \Delta^1$ within A to a map which factors through the projection $\partial\Delta^n \times \Delta^1 \to \partial\Delta^n$, since this map is a weak homotopy equivalence and A is a Kan complex. Using homotopy extension (first on one end within A, then for the rest within K), we obtain the desired homotopy rel boundary.

Now note that when freely homotoping such a map of pairs, we can first assume it to be constant on the sub-simplicial set $\Lambda^n_n \subseteq \Delta^n$, since that is contractible. The remaining map

$f : \partial_n \Delta^n \to A$ represents an element in homotopy, which in K is bounded by the map Δ^n, so f is zero in $\pi_{n-1}(K)$. But $\pi_{n-1}(A) \to \pi_{n-1}(K)$ is an isomorphism, so f is zero in $\pi_{n-1}(A)$. So we can in fact homotope any map $(\Delta^n, \partial\Delta^n) \to (K, A)$ to one which is constant on $\partial\Delta^n$. This therefore represents an element in $\pi_n(K)$, and since $\pi_n(A) \to \pi_n(K)$ is an isomorphism, it is homotopic to one with image in A. □

Theorem 3.11.2 (Simplicial Whitehead). *Given a map of Kan complexes $f : K \to L$ which induces isomorphisms $\pi_i(K) \to \pi_i(L)$ on all homotopy groups. Then it is a homotopy equivalence. In particular, the two possible notions of weak equivalence of simplicial sets (as in Definition 3.8.20, or as "map which induces isomorphisms on homotopy groups of Kan replacements") agree.*

Proof. The "simplicial mapping cylinder" $K \times \Delta^1 \amalg_f L$ has the property that the inclusion of L is anodyne, since it is a pushout of the inclusion $K \times \{1\} \to K \times \Delta^1$. So if we let L' be a Kan replacement of the mapping cylinder, we easily see that the original map f factors as $K \to L' \to L$, and $L \to L'$ is a homotopy equivalence. Since $K \to L'$ is an inclusion of Kan complexes, we have reduced to the case where f is injective.

In that case, by the simplicial compression lemma, the map $K \to L$ induces an isomorphism on all $[X, -]$. So an application of Yoneda shows the claim. □

This enables us to perform a very useful construction:

Lemma 3.11.3. *Let $K \to L$ be a map of Kan complexes which is an isomorphism on homotopy groups π_i for $i < n$, and surjective on π_n. Then there exists a diagram of Kan complexes*

$$\begin{tikzcd} K \arrow[r] \arrow[d, hook] & L \\ L' \arrow[ur, "\simeq"] & \end{tikzcd}$$

where the diagonal map is a homotopy equivalence and the vertical map is an inclusion which is bijective on n-skeleta.

Proof. Assume WLOG that everything is connected and pick basepoints. We build L' inductively from K by attaching simplices of dimension $\geq n+1$, making the map $L' \to L$ into an isomorphism on homotopy groups. We start with $L_n = K$. This comes with a map $L_n \to L$. Take generators of the kernel of $\pi_n(L_n) \to \pi_n(L)$, and form a pushout

$$\begin{tikzcd} \coprod \Delta^n/\partial\Delta^n \arrow[r] \arrow[d] & L_n \arrow[d] \\ \coprod \Delta^{n+1}/\Lambda^{n+1}_0 \arrow[r] & P_n. \end{tikzcd}$$

We let P_n' be a Kan replacement of P_n, which we can form by Lemma 3.9.5 by attaching only simplices of dimension $n+1$ and higher, since all $\leq (n+1)$-horns already possess fillers. In particular, this can also not introduce new elements in π_n, making the first of the maps

$$\pi_n(L_n) \to \pi_n(P_n') \to \pi_n(L)$$

surjective. But since the kernel of the first map by construction contains the kernel of the composite, we see that the second map is a bijection. We now form the simplicial set

$\bigvee \Delta^{n+1}/\partial\Delta^{n+1} \vee P'_n$, with one summand for each generator of $\pi_{n+1}(L)$, choose the map $\bigvee \Delta^{n+1}/\partial\Delta^{n+1} \vee P'_n \to L$ accordingly, and Kan-replace, again using Lemma 3.9.5, denoting the replacement by L_{n+1}. We have obtained a new Kan complex, still with the same n-skeleton as K, but such that the map $L_{n+1} \to L$ is bijective on homotopy in degrees $< n+1$ and surjective on π_{n+1}. This we can iterate. In the end, we let L' be the colimit of all L_k, and by Whitehead the map $L' \to L$ is a homotopy equivalence. □

3.12 Hurewicz

Recall that for a space X, we have a natural transformation

$$\pi_n(X) \to H_n(X).$$

Typically, this is far from an isomorphism (for example, $\pi_2(S^1 \times S^1) = 0$ but $H_2(S^1 \times S^1) = \mathbb{Z}$), but in Topology 1 we saw that for $n = 1$ and X connected, it is an isomorphism after abelianisation:

$$\pi_1(X)^{\mathrm{ab}} \xrightarrow{\cong} H_1(X).$$

In this section, we want to discuss a powerful generalisation of this fact:

Theorem 3.12.1 (Hurewicz theorem)**.** *Let $n \geq 2$ and let X be an $(n-1)$-connected CW-complex, i.e. $\pi_i(X) = 0$ for $i < n$. Then the natural map*

$$\pi_n(X) \to H_n(X)$$

is an isomorphism.

Remark 3.12.2. One way to remember this result is remembering that the lowest non-vanishing homotopy group agrees with homology after mofifying the homotopy group so that this statement parses:

1. $H_0(X)$ is the free abelian group on the set $\pi_0(X)$. This is the way to make a set into an abelian group.

2. If $\pi_0(X) = 0$ then $H_1(X)$ is the abelianization of $\pi_1(X)$. This is the way to make a group into an abelian group.

3. If $\pi_0(X) = \pi_1(X) = 0$ then $H_2(X) = \pi_2(X)$ etc.

Just like cellular approximation, we will prove this statement in the simplicial world. For that, let us set up the Hurewicz homomorphism.

Given a pointed Kan complex K and an element $f \in \pi_n(K)$ represented by a pointed map $\Delta^n/\partial\Delta^n \to K$, we can assign to f the image of the preferred generator of $\widetilde{H}_n(\Delta^n/\partial\Delta^n)$ (given by the identity simplex $\iota : \Delta^n \to \Delta^n/\partial\Delta^n$) under the induced map

$$f_* : \widetilde{H}_n(\Delta^n/\partial\Delta^n) \to \widetilde{H}_n(K).$$

This defines a map $\pi_n(K) \to \widetilde{H}_n(K)$, well-defined since homology is homotopy invariant. We can postcompose with the iso $\widetilde{H}_n(K) \to H_n(K)$, since $n > 0$, to land in absolute homology, but for some arguments reduced homology will be easier to work with.

To see that this is additive, suppose we are given $f, g : \Delta^n/\partial\Delta^n \to K$. Then there is a map $s : \Delta^{n+1} \to K$, defining the sum $(f+g)$, with $\partial_0 s = g \circ \iota$, $\partial_1 s = (f+g) \circ \iota$, $\partial_2 s = f \circ \iota$, and all other faces constant. So, in the reduced chain complex $C_*(K, k_0)$, we have

$$\partial[s] = [g \circ \iota] - [(f+g) \circ \iota] + [f \circ \iota],$$

so in homology, $(f+g)_*[\iota] = f_*[\iota] + g_*[\iota]$, and thus the Hurewicz map is a homomorphism.

We will prove the following

Theorem 3.12.3 (Simplicial Hurewicz). *Let $n \geq 2$. If K is a pointed Kan complex with $\pi_i(K) = 0$ for $i < n$. Then the Hurewicz homomorphism*

$$\pi_n(K) \to H_n(K)$$

is an isomorphism.

Note that the simplicial Hurewicz theorem implies the topological Hurewicz theorem 3.12.1, even without reference to Theorem 3.9.1, since for a CW complex X, the simplicial homotopy and homology of the Kan complex $\mathrm{Sing}(X)$ agree with the homotopy and homology of X by the adjunction.

The key ingredient in the proof will be the following lemma:

Lemma 3.12.4. *Let $n \geq 1$, K be a pointed Kan complex, and $f : \Delta^{n+1}/\mathrm{sk}^{n-1}\Delta^{n+1} \to K$ a pointed map. Then the restrictions $f|_{\partial_i \Delta^{n+1}}$ represent elements in $\pi_n(K)$, and*

$$\sum_{i=0}^{n+1} (-1)^i [f|_{\partial_i \Delta^{n+1}}] = 0$$

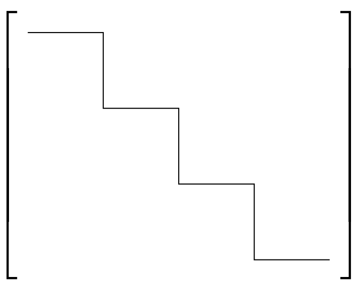
in $\pi_n(K)$ (abelianized if $n = 1$).

Proof. Draw picture for $n = 1$! We introduce some terminology. A list of maps $f_i : \Delta^n/\partial\Delta^n \to K$ for $0 \leq i \leq n+1$ defines a map

$$f : \partial\Delta^{n+1}/\mathrm{sk}^{n-1}\Delta^{n+1} \to K,$$

with $f_i = f|_{\partial_i \Delta^{n+1}}$, and we denote f by the tuple $(f_0, \dots, f_{n+1})$. We call $(f_0, \dots, f_{n+1})$ "fillable" if the associated map extends to all of $\Delta^{n+1}/\mathrm{sk}^{n-1}\Delta^{n+1}$. So the claim is that $(f_0, \dots, f_{n+1})$ fillable implies that $\sum_i (-1)^i [f_i] = 0$. We will actually prove that equivalence holds.

The plan is to use fillings for $(n+2)$-horns to show that certain tuples are fillable. For a family $f_{ij} : \Delta^n/\partial\Delta^n \to K$, indexed over $0 \leq i \leq n+1$ and $0 \leq j \leq n+2$, we say (f_{ij}) is "compatible" if, for every $i < j$, we have $f_{i,j} = f_{j-1,i}$. If we arrange the f_{ij} as a matrix, this is some kind of shifted symmetry along the broken diagonal drawn below.



These symmetries are coming from the simplicial identities $\partial_i \partial_j = \partial_{j-1}\partial_i$ for $i < j$. In fact, we immediately see: A compatible family (f_{ij}) defines a map

$$F : \mathrm{sk}^n \Delta^{n+2}/\mathrm{sk}^{n-1}\Delta^{n+2} \to K,$$

with $F|_{\mathrm{sk}^n(\partial_j \Delta^{n+2})/\mathrm{sk}^{n-1}}$ given by the map $\partial\partial_j \Delta^{n+2}/\mathrm{sk}^{n-1} \to K$ which corresponds to the tuple $(f_{ij})_{0\leq i \leq n+1}$.

Now say we have a compatible such family, and all except for one column define fillable tuples. Then the remaining column is also fillable. This is easy to see, because the fillings for the other tuples define a map

$$\Lambda^{n+2}_k/\mathrm{sk}^{n-1} \to K,$$

k being the index of the remaining column. Equivalently, this is a map $\Lambda^{n+2}_k \to K$, which is constant on the $n-1$-skeleton. K being a Kan complex, we can extend this to a map $\Delta^{n+2} \to K$, still necessarily constant on the $n-1$-skeleton, and thus a map $\Delta^{n+2}/\mathrm{sk}^{n-1}$. If we restrict to $\partial_k \Delta^{n+2}/\mathrm{sk}^{n-1}$, we found a filling for the k-th column.

Now we proceed as follows:

(1) Tuples $(0, 0, \ldots, 0, f, f, 0, \ldots, 0)$ are fillable. This is clear, since an explicit filling is given by $\sigma_k f$, k the correct index.

(2) Tuples $(f, g, h, 0 \ldots, 0)$ are fillable if and only if $[g] = [h] + [f]$, since this is the definition of the group structure. Note how this already proves the lemma in the case $n = 1$.

(3) Tuples $(f, f, 0, \ldots, 0, g, g, 0, \ldots, 0)$ are fillable. To see this, consider the compatible family

$$\begin{bmatrix} f & f & f & \cdots & & & & \\ f & f & f & \cdots & & & & \\ \vdots & \vdots & & & & g & g & \\ & & & & & g & g & \\ & & g & g & & & & \\ & & g & g & & & & \\ & & & & & & & \end{bmatrix}$$

Here all of the columns except for the third are fillable because of (1), so the third is fillable too. (The case for (f, f, g, g) is a bit degenerate, there we have only 6 g's, but the argument still goes through.)

(4) A tuple $(f_0, \ldots, f_{k-1}, f_k, f_{k+1}, 0, \ldots, 0)$ is fillable if and only if $(f_0, \ldots, f_{k-1}, f_k - f_{k+1}, 0, 0, \ldots, 0)$

is. To see this, consider the compatible family

$$\begin{bmatrix}
f_0 & f_0 & f_1 & f_2 & \cdots & f_{k-1} & f_k & f_{k+1} & 0 & \cdots & 0 \\
f_1 & f_1 & f_1 & f_2 & \cdots & f_{k-1} & (f_k - f_{k+1}) & 0 & 0 & \cdots & 0 \\
f_2 & f_2 & 0 & 0 & \cdots & 0 & -f_{k+1} & -f_{k+1} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & & & & & \vdots \\
f_{k-1} & f_{k-1} & 0 & 0 & \cdots & & & & & & \\
f_k & (f_k - f_{k+1}) & -f_{k+1} & 0 & \cdots & & & & & & \\
f_{k+1} & 0 & -f_{k+1} & 0 & \cdots & & & & & & \\
0 & 0 & 0 & 0 & \cdots & & & & & & \\
\vdots & \vdots & \vdots & \vdots & & & & & & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & & & & & \cdots & 0
\end{bmatrix}$$

Here all columns except for the first two are fillable because of (1), (2) and (3). So the first one is fillable if and only if the second one is fillable. (This argument goes through for $k \geq 2$), for smaller k the claim is a special case of (2).)

(5) The lemma now follows by inductively applying (4) to reduce the fillability of a family $(f_0, \ldots, f_n)$ to the fillability of the family $(\sum_i (-1)^i f_i, 0, \ldots, 0)$.

□

Proof of the simplicial Hurewicz theorem. By applying Lemma 3.11.3 to the map $\Delta^0 \to K$, we can replace K by a homotopy equivalent K' with $(n-1)$-skeleton a point. So we assume K to be of that form. In that case, we can now directly construct an inverse to the Hurewicz homomorphism. Given an n-simplex $\Delta^n \to K$, it factors uniquely through $\Delta^n/\partial\Delta^n \to K$, which defines an element of $\pi_n(K)$. Extending linearly, we obtain a map

$$C_n(K, \Delta^0) \to \pi_n(K),$$

which by Lemma 3.12.4 factors through $\widetilde{H}_n(K)$. One now checks directly that both composites

$$\pi_n(K) \to \widetilde{H}_n(K) \to \pi_n(K),$$
$$\widetilde{H}_n(K) \to \pi_n(K) \to \widetilde{H}_n(K)$$

are the identity. For the first one, this is immediate, for the second one, we check it on generators (single simplices). □

This concludes the proof of the simplicial and topological Hurewicz theorems. As a first application, we see:

Corollary 3.12.5. *$\pi_n(S^n) \cong \mathbb{Z}$, given by the mapping degree.*

Corollary 3.12.6. *A simply-connected CW-complex with trivial homology is contractible.*

Remark 3.12.7. There are examples of spaces with trivial homology and perfect, nontrivial π_1.

There is also a relative version, which we now want to sketch. For a CW pair (X, A), there is a relative Hurewicz homomorphism

$$\pi_n(X, A) \to H_n(X, A),$$

and we have

Theorem 3.12.8 (Relative Hurewicz theorem)**.** *Let $n \geq 2$, (X, A) be a CW pair, and $\pi_i(X, A) = 0$ for $i < n$. In addition, assume X and A to be connected and $\pi_1(A) = 0$. Then*

$$\pi_n(X, A) \to H_n(X, A)$$

is an isomorphism.

To phrase this simplicially, we need to introduce relative homotopy groups of a pair of Kan complexes.

Definition 3.12.9. For a pair (K, A) of pointed Kan complexes, with basepoint k_0 and $n \geq 1$, we let $\pi_n(K, A; k_0)$ denote the set of homotopy classes of maps of triples

$$(\Delta^n, \partial\Delta^n, \Lambda^n_n) \to (K, A, k_0).$$

This again gets a group structure for $n \geq 2$, abelian for $n \geq 3$. The sum $[f] + [g]$ is defined by extending along the anodyne inclusion

$$\bigcup_{i \neq 1, n+1} \partial_i \Delta^{n+1} \to \Delta^{n+1}$$

the map which is given by $g : \Delta^n \to K$ on $\partial_0\Delta^{n+1}$, f on $\partial_2\Delta^{n+1}$, constant maps on all other faces. Note the last one is not included in the list since this is where the composition on the 'boundary' happens. The relevance of relative homotopy groups for us is that there is a long exact sequence:

Lemma 3.12.10. *For a Kan complex pair (K, A), we have a long exact sequence*

$$\ldots \to \pi_n(A) \to \pi_n(K) \to \pi_n(K, A) \to \pi_{n-1}(A) \to \ldots$$

where the first two maps are the obvious induced ones, and the connecting homomorphism restricts a map $(\Delta^n, \partial\Delta^n, \Lambda^n_n) \to (K, A, k_0)$ to $\partial_n\Delta^n$.

We also have a relative simplicial Hurewicz homomorphism

$$\pi_n(K, A) \to H_n(K, A)$$

given by sending a representative $\Delta^n \to K$ of a relative homotopy class (in particular, the boundary $\partial\Delta^n$ lands in A!) to the homology class of the corresponding relative cycle in $C_n(X, A) = C_n(X)/C_n(A)$.

Theorem 3.12.11. *Let $n \geq 2$ and (K, A) be a pair of Kan complexes, and assume $\pi_i(K, A) = 0$ for $i < n$, A simply connected and K connected. Then*

$$\pi_n(K, A) \to H_n(K, A)$$

is an isomorphism.

Proof sketch. Using Lemma 3.11.3, we can replace the pair by one where A and K have the same $(n-1)$-skeleton. Then the crucial ingredient is that, as sets,

$$\pi_n(X, A) \cong [(\Delta^n, \partial\Delta^n), (K, A)].$$

This uses the simply-connectedness of A in addition to homotopy extension. (In general, there is a $\pi_1(A)$-action on $\pi_n(X, A)$, which we forget on the right.) We will prove this on the exercise sheet.

Now the map

$$[(\Delta^n, \partial\Delta^n), (K, A)] \to H_n(X, A)$$

is obviously surjective, since every n-simplex has its boundary in A. For injectivity, one needs a statement similar to Lemma 3.12.4 which we will omit here. □

The topological relative Hurewicz (Theorem 3.12.8) follows immediately, by application to the Kan complex pair $(\operatorname{Sing} X, \operatorname{Sing} A)$ for a CW pair (X, A).

Theorem 3.12.12 ("Whitehead for homology")**.** *If a map $X \to Y$ of simply connected Kan complexes or CW complexes is an equivalence on all homology groups, it is a homotopy equivalence.*

Proof. We reduce to the case of an inclusion of Kan complexes or a CW pair as in the proof of Whitehead (simplicial resp. topological). Now by assumption and the long exact sequence in homology, all relative homology groups vanish, so by relative Hurewicz, all relative homotopy groups vanish. So, by the long exact sequence, all maps on homotopy are isomorphisms, and thus the map $X \to Y$ is a homotopy equivalence by the simplicial or topological Whitehead theorem. □

One might think that for non-simply connected X, Y, it suffices to demand in addition that the map $X \to Y$ is also an isomorphism on π_1. This is not true (maybe we will discuss an example in the exercises), and the best one can do is:

Proposition 3.12.13. *If a map of connected CW-complexes $X \to Y$ is an isomorphism on π_1, and the map on universal covers $\widetilde{X} \to \widetilde{Y}$ is a homology equivalence, then $X \to Y$ is a homotopy equivalence.*

Proof. Consider the diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & \widetilde{Y} \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

By assumption, the upper horizontal map is a homology equivalence of simply-connected CW complexes. By Theorem, 3.12.12, this implies that it is an isomorphism on all homotopy groups. Since the vertical maps are isomorphisms on π_n for $n \geq 2$, the bottom map is an isomorphism on those π_n. By assumption, it is also an isomorphism on π_1, so by Whitehead, it is an equivalence. □

3.13 Proof of Theorem 3.9.1

The purpose of this section is to give a proof of Theorem 3.9.1 (of course without getting circular). We will first need an analogous statement to 3.12.13 for Kan complexes.

Definition 3.13.1. For a connected pointed Kan complex K, we define the *universal covering* $\widetilde{K}$ as follows:

$$\widetilde{K}_n = \{\text{Maps } \Delta^n \times \Delta^1 \to K \text{ which send } \Delta^n \times \{0\} \text{ to the basepoint } k_0\}/\sim,$$

where $\sim$ denotes homotopy rel $\Delta^n \times \partial\Delta^1$. There is an evident map $\widetilde{K} \to K$ which sends a map $\Delta^n \times \Delta^1 \to K$ to its restriction on $\Delta^n \times \{1\}$.

Informally, an n-simplex in $\widetilde{K}$ is given by an n-simplex in K together with a path connecting it to the basepoint of K. We refer to the preimage of a point $\Delta^0 \to K$ as the fiber of $\widetilde{K}$ over that point. If one is willing to use Hom simplicial sets, then this is a quotient of $\underline{\mathrm{Hom}}(\Delta^1, K)$.

Explicitly, the 0-simplices of the fiber over a point $p : \Delta^0 \to K$ are given by the set of homotopy classes rel $\partial\Delta^1$ of maps $\Delta^1 \to K$ which send $\{1\}$ to p and $\{0\}$ to the basepoint k_0. Similarly to the group structure on $\pi_1(K)$, we see that we have a free transitive action of $\pi_1(K)$ on this set.

Lemma 3.13.2. *$\widetilde{K}$ comes with a $\pi_1(K)$-action, free on $\widetilde{K}_n$ for all n, and K is exhibited as the quotient of that action by the map $\widetilde{K} \to K$.*

Proof. The homotopy classes of maps of pairs $(\Delta^n \times \Delta^1, \Delta^n \times \partial\Delta^1) \to (K, k_0)$ form a group via extension along the anodyne map $\Delta^n \times \Lambda^2_1 \to \Delta^n \times \Delta^2$. In fact, this coincides with $\pi_1(K)$, since the map

$$\Delta^0 \times \Delta^1 \cup \Delta^n \times \partial\Delta^1 \to \Delta^n \times \Delta^1$$

is anodyne.

By "precomposing", we get an action of this group on $\widetilde{K}_n$, which preserves the map $\widetilde{K}_n \to K_n$. In fact, if we consider the preimage of a given n-simplex $\Delta^n \to K$, it acts freely and transitively: Elements of the fiber are maps $\Delta^n \times \Delta^1 \to K$ which are constant on the left face and the given n-simplex on the right face, and given two different ones, we can glue them to a map $\Delta^n \times \Lambda^2_2 \to K$, which we can extend to $\Delta^n \times \Delta^2$.

Finally, the actions given here are compatible with all the simplicial structure maps (which induce isomorphisms on the group acting...) $\square$

We now want to prove that $\widetilde{K} \to K$ has nice lifting properties, similar to the classical situation of coverings. For that, let us consider the class $\mathcal{M}$ of all monomorphisms $A \hookrightarrow X$ such that in every square of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \widetilde{K} \arrow[d] \\ X \arrow[r] \arrow[ur, dashed] & K \end{tikzcd}$$

we have a *unique* dashed arrow making it commute. It is easy to see that this class is closed under retracts, pushouts and transfinite composition.

Lemma 3.13.3. *Any map $\Delta^0 \to \Delta^n$ is in the class $\mathcal{M}$.*

Proof. A point $\Delta^0 \to \widetilde{K}$ determines a homotopy class of a path, compatible with the n-simplex $\Delta^n \to K$, i.e. a map $\Delta^1 \amalg_{\{1\}} \Delta^n \to K$, unique up to homotopy rel $\{0\} \cup \Delta^n$. Now the pair

$$\Delta^n \amalg_{\{0\}} \Delta^1 \amalg_{\{1\}} \Delta^n \to \Delta^n \times \Delta^1$$

is anodyne, so the given map $\Delta^n \amalg_{\{0\}} \Delta^1 \amalg_{\{1\}} \Delta^n \to K$ (constant on the first n-simplex) extends uniquely up to homotopy rel $\Delta^n \times \partial\Delta^1$ to $\Delta^n \times \Delta^1$, i.e. determines a unique compatible n-simplex $\Delta^n \to \widetilde{K}$. □

Lemma 3.13.4. *Any horn inclusion $\Lambda^n_k \to \Delta^n$ is in $\mathcal{M}$. In particular, $\widetilde{K}$ is a Kan complex. It also follows that all anodyne maps are in $\mathcal{M}$.*

Proof. The map $\Lambda^n_k \to \widetilde{K}$ is the unique lift of the composite $\Lambda^n_k \to K$ compatible with the map $\{k\} \to \widetilde{K}$ (applying Lemma 3.13.3 to all the $n-1$-simplices of Λ^n_k, which all contain $\{k\}$). Now the map $\Delta^n \to K$ determines a unique lift compatible with the point $\{k\} \to \widetilde{K}$ and thus the desired horn-filler upstairs.

This shows in particular that we can fill any horn $\Lambda^n_k \to \widetilde{K}$. □

But $\mathcal{M}$ is actually *larger* than the class of anodyne maps:

Lemma 3.13.5. *The inclusions $\partial\Delta^n \to \Delta^n$ for $n \geq 2$ are in $\mathcal{M}$. In particular, any inclusion $A \to X$ which is an isomorphism on 1-skeleta is in $\mathcal{M}$.*

Proof. Given a map $\partial\Delta^n \to \widetilde{K}$ and a filler $\Delta^n \to K$ downstairs, we can obtain a lift $\Delta^n \to \widetilde{K}$ compatible with any restriction to a horn Λ^n_k. Since $\partial_i\Delta^n \to \Delta^n$ is in $\mathcal{M}$, all these lifts agree, since any two horns intersect in a face (here, $n \geq 2$ enters). So the lift is actually compatible with all of $\partial\Delta^n \to \widetilde{K}$.

For the more general claim, we just observe that all $A \cup X^{k-1} \to A \cup X^k$ for $k \geq 2$ are in $\mathcal{M}$, since they are iterated pushouts against $\partial\Delta^k \to \Delta^k$. Thus their transfinite composite $A \to X$ is in $\mathcal{M}$. □

Corollary 3.13.6. *The map $\pi_n(\widetilde{K}) \to \pi_n(K)$ is an isomorphism for $n \geq 2$.*

Proof. For surjectivity, we use that $\partial\Delta^n \to \Delta^n$ is in $\mathcal{M}$, with the following diagram

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "\text{const}"] \arrow[d] & \widetilde{K} \arrow[d] \\
\Delta^n \arrow[r, "\alpha"] & K
\end{tikzcd}$$

For injectivity, we use that $\partial(\Delta^n \times \Delta^1) \to \Delta^n \times \Delta^1$ is in $\mathcal{M}$ (Lemma 3.13.5). □

Proposition 3.13.7. *$\widetilde{K}$ is simply connected.*

Proof. En element in $\pi_1(\widetilde{K})$ is represented by a map $\Delta^1 \times \Delta^1 \to K$ which sends $\Delta^1 \times \{0\}$ to the basepoint as well as $\partial\Delta^1 \times \Delta^1$ to paths that are homotopic to constant path. Using homotopy extension we may assume that it is actually also send to a constant path. But then this map is homotopic rel $\Delta^1 \times \{0\} \cup \partial\Delta^1 \times \Delta^1$ to the constant map, since the inclusion

$$\Delta^1 \times \{0\} \cup \partial\Delta^1 \times \Delta^1 \to \Delta^1 \times \Delta^1$$

is anodyne. □

Lemma 3.13.8. *For a pointed connected CW complex X with universal covering $\widetilde{X}$, $\operatorname{Sing}\widetilde{X} \to \operatorname{Sing} X$ is the universal covering of $\operatorname{Sing} X$. For a pointed connected Kan complex K, $|\widetilde{K}| \to |K|$ is a universal covering of CW complexes. Additionally, the natural map $K \to \operatorname{Sing}|K|$ is an isomorphism on π_1.*

Proof. For the first statement, observe that a map $|\Delta^n \times \Delta^1| \to X$ which sends $|\Delta^n \times \{0\}|$ to the basepoint of X lifts uniquely to $\widetilde{X}$, thus determines a map $|\Delta^n| \to \widetilde{X}$ and thus an n-simplex. This only depends on the homotopy class of $|\Delta^n \times \Delta^1| \to X$ rel $|\Delta^n \times \partial\Delta^1|$. The other way around, an n-simplex $|\Delta^n \times \{1\}| \to \widetilde{X}$ extends to a map $|\Delta^n \times \Delta^1| \to \widetilde{X}$ which sends $|\Delta^n \times \{0\}|$ constantly to the basepoint. This map is determined uniquely up to homotopy rel $|\Delta^n \times \partial\Delta^1|$ since $\widetilde{X}$ is simply-connected, and postcomposition gives an unique such homotopy class of maps $|\Delta^n \times \Delta^1| \to X$. We have thus established a bijection between the n-simplices of $\operatorname{Sing}\widetilde{X}$ and the n-simplices of the universal covering of $\operatorname{Sing} X$, compatible with the simplicial structure maps and the map to $\operatorname{Sing} X$.

For the second statement, observe that the action of $\pi_1(K)$ on $\widetilde{K}$ gives a properly discontinuous free action on $|\widetilde{K}|$, so that the quotient map $|\widetilde{K}| \to |\widetilde{K}|/\pi_1(K) \cong |K|$ is a covering map. We can also check that $|\widetilde{K}|$ is simply-connected: Since $\widetilde{K}$ is simply-connected, Lemma 3.11.3 shows that $\widetilde{K}$ is homotopy equivalent to a Kan complex with trivial 1-skeleton, so its realisation is homotopy equivalent to a CW complex with trivial 1-skeleton, thus simply-connected. This shows that $|\widetilde{K}| \to |K|$ is the universal covering, and furthermore that $\pi_1(K) = \pi_1|K| = \pi_1 \operatorname{Sing}|K|$. □

We can now use this to prove Theorem 3.9.1.

Proof of Theorem 3.9.1. Firstly, for any simplicial set K, the map $K \to \operatorname{Sing}|K|$ is a homology equivalence. We have a map of long exact sequences (using the adjunction to write $H_k(\operatorname{Sing} X) = H_k(X)$)

$$\begin{tikzcd}
\dots \arrow[r] & H_k(K^{n-1}) \arrow[r] \arrow[d] & H_k(K^n) \arrow[r] \arrow[d] & H_k(K^n, K^{n-1}) \arrow[r] \arrow[d] & \dots \\
\dots \arrow[r] & H_k(|K^{n-1}|) \arrow[r] & H_k(|K^n|) \arrow[r] & H_k(|K^n|, |K^{n-1}|) \arrow[r] & \dots
\end{tikzcd}$$

and thus by the five-lemma and induction we will be done once we see that the vertical maps between relative terms are isomorphisms. Since we have a pushout

$$\begin{tikzcd}
\coprod \partial\Delta^n \arrow[r] \arrow[d] & K^{n-1} \arrow[d] \\
\coprod \Delta^n \arrow[r] & K^n,
\end{tikzcd}$$

and same for CW complexes after realisation, by excision and additivity we are left to show that the map

$$H_k(\Delta^n, \partial\Delta^n) \to H_k(|\Delta^n|, |\partial\Delta^n|)$$

is an isomorphism. We know both sides are generated by a single degree n generator, which is represented by the tautological n-simplex on the left, it is easy to see that that goes to a generator under realisation. In fact, we note that we actually see that the map

$$H_k(K) \to H_k(\operatorname{Sing}|K|) = H_k(|K|)$$

is the inclusion cellular homology with respect to the CW structure on the realisation.

In the natural diagram

$$\begin{tikzcd} \widetilde{K} \arrow[r] \arrow[d] & \operatorname{Sing}|\widetilde{K}| \arrow[d] \\ K \arrow[r] & \operatorname{Sing}|K| \end{tikzcd}$$

we know by Lemma 3.13.8 that the right vertical map is a universal covering of $\operatorname{Sing}|K|$, and the bottom map is a π_1-isomorphism.

Now the top map is a homology equivalence between simply-connected Kan complexes, so by Theorem 3.12.12, it is an equivalence, thus an isomorphism on π_n for $n \geq 2$. By Corollary 3.13.6, the vertical covering maps are also isomorphisms on π_n for $n \geq 2$, and thus the lower horizontal map is an isomorphism on all homotopy groups, thus by the simplicial Whitehead theorem it is an equivalence. □

3.14 Eilenberg-MacLane spaces and Brown representability

Definition 3.14.1. A pointed, connected space/Kan complex (X, x) is called Eilenberg-MacLane space if we have $\pi_i(X, x) = 0$ for all $i \neq n$. An Eilenberg-MacLane space $K(G, n)$ for G a group which is abelian if $n \geq 2$ is an Eilenberg-MacLane space X together with an isomorphism $\pi_n(X, x) \cong G$. We will write $K(G, n)$ for X in this case.

Example 3.14.2. The space $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}/2, 1)$. Every connected, aspherical space X is a $K(\pi_1(X), 1)$. For example all Σ_g for $g \geq 1$ are Eilenberg-MacLane spaces. The space $\mathbb{C}P^\infty$ is a $K(\mathbb{Z}, 2)$ (we have not yet shown this).

Proposition 3.14.3. *For every n and G (which is abelian for $n \leq 2$) there exists an Eilenberg-Mac Lane space $K(G, n)$. Any two $K(G, n)$'s are weakly homotopy equivalent.*

Proof. We will work in the category of CW complexes. First we want to show the existence of a $K(G, n)$ for each n and each G. We will construct such a space inductively. As a first step we choose generators $(g_i)_{i \in I}$ for G as a group. Then we form the space

$$X_0 := \bigvee_I S^n \ .$$

The space X_0 has vanishing homotopy in degrees $< n$ and the elements g_i induce a map

$$\pi_n(X_0) \to G.$$

Let $K_0 \subseteq \pi_n(X_0)$ be the kernel of this map and choose generators $(k_j)_{j \in J}$ for K_0. Then we form the CW X_1 obtained from X_0 by attaching an $(n+1)$-cell along each of the generators

k_j, i.e. the pushout

$$\begin{tikzcd} \bigvee_J S^n \arrow[r] \arrow[d] & X_0 \arrow[d] \\ \bigvee_J D^{n+1} \arrow[r] & X_1 \end{tikzcd}$$

The space X_1 is n-connective and we claim that $\pi_n(X_1) \cong G$. For $n = 1$ this is a consequence of Seifert-van-Kampen (as we have already seen in Topology 1 and the exercises). For $n > 2$ we use that by Hurewicz it suffices to show that $H_n(X_1) = G$ which is immediate from the Mayer-Vietoris sequence.

Now the space X_1 has potentially higher homotopy groups which we want to 'kill' by attaching higher cells. Thus pick generators of $\pi_{n+1}(X_1)$ and attach $(n+2)$-cells to X_1, i.e. form the CW complex X_2 obtained as the pushout

$$\begin{tikzcd} \bigvee S^{n+1} \arrow[r] \arrow[d] & X_1 \arrow[d] \\ \bigvee D^{n+2} \arrow[r] & X_2 \end{tikzcd} .$$

Then the map $\pi_k(X_1) \to \pi_k(X_2)$ is an isomorphism in degrees $\leq n$ by cellular approximation and by construction the map $\pi_{n+1}(X_1) \to \pi_{n+1}(X_2)$ sends the generators to 0. Thus this map is zero. But by cellular approximation every element in $\pi_{n+1}(X_2)$ lies in the image of this map. We conclude that $\pi_{n+1}(X_2) = 0$. Now we repeat the last step inductively to get a sequence of CW inclusions

$$X_1 \to X_2 \to X_3 \to ...$$

such that for all these π_n is G and the more and more higher homotopy groups vanish. The colimit X is then a CW complex which is a $K(G, n)$.

This shows the existence part. For the uniqueness part let Y be another $K(G, n)$. We want to construct a map $X \to Y$ that is a weak equivalence. The generators g_i of G give us a map $f_0 : X_0 = \bigvee_I S^n \to Y$ that is an isomorphism on π_k for $k < n$ and surjective for $k = n$. More precisely this maps sends in degree n precisely the elements named k_j above to zero. If follows that we can extend f_0 to a map $X_1 \to Y$ by choosing nullhomotopies of the corresponding maps $S^n \to Y$. The resulting map $X_1 \to Y$ is thus an isomorphism in degree n. We now proceed inductively to factor this map to a map $X_2 \to Y$ which is possible since all elements in $\pi_{n+1}X_1$ are clearly mapped to 0 in $\pi_{n+1}Y$. Thus we iteratively get maps $f_n : X_n \to Y$ that all induce an isomorphism in more and more degrees. Thus the colimit of these maps gives rise to a map $f : X \to Y$ that is an isomorphism on all homotopy groups, i.e. a weak equivalence.

For two abstract $K(G, n)$'s Y and Z we simply construct a zig-zag of weak equivalences by choosing weak equivalences $X \to Y$ and $X \to Z$ for our 'standard model' X. $\square$

We will use the uniqueness statement in the last Theorem to use the generic notation $K(G, n)$ for Eilenberg Mac Lane spaces. We can (and will) always assume these are CW complexes, respectively Kan complexes when working simplicially.

Proposition 3.14.4. *We have the following facts about low-degree homology and cohomology groups of Eilenberg-MacLane spaces:*

1. *We have* $H_i(K(G,n)) = 0$ *for* $0 < i < n$, $H_n(K(G,n)) = G$, *and if* $n \geq 2$, $H_{n+1}(K(G,n)) = 0$.

2. *We have* $H^i(K(G,n);G') = 0$ *for* $0 < i < n$, $H^n(K(G,n);G') = \mathrm{Hom}(G,G')$, *and if* $n \geq 2$, $H^{n+1}(K(G,n);G') = \mathrm{Ext}(G,G')$.

Proof. The construction of $K(G,n)$ in the proof of Proposition 3.14.3 uses n-cells corresponding to the generators and $n+1$-cells to kill the kernel of the corresponding map $\pi_n(\bigvee S^n) \to G$. Since the kernel is a subgroup of a free abelian group, it is free, and we can choose the $n+1$-cells to correspond to the basis of this kernel. The cellular complex of the resulting $K(G,n)$ then is given in degrees $n-1, n, n+1$ by:

$$\ldots \leftarrow 0 \leftarrow C_n(K(G,n)) \hookleftarrow C_{n+1}(K(G,n)) \leftarrow \ldots,$$

where the differential is injective (if $n \geq 2$) and has cokernel G, by construction. The claims about homology follow.

Using the universal coefficients theorem (Corollary 2.5.7) we conclude the statements about cohomology. □

In particular we get a canonical element $\iota_n \in H^n(K(G,n),G)$ corresponding to the identity map $G \to G$. Then for every map $X \to K(G,n)$ we get an induced element $f^*(\iota_n) \in H^n(X;G)$. This defines a map

$$[X, K(G,n)] \to H^n(X;G)$$

Theorem 3.14.5. *For every CW complex X and every abelian group G this map*

$$[X, K(G,n)] \to H^n(X;G)$$

is a bijection.

We will deduce this Theorem from an abstract representability result for cohomology theories. Note that for general spaces we can simply replace $[X, K(G,n)]$ by weak homotopy classes of maps (i.e. morphisms in the homotopy category obtained by inverting weak equivalences) and the statement remains true. Of course the simplicial version of the Theorem is true as well.

Corollary 3.14.6. *We have that*

$$[K(G,n), K(G',n)] = \mathrm{Hom}(G,G')$$
$$[K(G,n), K(G',n+1)] = \mathrm{Ext}(G,G').$$

In particular, the full subcategory of $\mathrm{Ho}(CW)$ *on all* $K(G,n)$ *for a fixed* $n \geq 2$ *is equivalent to* Ab*, implying that there is a functor* $\mathrm{Ab} \to \mathrm{Ho}(CW)$ *sending* $G \mapsto K(G,n)$.

Proof. This follows immediately from Theorem 3.14.5 and Proposition 3.14.4. □

Now we will formulate the abstract result from which we can deduce Theorem 3.14.5.

Theorem 3.14.7 (Brown representability I)**.** *Let (E^*, δ) be a generalized cohomology theory (as defined in 2.2.7). Then there exist pointed CW complexes E_n for every integer $n \in \mathbb{Z}$ together with natural isomorphism*

$$E^n(X,A) \cong [X/A, E_n]_*$$

for every CW pair (X, A).

Proof of Theorem 3.14.5 from Theorem 3.14.7. Using Brown representability we get for every n and every G we get a space E_n together with a natural isomorphism

$$H^n(X,A;G) = [X/A, E_n]_*$$

We claim that E_n is an Eilenberg-MacLane space $K(G, n)$. To see this it suffices to compute the homotopy groups. But we get that

$$\pi_k(E_n) = [S^k, E_n]_* = \tilde{H}^k(S^n, G) = \begin{cases} G & \text{for } k = n \\ 0 \text{ otherwise} \end{cases}$$

Thus we have shown that there is an abstract, natural isomorphism

$$H^n(X,A;G) = [X/A, K(G,n)]_*$$

for every CW pair (X, A). In particular for a space X we have an isomorphism

$$H^n(X;G) = H^n(X,\emptyset;G) = [X_+, K(G,n)]_* = [X, K(G,n)]$$

where we have used that $X/\emptyset = X_+$. This isomorphism is natural in X.

Now we are almost done, it only remains to show that the isomorphism $H^n(X;G) \cong [X, K(G,n)]$ has to be of the form claimed in Theorem 3.14.5, namely by pullback of the universal class α. To see this we observe that by Yoneda any natural transformation has to be given by pullback along some class in $\alpha' \in H^n(K(G,n),G) = \mathrm{Hom}(G,G)$. Since the natural transformation is an isomorphism we can conclude that it has to correspond to an isomorphism $G \to G$. Now we simply change the map $\pi_n(K(G,n)) \cong G$ by this automorphism.[6] □

Before we go on to prove Brown representability let us recall that for every cohomology theory E and every CW pair (X, A) we have a natural isomorphism

$$E^n(X,A) \cong \tilde{E}^n(X/A)$$

so that it suffices to produce a pointed space E_n such that for any other pointed space X we have

$$\tilde{E}^n(X) = [X, E_n]_*$$

In other words: we try to find an object in the homotopy category that represents the functor

$$\tilde{E}^n : \mathrm{Ho}(\mathrm{CW}_*)^{op} \to \mathrm{Set}.$$

[6] Here we really use that an Eilenberg MacLane space is given by a pair of a space and an isomorphism $\pi_n \cong G$ and not just an abstract space.

In general a functor $F : C^{op} \to \mathrm{Set}$ is called representable if there exists an object $c \in C$ and a natural isomorphism $F(x) \cong \mathrm{Hom}_C(x, c)$. By Yoneda the object c is by this requirement uniquely determined and there is a universal element $\alpha \in F(c)$ such that the map

$$\mathrm{Hom}_C(x, c) \to F(x)$$

is given by sending $\varphi : x \to c$ to the element in $F(x)$ obtained as the image of α under $F(\varphi) : F(c) \to F(x)$. Thus one should more precisely say that the pair $c \in C$ and $\alpha \in F(c)$ represents the functor F. Using this terminology we can state Brown representability as saying that for every cohomology theory the functors E^n are representable. The statement of Theorem 3.14.5 can then by summarized by saying that the pair $K(G, n)$ and α represents $H^n(-; G)$.

In order to prove Brown representability we would like to formulate general criteria for a functor $\mathrm{Ho}(\mathrm{CW}_*)^{op} \to \mathrm{Set}$ to be representable. For technical reasons we have to restrict to the full subcategory $\mathrm{Ho}(\mathrm{CW}_*^{\mathrm{con}})$ of pointed connected spaces and we will assume eveything is well-pointed (or work simplicially).

Theorem 3.14.8 (Brown representability II). *A functor $F : \mathrm{Ho}(\mathrm{CW}_*^{\mathrm{con}})^{op} \to \mathrm{Set}$ is representable precisely if it satisfies the following conditions*

1. *It sends wedge of connected space $\mathrm{Ho}(\mathrm{CW}_*^{\mathrm{con}})^{op}$ to products of sets, i.e. the canonical map*
$$F(\bigvee X_i) \to \prod F(X_i)$$
induced by the inclusions $X_i \to \bigvee X_i$ is an isomorphism.

2. *It sends pushouts along CW inclusions in CW to weak pullbacks in* Set*. That is for every pushout diagram*
$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$
where $A \subseteq X$ is a CW inclusion (and therefore also $B \subseteq Y$) the induced map
$$F(Y) \to F(B) \times_{F(A)} F(X)$$
is surjective.

Remark 3.14.9. The second assertion is also sometimes called the *Mayer-Vietoris property.* It can be stated as saying that for given elements $b \in F(B)$ and $x \in F(X)$ such that the images in $F(A)$ agree there is an element $y \in F(Y)$ that maps to x and b respectively.

One can also state the second assertion using an open cover by a topological space, i.e. consider the square
$$\begin{tikzcd} U_0 \cap U_1 \arrow[r] \arrow[d] & U_0 \arrow[d] \\ U_1 \arrow[r] & U_0 \cup U_1 \end{tikzcd}$$
More generally, the correct generality is to consider homotopy pushout squares. We will introduce this notion soon. Then one would say that F sends homotopy pushouts to weak pullbacks.

Proof of Theorem 3.14.7 from Theorem 3.14.8. Let E be a cohomology theory. As we have argued before, we just need to see that the functor

$$\tilde{E}^n : \mathrm{Ho}(\mathrm{CW}_*)^{op} \to \mathrm{Set}$$

is representable. We restrict this functor to the subcategory of pointed connected space, i.e. consider the restriction

$$\tilde{E}^n : \mathrm{Ho}(\mathrm{CW}_*^{\mathrm{con}})^{op} \to \mathrm{Set}$$

and we claim that it satisfies the above conditions: it clearly commutes with wedge products (thats why we had to take reduced cohomology so that it vanishes in the point) and for a given pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

we get a Mayer-Vietoris sequence

$$... \to E^n(Y) \to E^n(X) \oplus E^n(B) \to E^n(A) \to ...$$

which shows that $E^n(Y)$ surjects onto the pullback.

Thus applying Theorem 3.14.8 we get a pointed connected space E'_n representing this functor, i.e so that

$$\tilde{E}^n(X) = [X, E'_n]_*$$

for all pointed connected spaces X. It is however not true that this also holds for non-necessarily connected spaces X. We however consider the space

$$E_n := \Omega E'_{n+1}$$

and claim that this represents the functor $\tilde{E}^n$ on all pointed spaces X. To this see we note that we have the suspension isomorphism

$$\tilde{E}^n(X) = \tilde{E}^{n+1}(\Sigma X)$$

and furthermore, since ΣX is always connected we get

$$\tilde{E}^{n+1}(\Sigma X) = [\Sigma X, E'_{n+1}] = [X, \Omega E'_{n+1}]$$

Together this shows that $\tilde{E}^n$ is represented by E_n and thus the claim. □

Warning 3.14.10. Note that we had to argue a bit indirect in the proof of Theorem 3.14.7 since in Theorem 3.14.8 we only give a criterion for functors on connected spaces to be representable. It would be easier if we had a version of Theorem 3.14.8 for functors $\mathrm{Ho}(\mathrm{CW}_*)^{op} \to \mathrm{Set}$ or possibly even for functors $\mathrm{Ho}(\mathrm{CW})^{op} \to \mathrm{Set}$. Unfortunately the obvious guesses for such a Theorem are all false and there is no easy characterisation of such representable functors. We encourage everyone to think about this carefully!

Proof. We first shown that the conditions are necessary. Thus let us consider the functor F given by $F(X) = [X, E]_*$ for some pointed connected space E. Then we find that

$$[\bigvee_i X_i, E]_* = \prod_i [X_i, E]_*$$

Moreover for a given cellular pushout

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

we have to show that every pair of pointed maps $f : B \to E$ and $g : X \to E$ such that restrictions $A \to E$ are pointed homotopic, we can find a map $Y \to E$ such that the restrictions to B and X are homotopic to f and g, or on diagramatic language

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, "f"] & \\ X \arrow[r] \arrow[drr, "g"'] & Y \arrow[dr, dashed] & \\ & & E \end{tikzcd}$$

Here all the triangles are possibly filled by homotopies. We could obviously find such an extension if f and g where equal when restricted to A and not just homotopic and we want to reduce to this situation. The claim is clearly invariant under replacing g by a homotopic map g'. Thus we simply take the restriction $f|_A$ and the homotopy to $g|_A$ and use the homotopy extension property to be able to find such a function g' extending $f|_A$ to X and which is homotopic to g. This finishes one direction of the statement.

Now we want to prove that conditions (1) and (2) are also necessary for the representability of a functor. The idea is to construct a representing object $E \in \mathrm{CW}_*^{\mathrm{con}}$ similar to the construction of Eilenberg-MacLane spaces. To this end assume that we have F satisfiying these two conditions. We first note that since spheres S^n for $n \geq 1$ are cogroup objects in $\mathrm{CW}_*^{\mathrm{con}}$ with the pinch maps

$$S^n \to S^n \vee S^n \qquad\qquad S^n \to \mathrm{pt}$$

it follows that $F(S^n)$ is a group for every such functor (only using that it sends wedges to products) and that every natural transformation $F' \to F$ of such functors induces group homomorphisms $F'(S^n) \to F(S^n)$. We now will proceed in two steps:

- We first construct inductively a pointed, connected CW complex E together with a transformation
$$\eta : [X, E]_* \to F(X)$$
such that η is a group isomorphism for X given by a sphere S^n for all $n \geq 1$.

- Secondly we show that such a morphism η inducing isomorphisms on spheres is a natural isomorphism.

For the first step we start by seting

$$E_1 := \bigvee_{n \geq 1, x \in F(S^n)} S^n$$

We construct a natural map

$$\eta_1 : [X, E_1]_* \to F(X)$$

which is under the isomorphism of the Yoneda lemma induced by the element in

$$F(E_1) = F\left(\bigvee_{n\geq 1, x\in F(S^n)} S^n\right) = \prod_{n\geq 1, x\in F(S^n)} F(S^n)$$

given by the family $(x \in F(S^n))_{x\in E(S^n)}$. Now we claim that η_1 is surjective for each sphere: to see this we see that η_1 evaluated on a k-sphere S^k is given by the obvious map

$$\left[S^k, \bigvee_{n\geq 1, x\in F(S^n)} S^n\right]_* \to F(S^k)$$

sending a map $S^k \to \bigvee_{n\geq 1, x\in F(S^n)} S^n$ to the image of the vector $(x)_{n\geq 1, x\in F(S^n)}$ under the induced map

$$F\left(\bigvee_{n\geq 1, x\in F(S^n)} S^n\right) = \prod_{n\geq 1, x\in F(S^n)} F(S^n) \to F(S^k)$$

and so is surjective as $x \in F(S^k)$ is hit by the unit vector in the coorinate $n = k, x = x$.

In the next step we attach for each element y in the kernel $K_n = \ker([S^n, E_1]_* \to F(S^n))$ for varying n a cell to E_1. More precisely we form the pushout

$$\begin{tikzcd} \bigvee_{n\geq 1, K_n} S^n \arrow[r] \arrow[d] & E_1 \arrow[d] \\ \bigvee_{n\geq 1, K_n} D^n \arrow[r] & E_2 \end{tikzcd}$$

We use the property (2) of the theorem for the functor F to extend the map η_1 over E_2, to obtain a map

$$\eta_2 : [X, E_2]_* \to F(X).$$

This map has the property that it is still surjective for spheres S^k (as it extends η_1) and also that the elements in the kernel K_n of η_1 get mapped to zero under the map

$$[S^k, E_1]_* \to [S^k, E_2]_*$$

We proceed by induction to obtain a sequence of CW complexes

$$E_1 \to E_2 \to E_3 \to \ldots$$

and define E as the colimit. We claim that $F(E)$ surjects onto $\varprojlim F(E_i)$. To see this note that we can write E also as the pushout

$$\begin{tikzcd} \bigvee E_n \arrow[r] \arrow[d] & \bigvee E_{2n} \arrow[d] \\ \bigvee E_{2n+1} \arrow[r] & E \end{tikzcd}$$

and we can apply property 2. Using this observation we can get a map

$$\eta : [X, E]_* \to F(X)$$

extending the maps η_n. This map is still surjective on spheres S^k. We claim that is is also injective. To this end note that by compactness every element in the kernel has to factor trough some finite stage $E_k \subseteq E$ and thus is killed in the next step.

Now we have produced a map η with the desired property. We want to finish the proof by showing that the map $\eta : [X, E]_* \to F(X)$ is an isomorphism for each CW complex X and not merely for spheres. For this we will not use the specific form of E but merely that the transformation η is an isomorphism on spheres.

Assume that $f, g : X \to E$ map to the same element in $F(X)$ under η. Then form the double mapping cone $C_{f,g}$ defined as the pushout

$$\begin{tikzcd} X \vee X \arrow[r] \arrow[d] & E \arrow[d] \\ X \wedge \Delta^1_+ \arrow[r] & C_{f,g} \end{tikzcd}$$

We can extend the transformation η to a transformation $\eta' : [-, C_{f,g}] \to F(-)$ using property 2 and the fact that f and g are equalized in η. Now we proceed as in step 1 to find a map $C_{f,g} \to E'$ together with an extension $\eta'' : [-, E'] \to F(-)$ that is an isomorphism on spheres. It follows that $E \to C_{f,g} \to E'$ is an isomorphism on homotopy groups and thus by Whitehead a homotopy equivalence. Thus the maps $f, g : X \to E$ were already homotopic.

To see that η is surjective we argue similar: for a given element $z \in F(X)$ we get a transformation $\eta' : [-, E \vee X] \to F(-)$ extending η and with z in the image. Then we form $E \vee X \to E'$ and a transformation extending η' that is an isomophism on spheres so that we conclude that $E \to E'$ is a homotopy equivalence and that z was already in the image. □

Finally we want to see a bit cleaner what Brown Representability (or more specifically Theorem 3.14.7) gives us for a cohomology theory. We have already seem that for every integer n we get pointed spaces E_n together with natural isomorphisms

$$\tilde{E}^n(X) = [X, E_n]_*$$

Not the suspension isomorphism gives isomorphisms

$$[X, \Omega E_{n+1}]_* = [\Sigma X, E_{n+1}] = \tilde{E}^{n+1}(\Sigma X) = \tilde{E}^n(X) = [X, E_n]_*$$

By the Yoneda Lemma these are induced by a homotopy equivalence $E_n \simeq \Omega E_{n+1}$. Thus we also get a (homotopy class) of a homotopy equivalence as such from a cohomology theory. In particular we see that E_n for $n < 0$ can be recovered from E_0 by taking iterated loopspaces and is thus a ‘redundant’ datum. This motivated the following Definition

Definition 3.14.11. A spectrum is a sequence of pointed spaces E_n for every $n \in \mathbb{N}$ together with homotopy equivalences $E_n \to \Omega E_{n+1}$ for every $n \in \mathbb{N}$.

Brown representability implies that we can get a spectrum from every cohomology theory. Conversely if we start with a spectrum E we claim that we can built a cohomology theory as follows: for every pointed space we define

$$\tilde{E}^n(X) := [X, E_n]_*$$

where E_n for $n < 0$ is defined to be $\Omega^{-n} E_0$. A priori this is only a set, but it inherits an abelian group structure from the fact that $E_n = \Omega^2 E_{n+2}$ and composition of loops (for the group structure a single loop is sufficient, but for it to be abelian we need two). We also have the suspension isomorphism $\tilde{E}^{n+1}(X) \cong \tilde{E}^n(X)$ induced from the homotopy equivalence $E_n \simeq \Omega E_{n+1}$ which is clearly compatible with the loop structures, as the map $E_n \to \Omega E_{n+1}$ is itself the looping of the map $E_{n+1} \to \Omega E_{n+2}$ as the diagram

$$\begin{tikzcd} E_n \arrow[r] \arrow[d] & \Omega E_{n+1} \arrow[d] \\ \Omega E_{n+1} \arrow[r] & \Omega^2 E_{n+2} \end{tikzcd}$$

clearly commutes. It is not not hard to check that this satisfies all the axioms of a homology theory [7] It is clear that these two constructions are 'inverse' to each other so that spectra are the 'same' as cohomology theories. Lets make that precise.

Definition 3.14.12. Two spectra E and E' are called equivalent if there are homotopy equivalences $E_n \to E'_n$ that are compatible with the structure maps up to homotopy.

Theorem 3.14.13 (Brown representability 3)**.** *The assignment from spectra to comology theories induces a bijection between isomorphism classes of coomology theories and equivalence classes of spectra.*

Remark 3.14.14. We will later introduce a category of spectra in which the isomorphism classes of objects are precisely the equivalence classes of spectra. But it will not be the case that this category is equivalent to the category of cohomology theories. This has to do with the existence of so-called phantom maps.

Examples 3.14.15.

1. For every abelian group A we have the Eilenberg Mac Lane spectrum HA given by the sequence of spaces

$$(K(A,0), K(A,1), K(A,2), ...)$$

where $K(A,0)$ just means A with the discrete topology together with the homotopy equivalences $K(A,n) \simeq \Omega K(A,n+1)$ inducing the identity on π_n. This spectrum represents ordinary A-valued homology.

2. The arguably most important (and in some sense most complicated) spectrum is the sphere spectrum $\mathbb{S}$. The sphere spectrum is in degree n given by the space

$$\varinjlim_{k\to\infty} \Omega^k S^{n+k} = Q(S^n) \ .$$

Here the colimit is taken over the maps induced by the equator maps. For the transition maps we take the map $S^n \to S^{n+1}$ given by the equator inclusion and take the induced map $S^n \to \Omega S^{n+1}$. The cohomology theory it represents is called stable cohomotopy and written as

$$\mathbb{S}^*(X) = [X, \varinjlim_{k\to\infty} \Omega^k S^{n+k}]_*$$

for a space X.

[7] Here it is cleaner to work with cohomology theories defined on pointed spaces as opposed to pairs of spaces and take the suspension isomorphism as part of the datum instead of the connecting homomorphism.

3. Generalizing the last example we can take any pointed space X and form the suspension spectrum $\Sigma^\infty X$ as follows. The n-th space is given by
$$Q(\Sigma^n X) = \underrightarrow{\mathrm{colim}}_{k\to\infty}\Omega^k\Sigma^{n+k}X \ .$$
So we have that $\mathbb{S} = \Sigma^\infty S^0$.

Definition 3.14.16. For a given spectrum E and any integer $n \in \mathbb{Z}$ we define the n-th homotopy group of E to be
$$\pi_n(E) := \pi_k(E^{n+k}) \qquad \text{for some } k >> 0.$$
This is clearly independent of the choice of k. The stable homotopy groups $\pi_n^{\mathbb{S}}(X)$ of a pointed space X are the homotopy groups of the suspension spectrum $\Sigma^\infty X$.

Lemma 3.14.17. *The stable homotopy groups of a space X are given by the colimit*
$$\pi_n^{\mathbb{S}}(X) = \underrightarrow{\mathrm{colim}}\left(\pi_n(X) \to \pi_{n+1}(\Sigma X) \to \pi_{n+2}(\Sigma^2 X) \to ...\right).$$
For negative n they vanish.

Proof. We have
$$\pi_n(Q(X)) = \pi_n\left(\underrightarrow{\mathrm{colim}}_{k\to\infty}\Omega^k\Sigma^k X\right) = \underrightarrow{\mathrm{colim}}_{k\to\infty}\pi_n(\Omega^k\Sigma^k X) = \underrightarrow{\mathrm{colim}}_{k\to\infty}\pi_{n+k}(\Sigma^k X)$$
proving the first claim. For negative n we use that Σ^k is k-connective, i.e. all homotopy groups below k vanish (by cellular approximation). $\square$

Theorem 3.14.18 (Freudenthal suspension Theorem)**.** *If X is a n-connected CW complex then the suspension map*
$$\pi_i(X, x) \to \pi_{i+1}(\Sigma X, x) \qquad (S^n \to X) \mapsto (S^{n+1} \to \Sigma X)$$
is an isomorphism for $i \leq 2n$ and a surjection for $i = 2n+1$.

Proof. This will follows from Blakers-Massey (that we will discuss later). $\square$

As a special case we get that the morphism
$$\pi_i(S^n) \to \pi_{i+1}(S^{n+1})$$
is an isomorphism for $i < 2n-1$ and a surjection for $i = 2n-1$. We again can deduce the computation of $\pi_n(S^n)$ from that:

Corollary 3.14.19. *We have that $\pi_n(S^n) \cong \mathbb{Z}$ for every n and thus also $\pi_n(\mathbb{S}) = \mathbb{Z}$*

Proof. From the preceeding statement we get that in the sequence
$$\pi_1(S^1) \to \pi_2(S^2) \to \pi_3(S^3) \to ...$$
the first map is surjective and all the others are isos. We know that $\pi_1(S^1) \cong \mathbb{Z}$ and thus $\pi_2(S^2)$ can only be $\mathbb{Z}$ or $\mathbb{Z}/n$. But we already know that the degree $\pi_2(S^2) \to \mathbb{Z}$ is surjective, thus it can not be $\mathbb{Z}/n$. $\square$

Corollary 3.14.20. *For any pointed space X we have that*
$$\pi_i^s(X) = \pi_{2i+1}(\Sigma^{i+1}X)$$

3.15 Homotopy pullbacks and fibre sequences

In this section we want to introduce the important concept of homotopy limits and colimits. We begin with the case of pullbacks. We will work in the category of Kan complexes, but most of what we do can equally well be carried out in the category of topological spaces (except that some of the limits will take us out of CW complexes so that we have to sprinkly in CW replacements). We will freely use the simplicial path space X^{Δ^1} for a Kan complex X given as

$$X_n^{\Delta^1} = \mathrm{Hom}(\Delta^n \times \Delta^1, X)$$

which has the universal property that

$$\mathrm{Hom}(Y, X^{\Delta^1}) = \mathrm{Hom}(Y \times \Delta^1, X)$$

for any other simplicial set Y. This is a Kan complex by Lemma 3.8.4 since lifting agains a horn $\Lambda_k^n \to \Delta^n$ is by adjunction the same as lifting X against $\Lambda_k^n \times \Delta^1 \to \Delta^n \times \Delta^1$ which is anodyne. Moreover we have for any topological space Z that

$$\mathrm{Sing}(Z^{[0,1]}) = \mathrm{Sing}(Z)^{\Delta^1}$$

and therefore also that $|X^{\Delta^1}| \simeq |X|^{[0,1]}$.

Let us consider a diagram

$$\begin{tikzcd} & X \arrow[d, "p"] \\ Y \arrow[r, "q"] & Z \end{tikzcd} \tag{3.1}$$

of Kan complexes.

Definition 3.15.1. The homotopy pullback of the diagram (3.1) is given by the Kan complex

$$X \times_Z^h Y := X\,{}_p{\times}_{Z,\mathrm{ev}_0}\, Z^{\Delta^1}\,{}_{Z,\mathrm{ev}_1}{\times}_q\, Y$$

Note that the 0-simplices of this pullback are given by 'points' $x \in X, y \in Y$ together with a path connecting the images in Z. In the topological picture this becomes slighthly cleaner: the homotopy pullback of spaces X and Y is the space whose points are given by triples (x, y, γ) where $x \in X$, $y \in Y$ and γ is a path from $p(x)$ to $q(y)$. The singular complex will literally take this to the homotopy pullback of Kan complexes (using Lemma 3.8.8).

Example 3.15.2. The homotopy pullback of the diagram

$$\begin{tikzcd} & \mathrm{pt} \arrow[d] \\ \mathrm{pt} \arrow[r] & X \end{tikzcd}$$

for a pointed space X is given by the loopspace ΩX. Note that the actual pullback of this diagram is the point.

Lemma 3.15.3. *The simplicial set $X \times_Z^h Y$ is indeed a Kan complex.*

Proof. We write the homotopy pullback as the pullback

$$\begin{tikzcd} X \times_Z^h Y \arrow[r] \arrow[d] & Z^{\Delta^1} \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ X \times Y \arrow[r, "p\times q"] & Z \times Z \end{tikzcd}$$

We now claim that the right hand map is a *Kan fibration*, that is every diagram of the form

$$\begin{tikzcd} \Lambda_i^n \arrow[r] \arrow[d] & Z^{\Delta^1} \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ \Delta^n \arrow[r] & Z \times Z \end{tikzcd}$$

admits a lift as indicated by the dottet arrow

$$\begin{tikzcd} \Lambda_i^n \arrow[r] \arrow[d] & Z^{\Delta^1} \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Z \times Z \end{tikzcd}$$

(and such that everything commutes). To see this we observe that by the universal properties a lift in this diagram is the same as lift in the diagram

$$\begin{tikzcd} (\Lambda_i^n \times \Delta^1) \times_{(\Lambda_i^n \times \partial\Delta^1)} (\Delta^n \times \partial\Delta^1) \arrow[r] \arrow[d] & Z \\ \Delta^n \times \Delta^1 \arrow[ur, dashed] & \end{tikzcd}$$

which is possible since Z is a Kan complex and the left hand map is anodyne by Lemma 3.8.4. Thus we find that the pullback of this map $X \times_Z^h Y \to X \times Y$ is also a Kan fibration. But the map $X \times Y \to \mathrm{pt}$ is also a Kan fibration (since products of Kan complexes are also Kan complexes). But then the the composition

$$X \times_Z^h Y \to X \times Y \to \mathrm{pt}$$

is a Kan fibration as well, which shows that the source is a Kan complex. □

Note that by construction the homotopy pullback comes with a diagram

$$\begin{tikzcd} X \times_Z^h Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

That does *not* commute but for which we have a canonical homotopy from the clockwise to the counterclockwise composition (draw in the diagram). Conversely for any diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

filled by a homotopy we get an induced map $P \to X \times_Z^h Y$ and we say that this diagram exhibits P as the homotopy pullback if this map is a (weak) homotopy equivalence. Note that the direction of the homotopy in the diagram does not really matter as we can easily take the 'inverse' homotopy. Therefore we will usually not mention this explicitly.

We now want to make the universal property slightly more explict, in other words we try to understand which functor

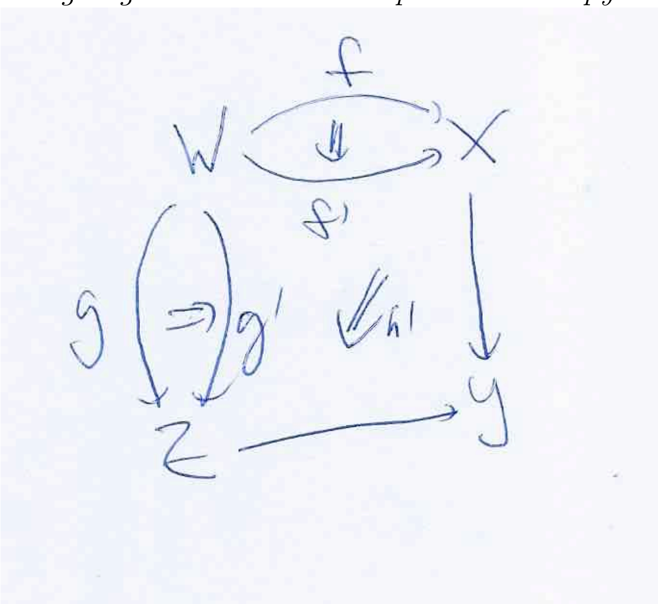
$$\mathrm{Ho}(\mathrm{Kan})^{op} \to \mathrm{Set}$$

the homotopy pullback $X \times_Z^h Y$ represents.

Proposition 3.15.4. *For any Kan complex W we have a natural bijection between the set $\mathrm{Hom}_{\mathrm{Ho}(\mathrm{Kan})}(W, X \times_Z^h Y)$ and the following set*

$$\frac{\{(f,g,h) \mid f : W \to X, g : W \to Y, h : pf \simeq qg\}}{\text{homotopy}}$$

Here two triples (f,g,h) and (f',g',h') are called homotopic if there are homotopies $f \simeq f'$ and $g \simeq g'$ such that the composite homotopy $W \times \Delta^1 \to Z$ in the diagram



is homotopic relative $W \times \partial\Delta^1$ to h.

Proof. Unfolding the definitions, concretely analyse what maps into $X \times_Z Z^{\Delta^1} \times_Z Y$ are. $\square$

Remark 3.15.5. It follows that for any homotopy pullback the same holds. In fact one could (and should probably) us the above description as a definition: the homotopy pullback is given by a homotopy commutative square (with a chosen fillers) such that P represents the above functor.

Corollary 3.15.6. *1. We have a canonical homotopy equivalence $X \times_Z^h Y \simeq Y \times_Z^h X$.*

2. Assume we are given a diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ D \arrow[r] & E \arrow[r] & F \end{tikzcd}$$

with filling homotopies and such that the right hand square is a homotopy pullback. Then the outer square with the composite homotopy is a homotopy pullback, if and only if the left hand side is one.

Proof. Use universal property and Yoneda. □

Example 3.15.7. The equivalence $X \times^h_Z Y \simeq Y \times^h_Z X$ gives in the case $X = Y = \mathrm{pt}$ the self equivalence $\Omega Z \to \Omega Z$ which takes a path γ to $-\gamma$.

A priori the homotopy pullback $X \times^h_Z Y$ is functorial in maps of diagrams

$$\begin{tikzcd} & X \arrow[d] \arrow[dr] & \\ Y \arrow[r] \arrow[dr] & Z \arrow[dr] & X' \arrow[d] \\ & Y' \arrow[r] & Z' \end{tikzcd}$$

that are strictly commutative. Now we claim that it is also functorial in diagrams with fillers

$$\begin{tikzcd} & X \arrow[d] \arrow[dr] & \\ Y \arrow[r] \arrow[dr] & Z \arrow[dr] & X' \arrow[d] \\ & Y' \arrow[r] & Z' \end{tikzcd}$$

To see this we simply observe that extending this diagram

$$\begin{tikzcd} X \times^h_Z Y \arrow[r] \arrow[d] & X \arrow[d] \arrow[dr] & \\ Y \arrow[r] \arrow[dr] & Z \arrow[dr] & X' \arrow[d] \\ & Y' \arrow[r] & Z' \end{tikzcd}$$

and considering the outer diagram with the respective fillers gives the desired map $X \times^h_Z Y \to X' \times^h_{Z'} Y'$. It easily follows from the universal property that this is indeed functorial, that is for compositions ...

Corollary 3.15.8. *Assume that we have a diagram*

$$\begin{tikzcd} & X \arrow[d] \arrow[dr] & \\ Y \arrow[r] \arrow[dr] & Z \arrow[dr] & X' \arrow[d] \\ & Y' \arrow[r] & Z' \end{tikzcd}$$

(with possible homotopies as fillers) in which all the diagonal maps are homotopy equivalences. Then so is the induced map $X \times^h_Z Y \to X' \times^h_{Z'} Y'$.

Proof. Clear, just pick the inverses. □

Remark 3.15.9. Corollary 3.15.8 for strictly commutative diagrams is of course completely false for ordinary pullbacks, e.g. consider the case of the map

$$\begin{tikzcd} & \mathrm{pt} \arrow[d] \arrow[dr] & \\ \mathrm{pt} \arrow[r] \arrow[dr] & X \arrow[dr] & PX \arrow[d] \\ & \mathrm{pt} \arrow[r] & X \end{tikzcd}$$

where X is a pointed space and PX is the space of paths ending in the basepoint. This space is clearly contractible, so all diagonal maps are homotopy equivalences. The actual pullback $pt \times_X \mathrm{pt}$ is of course the point and the pullback $\mathrm{pt} \times_X PX$ is the loopspace ΩX. These two are not homotopy equivalent.

We also see how the homotopy pullback resolves this problem, as we get in both cases the loopspace. In fact, very often, this invariance property of the homotopy pullback is taken as the defining propery and design criterion. In our approach this is just a consequence of making everything functorial up to homotopy.

Definition 3.15.10. For a map $p : E \to B$ and a point $b \in B$ the homotopy fibre $\mathrm{fib}(p) = \mathrm{fib}_b(p) = E^h_b$ is given by the homotopy pullback

$$\begin{tikzcd} \mathrm{fib}(p) \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d, "b"] \\ E \arrow[r] & B \end{tikzcd}$$

We say that a pair of maps $F \to E \to B$ together with a nullhomotopy of the composite (i.e. a homotopy to the map $F \to B$ projecting to the basepoint of B) is a fibre sequence of the induced diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d, "b"] \\ E \arrow[r] & B \end{tikzcd}$$

is a homotopy pullback.

If the basepoint b is clear we shall drop it from the notation and leave it implicit We warn the reader that one very often speaks of fibre sequences $F \to E \to B$ and leaves the choice of nullhomotopy of the composite implicit. But the nullhomotopy is part of the datum of a fibre sequence and this abusive notation can lead to mistakes if one is not careful!

Example 3.15.11.

1. For a pointed space B the homotopy fibre of $\mathrm{pt} \to B$ is the loop space ΩB.
2. For a fibre sequence $F \to E \to B$ (of pointed spaces) it follows from the pasting law that the fibre of the map $F \to E$ is given by ΩB. Thus we get a sequence of maps
$$\Omega B \to F \to E \to B$$
in which each double composition has a canonical nullhomotop that exhibits it as a fibre sequence.

3. We can continue the sequence to the left by iterating the procedure:

$$... \to \Omega^2 B \to \Omega F \xrightarrow{-\Omega i} \Omega E \xrightarrow{-\Omega p} \Omega B \to F \xrightarrow{i} E \xrightarrow{p} B$$

Here we get the sign since the canonical map has flipped the two points on Ω.

Theorem 3.15.12. *For a fibre sequence $F \to E \to B$ we get a long exact sequence*

$$... \to \pi_1(E) \to \pi_1(B) \to \pi_0(F) \to \pi_0(E) \to \pi_0(B)$$

(not necessarily surjective at the very right).

Proof. We first proof exactness at $\pi_0(E)$. Since $F \to E \to B$ is nullhomotopic it follows immediately that the image is contained in the kernel (which for pointed sets is simply the preimage of the basepoint). For the converse we note that by the universal property an element of $\pi_0(F)$ is given by a point in E together with a path connecting it to the basepoint in B. But an element in the kernel of $p_* : \pi_0(E) \to \pi_0(B)$ clearly admits such a lift. Now we simply apply this observation to the sequence of the last example together with the observation that $\pi_0(\Omega^n B) = \pi_n B$. □

Example 3.15.13. For $E \to B$ a covering space we want to identify the homotopy fibre F (here we work in topological spaces). We observe that by definition F is given by pairs consisting of a point $e \in E$ and a path $\gamma p(e) \to b$. Using the unique path lifting property we get that this is equal to the space of path in E ending in the preimage $p^{-1}b$ (and starting in some point e), i.e.

$$F = \{\gamma : [0,1] \to E \mid p\gamma(1) = b\}$$

Contracting every path to its endpoint, this space has an obvious deformation retract given by constant path, i.e. is homotopy equivalent to the discrete set $p^{-1}(b)$. Thus we conclude that

$$F \simeq p^{-1}b \ .$$

The long exact sequence thus takes the form

$$\pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F)$$

and since $\pi_k(F) = 0$ for $k \geq 1$ we get back the result that $\pi_n(E) \cong \pi_n(B)$ for $n \geq 2$.

Recall that we also had a long exact sequence of relative homotopy groups

$$... \to \pi_1(A) \to \pi_1(X) \to \pi_1(X, A) \to \pi_0(A) \to \pi_0(X)$$

shown in Theorem 3.3.5. This turns out to be a special case of the last proposition by virtue of the next result.

Proposition 3.15.14. *For an inclusion $i : A \subseteq X$ of topological spaces/Kan complexes we we have a canonical identification $\pi_{n+1}(X, A) = \pi_n(\mathrm{fib}(i))$ for each $n \geq 0$.*

Proof. We use the universal property to see that an element in $\pi_n(\mathrm{fib}(i)) = [S^n, \mathrm{fib}(i)]_*$ is the same as a homotopy class of pairs consisting of a map $S^n \to A$ together with a nullhomotopy of the composition $S^n \to X$. But this is clearly the same as a map $D^{n+1} \to X$ that sends $S^n = \partial D^{n+1}$ to A. □

This also explains precisely why $\pi_0(X,A)$ is not defined, $\pi_1(X,A)$ is only a pointed set, $\pi_2(X,A)$ a groups and $\pi_k(X,A)$ for $k \geq 3$ abelian.

We end this chapter by a criterion that is sometimes useful to identify homotopy pullbacks/fibres.

Definition 3.15.15. A map $E \to B$ of simplicial sets is called a Kan fibration if every diagram of the form

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & E \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

admis a filler. A map $E \to B$ of topological spaces is called Serre fibration if every diagram of the form

$$\begin{tikzcd} D^n \arrow[r] \arrow[d, "(\mathrm{id},0)"'] & E \arrow[d] \\ D^n \times I \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

admits a filler.

Warning 3.15.16. The notion of Kan/Serre fibrations are not homotopy invariant concepts. In fact, up to equivalence each map can be replace by a Kan/Serre fibration. So these are really point set notions that are important in the respective models.

Example 3.15.17. Every covering space $E \to B$ is a Serre fibration by the usual covering theory.

Proposition 3.15.18. *A map $E \to B$ of topological spaces is a Serre fibration precisely* $\operatorname{Sing} E \to \operatorname{Sing} B$ *is a Kan fibration.*

Proof. For $\operatorname{Sing} E \to \operatorname{Sing} B$ to be a Kan fibration we must be able to solve the lifting problem

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & \operatorname{Sing} E \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \operatorname{Sing} B \end{tikzcd}$$

Such a lift is by adjunction the same as a lift in the adjoint diagram

$$\begin{tikzcd} |\Lambda^n_i| \arrow[r] \arrow[d] & E \arrow[d] \\ |\Delta^n| \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

But the map $|\Lambda^n_i| \to |\Delta^n|$ is homeomorphic to the map $D^{n-1} \to D^{n-1} \times I$. $\square$

A map of topological spaces $E \to B$ is called fibre bundle of there exists an open covering $\{U_i\}_{i\in I}$ of B such that for every i there is a space F_i and a homeomorphism

$$\begin{tikzcd} E|_{U_i} \arrow[rr, "\simeq"] \arrow[dr] & & U_i \times F_i \arrow[dl] \\ & U_i & \end{tikzcd}$$

An example is a Covering space in which case all F_i are discrete sets.

Proposition 3.15.19. *Every fibre bundle is a Serre fibration.*

Proof. A fibre bundle $E \to B$ is called trivial if it is of the form $B \times F \xrightarrow{pr} B$. We first treat the case of a trivial bundle. In this cae □

First induction on n. Then subdivide and reduce to the trivial bundle.

Finally the main reason to consider fibrations here is the following result:

Theorem 3.15.20. *Assume that $E \to B$ is a Kan fibration (Serre fibration) and assume that we have a pullback square*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & E \arrow[d] \\ Y \arrow[r] & B \end{tikzcd}$$

of Kan complexes (top. Spaces). Then this square (with the identity homtopy) is also a homotopy pullback square.

Proof. Exercise. □

Corollary 3.15.21. *For a Kan fibration $p : E \to B$ the homotopy fibre is equivalent to the actual fibre $p^{-1}(b)$ and we get a corresponding long exact sequence on homotopy groups.*

Corollary 3.15.22. *For a Kan fibration $E \xrightarrow{p} B$ we get a long exact seuqence in homotopy groups:*

$$... \to \pi_2(p^{-1}(b)) \to \pi_2(E) \to \pi_2(B) \to \pi_1(p^{-1}(b)) \to ...$$

Example 3.15.23. Consider the Hopf fibration $S^3 \to S^2$ with preimage S^1. This is a fibre bundle, thus a Serre fibration. We find that

$$\pi_n(S^3) \cong \pi_n(S^2)$$

for $n \geq 3$ since $\pi_n(S^1) = 0$. In particular we find that $\pi_3(S^2) = \mathbb{Z}$.

3.16 Postnikov and Whitehead towers

Definition 3.16.1. Let X be a Kan complex. We say that a map $X \to X'$ is an n-truncation if X' is n-truncated (that is $\pi_k(X) = 0$ for $k > n$ all choices of basepoints) and it is universal with respect to this: that is for any other map $X \to Y$ with Y n-truncated there exists a unique factorization

$$X \to X' \to Y$$

in Ho(Kan) (that is it only factors up to homotopy and is only unique up to homotopy). In this case we also write $X' = \tau_{\leq n} X$.

Clearly, the truncation is uniquely determined if it exists.

Example 3.16.2. The map $X \to \pi_0(X)^\delta$ exhibits $\pi_0(X)$ as the 0-truncation of X. To see this note that for any space with vanishing higher homotopy groups is homotopy equivalent to a discrete Kan complex on a set. Thus any map factors uniquely through $\pi_0(X)$ (in other words: $\pi_0(X)$ is left adjoint to the inclusion of sets into simplicial sets).

Proposition 3.16.3. *For every space X there exist all truncations $\tau_{\leq n}X$. The map $X \to \tau_{\leq n}X$ is an iso on π_k for $k \leq n$.*

Proof. We first construct for every space X a map $X \to X'$ such that X' is n-truncated and such that the map is an n-equivalence, i.e. an isomorhism on homotopy groups π_k for $k \leq n$. This is easy to achieve by attaching cells to X killing the higher homotopy groups. In particular we get that (X', X) is a CW pair with cells of dimension $> n$. If follows by the compression Lemma 3.3.9/the simplicial analogue that any given map $X \to Y$ for Y n-connective can be uniquely (up to homotopy) extended over X' and thus that X' satisfies the universal property of the n-truncation. □

Definition 3.16.4. A map $X' \to X$ of pointed spaces is called n-connective (or $(n-1)$-connected) covering if X' is n-connective (i.e. $\pi_k(X) = 0$ for $k < n$) and it is universal, that is for any other map of pointed spaces $Y \to X$ such that Y is n-connective there exists a unique factorizaton

$$Y \to X' \to X \ .$$

up to homotopy We also write $X' = \tau_{\geq n}X$.

Example 3.16.5. Let X be a pointed spaces. A connected covering is simply the inclusion of the basepoint component $\tau_{\geq 1}X = X_0 \subseteq X$. A 1-connected covering is the universal covering of X_0.

Proposition 3.16.6. *For every pointed homotopy type X and every $n \geq 0$ there exists an n-connective covering $\tau_{\geq n}X$ and we have that the map $\tau_{\geq n}X \to X$ is an isomorphism on homotopy groups π_k for $k \geq n$. Moreover there is a fibre sequence*

$$\tau_{\geq n}X \to X \to \tau_{\leq n-1}X \ .$$

Proof. For a given pointed homotopy type X we define our candidate for an n-connective cover as the fibre of the $n-1$-truncation so that there is a fibre sequence

$$\tau_{\geq n}X \to X \to \tau_{\leq n-1}X \ .$$

In order to verify the universal propery we observe that for every n-connective pointed space Y and a given map $Y \to X$ the composite $Y \to \tau_{\leq n-1}X$ is (uniquely) nullhomotopic since such a map has to factor through $\tau_{\leq n-1}Y \simeq \mathrm{pt}$. As a result we get a (unique) factorization through the fibre. This verifies the universal propery. The long exact sequence on homotopy groups then gives the claim about the homotopy groups. □

Definition 3.16.7. We now get for every space X two sequences of spaces:

$$... \to \tau_{\leq 2}X \to \tau_{\leq 1}X \to \tau_{\leq 0}X$$

called the *Postnikov Tower* and

$$\to \tau_{\geq 2}X \to \tau_{\geq 1}X \to \tau_{\geq 0}X = X \ .$$

called the *Whitehead tower* of X.

What are the fibres of these towers? We find that the fibre of $\tau_{\leq n}X \to \tau_{\leq n-1}X$ is equivalent to $\tau_{\geq n}(\tau_{\leq n})X$. This space has a single homotopy group in degree n which is given by π_nX. Thus it is an Eilenberg-MacLane space. More generally we get

Proposition 3.16.8. *We have for every n fibre sequences*

$$K(\pi_n X, n) \to \tau_{\leq n} X \to \tau_{\leq n-1} X$$

and

$$K(\pi_n X, n-1) \to \tau_{\geq n+1} X \to \tau_{\geq n} X \ .$$

Finally we want to understand the limit of these towers. Informally it should be clear that the limit of the Postnikov tower should be X and the limit of the Whitehead tower should be the point. In order to make this precise we need to introduce inverse homotopy limits.

Definition 3.16.9. The Homotopy equalizer of two maps $f, g : X \to Y$ is defined as the homotopy pullback

$$\begin{tikzcd} \mathrm{hEq}(f,g) \arrow[r] \arrow[d] & Y \arrow[d, "\Delta"] \\ X \arrow[r, "{(f,g)}"] & Y \times Y \end{tikzcd}$$

From the Definition of the homotopy equalizer as a homotopy pullback we can deduce that it is functorial in maps of diagrams that commute up to homotopy (in the obvious sense) and that it is invariant under homotopy equivalence. One can also deduce a universal property, namely that a map into $\mathrm{hEq}(f,g)$ is given by a map into X together with a homotopy between the two composites to Y. In particular the points of $\mathrm{hEq}(f,g)$ are given by pairs consisting of a point $x \in X$ together with a path between $f(x)$ and $g(x)$.

Proposition 3.16.10. *Assume that X, Y and both maps f, g are pointed. Then we have a fibre sequence*

$$\Omega Y \to \mathrm{hEq}(f,g) \to X$$

and thus a long exact sequence

$$... \to \pi_1 X \xrightarrow{fg^{-1}} \pi_1 Y \to \pi_0 \mathrm{hEq}(f,g) \to \pi_0(X)$$

Proof. We consider the diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \mathrm{hEq}(f,g) \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathrm{pt} \arrow[r] & X \arrow[r] & Y \times Y \end{tikzcd}$$

which shows that the fibre of the map is given by the outer pullback, i.e. the fibre of the diagonal map $Y \to Y \times Y$. But this fibre is equivalent to ΩY since it is given by two paths in Y starting in the basepoint and ending at the same point. The map $\Omega Y \to \mathrm{hEq}(f,g)$ sends a loop to the triple consisting of the basepoint of X together with the two path given. by the respective halfs of the loop (having a common point in Y). Finally we extend to a fibre sequence

$$\Omega X \to \Omega Y \to \mathrm{hEq}(f,g) \to X$$

using the diagram

$$\begin{tikzcd}
\Omega X \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d] & \\
F \arrow[r] \arrow[d] & \mathrm{hEq}(f,g) \arrow[r] \arrow[d] & Y \arrow[d] \\
\mathrm{pt} \arrow[r] & X \arrow[r] & Y \times Y
\end{tikzcd}$$

If we go through the idenfitifications we see that the map $\Omega X \to \Omega Y$ is given by sending a loop in X to the loop in Y, where we apply f to the loop and then compose with the inverse of applying Y to that loop. □

Note that the map $\pi_1 X \xrightarrow{fg^{-1}} \pi_1 Y$ is not a group homomorphism! It is the map induced from the map $\Omega X \to \Omega Y$ that is not a loop map.

Definition 3.16.11. Consider a diagram of spaces

$$... \to X_2 \xrightarrow{f} X_1 \xrightarrow{f} X_0$$

Then the sequential homotopy inverse limit $\underleftarrow{\mathrm{holim}}_n(X_n)$ is defined as the homotopy equalizer of the two maps

$$\prod_{n\geq 0} X_n \to \prod_{n \geq 0} X_n$$

given by the identity and the map that sends a sequence $(x_n)_{n\geq 0}$ to the sequence $(f(x_n))_{n\geq 1}$.

Again the points of homotopy inverse limit can be described as tuples (x_n, γ_n) consiting of points $x_n \in X_n$ and paths γ_n in X_n connecting x_n to $f(x_{n+1})$. As before we see that the homotopy limit is functorial in maps of diagrams that commute up to homotopy and it has the following universal property: giving a map $Z \to \underleftarrow{\mathrm{holim}}\, X_n$ is the same as giving a map into each X_n together with homotopies witnessing that the diagrams

$$\begin{tikzcd}
Z \arrow[r] \arrow[dr] & X_{n+1} \arrow[d] \\
 & X_n
\end{tikzcd}$$

commute up to homotopy.

Remark 3.16.12. As before one gets that if all the maps $X_n \to X_{n-1}$ are fibrations (in the respective sense) then the homotopy inverse limit can be computed by the actual limit. We have in the definition of the sequential inverse limit (as well as the equalizer) implicitly used that the 'homotopy product' can be defined as the actual product since there are no possible homotopies are coming up and the product is already homotopy invariant.

Proposition 3.16.13 (Milnor sequence)**.** *For a diagram of pointed spaces*

$$... \to X_2 \xrightarrow{f} X_1 \xrightarrow{f} X_0$$

and $k \geq 1$ we have a short exact sequence

$$0 \to \underleftarrow{\lim}^1_n \pi_{k+1}(X_n) \to \pi_k(\underleftarrow{\mathrm{holim}}_n X_n) \to \underleftarrow{\lim}_n \pi_k(X_n) \to 0 \ .$$

Proof. We consider the definition of the homotopy limit as the homotopy equalizer of two pointed maps

$$\mathrm{id}, f : \prod_{n\geq 0} X_n \to \prod_{n\geq 0} X_n$$

Then we get a long exact sequence from Proposition 3.16.10 that takes the form

$$\prod_{n\geq 0} \pi_{k+1}(X_n) \to \prod_{n\geq 0} \pi_{k+1}(X_n) \to \pi_k(\underleftarrow{\mathrm{holim}}_n X_n) \to \prod_{n\geq 0} \pi_k(X_n) \to \prod_{n\geq 0} \pi_k(X_n)\ .$$

This induces a short exact sequence as desired. □

Corollary 3.16.14. *The inverse homotopy limit of the Whithead tower of X is* pt *and the inverse homotopy limit of the Postnikov tower is X.*

Proof. There is always a canonical map $\mathrm{pt} \to \underleftarrow{\mathrm{holim}}\, \tau_{\geq n} X$ and $X \to \underleftarrow{\mathrm{holim}}\, \tau_{\leq n} X$. We have to show that they are isomorphisms on homotopy groups for all choices of basepoints. This immediately follows from the Milnor sequence for higher homotopy groups since the sequence of homotopy groups stabilize. It remains to understand the case of π_0. □

One should see the last result as saying that the Whithead tower induces a 'complete, exhaustive filtration' of X with fibres given by Eilenberg-MacLane spaces. Or said differently: every space can be build out of Eilenberg MacLane spaces.

We end this section by remarking that there is a definition of homotopy limit for any diagram $I \to \mathrm{Kan}$. We have only treated the special case where I is in some sense 1-dimensional (more precisely the classifying space of I is 1-dimensional). In the general case the universal propery is harder to articulate since there are all sorts of 'higher coherences' coming up. Still, one can in fact reduce the case of arbitrary homotopy limits to iterations of pullbacks and infinite products. We will not be able to cover this here.

3.17 Homotopy pushouts and colimits

In this section we are briefly going to discuss the concept of homotopy pushouts and colimits which is 'dual' to limits.

Definition 3.17.1. A square

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

filled by a homotopy is called a homotopy pushout square (and D the homotopy pushout) if for any space E the map

$$[D, E] \to \frac{\{(f, g, h) \mid f : B \to E, g : C \to E, h : fi \simeq gj\}}{\text{homotopy}}$$

is a bijection. For a map $X \to Y$ the pushout $\mathrm{pt} \amalg_X Y$ is also called the cofibre of f and written as $\mathrm{cof}(f)$.

Proposition 3.17.2. *For every diagram*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"] & B \\ C & \end{tikzcd}$$

there exists a pushout $D = B \amalg^h_A C$ given by the space

$$B \amalg_{A\times\{0\}} (A \times I) \amalg_{A\times\{1\}} C \ .$$

Moreover we have a pasting law for homotopy pushout diagrams and they are invariant under homotopy equivalence.

Proof. Clear. □

Example 3.17.3. 1. The pushout $\mathrm{pt} \amalg_X \mathrm{pt}$ is given by ΣX (the unreduced suspension).

2. The cofibre of a map $f : X \to Y$ is given by the mapping cone

$$\mathrm{cof}(X \to Y) = ((X \times I) \amalg_{X\times 1} Y) /(X \times 0) \ .$$

   Bild.

3. Dual to the sequence for iterated fibres we get for every map $X \to Y$ a sequence

$$X \to Y \to \mathrm{cof}(f) \to \Sigma X \to \Sigma Y \to \Sigma\mathrm{cof}(f) \to \Sigma^2 X \to ...$$

4. The cofibre of a map $S^n \to X$ is given by pushout $D^{n+1} \amalg_{S^n} X$, i.e. the space obtained by attaching a cell. More generally all the pushouts occuring in CW complexes

$$\begin{tikzcd} \coprod_I S^n \arrow[r] \arrow[d] & X_n \arrow[d] \\ \coprod_I D^{n+1} \arrow[r] & X_{n+1} \end{tikzcd}$$

   are homotopy pushouts. Thus we can just describe X_{n+1} as a pushout of X_n against points.

Proposition 3.17.4. *If we have a pushout square of simplicial sets*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

then it is a homotopy pushout provided that one of the maps $A \to B$ or $B \to C$ is a monomorphism. If it is a square of topological spaces then it is enough that one of the maps is a subspace inclusion that has the HEP (e.g. is a relative CW pair), see Definition 3.3.7.

It is also a homotopy pushout if it is an excisive triad, that is B, C are subspaces of D and $A = B \cap C$ (that is the square is also a pullback) and that the open cores of B and C still cover D.

Proof. Skipped but not hard. □

Remark 3.17.5. And example of an excisive triad is an open cover. Most of the theorems that we have shown for open covers / cellular pushouts remain true for arbitrary homotopy pushouts. For example the Mayer-Vietoris sequence in homology or the Brown-representability theorem.

Recall that for homotopy groups we had a sort of Mayer-Vietoris sequence for pullbacks.

Theorem 3.17.6 (Blakers-Massey)**.** *Assume that we have a homotopy pushout square*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

where $\mathrm{fib}(i)$ *is* n*-connected and* j *is* m*-connected for* $n, m \geq 0$*. Then the fibre of the map* $A \to B \times^h_D C$ *is* $(m+n)$*-connected.*

Proof. Skipped. There are purely homotopy theoretic proofs! □

Note that the fibre of a map $X \to Y$ is n-connected precisely if $\pi_k(X) \to \pi_k(Y)$ is an iso for $k < n+1$ and a surjection for $k = n$ by the long exact sequence.

Corollary 3.17.7 (Freudenthal suspension Theorem, see 3.14.18)**.** *If* X *is a* $(n-1)$*-connected CW complex. Then the suspension map*

$$\pi_i(X, x) \to \pi_{i+1}(\Sigma X, x) \qquad (S^n \to X) \mapsto (S^{n+1} \to \Sigma X)$$

is an isomorphism for $i < 2n-1$ *and a surjection for* $i = 2n-1$*.*

Proof. Consider the homotopy pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d] \\ \mathrm{pt} \arrow[r] & \Sigma X \end{tikzcd}$$

The fibres here are $(n-1)$-connected. The homotopy pullback is given by $\Omega\Sigma X$ and thus we conclude that the map

$$X \to \Omega\Sigma X$$

is an isomorphism in degrees through $2n-2$ and a surjection in degree $(2n-1)$. This implies the claim. □

Definition 3.17.8. For a diagram

$$X_0 \to X_1 \to X_2 \to \ldots$$

the homotopy colimit is given by a universal space X_∞ together with maps $X_i \to X_\infty$ and homotopies filling the diagrams

$$\begin{tikzcd} X_i \arrow[r] \arrow[dr] & X_{i+1} \arrow[d] \\ & X_\infty \end{tikzcd}$$

(here universal means that every other such space admits a unique (up to homotopy) map to X_∞).

Proposition 3.17.9. *The sequential homotopy colimit exists and is given by the mapping telescope*

$$\frac{\coprod (X_i \times I)}{(x,1) \sim (f(x),0)}$$

which is also equivalent to the homotopy pushout of the diagram

$$\begin{tikzcd} (\coprod X_i) \amalg (\coprod X_i) \arrow[r, "(\mathrm{id},f)"] \arrow[d, "\nabla"] & \coprod X_i \\ \coprod X_i & \end{tikzcd}$$

The latter is an explict form of the homotopy coequalizer of two maps

$$f, \mathrm{id} : \coprod X_i \to \coprod X_i \ .$$

Proposition 3.17.10. *For a sequential homotopy colimit $X_0 \to X_1 \to ... \to X_\infty$ we get that the induced map*

$$\varinjlim_i \pi_k(X_i) \to \pi_k(X_\infty)$$

is an isomorphism.

Proof. We show that the map is injective and surjective (similar as we have seen this for CW complexes and increasing unions of open sets before). Both follow from the fact that every map from a compact space K to X_∞ has to factor (up to homotopy) through a finite inclusion $X_i \to X_\infty$ which follows by covering X_∞ by open sets containing the X_i. The rest proceeds as before. □

Corollary 3.17.11. *Every CW complex is the homotopy colimit of its skeleta. In particular every CW complex can be obtained by iterative cuproducts, pushouts and sequential colimits from the one point space* pt.

In that way the collection of all homotopy types is 'generated' from the point.

Remark 3.17.12. This should be seen as the 'dual' of the Milnor sequence on homotopy for homotopy inverse limits, where the dual of the $\lim^1$-term vanishes. In fact, the previous result more generally is true for any filtered colimit, not necessarily directed.

Corollary 3.17.13. *For every sequence of simplicial sets*

$$X_0 \to X_1 \to ...$$

the actual colimit is (weakly) equivalent to the homotopy colimit.

Proof. We first note that the homotopy groups of the colimit are also isomorphic to the colimit of the homotopy groups. To see this, we simply observe that $|\mathrm{colim} X_i| = \mathrm{colim}|X_i|$ and that for maps of CW complexes we have that homotopy groups commute with sequential colimits (for arbitrary diagrams of spaces this is false though...).

Now we get a map from the homotopy colimit to the actual colimit by abstract nonsense and we claim that this is an isomorphism on homotopy groups. □

3.18 The category of spectra

We end this lecture by an overview of the category of spectra. We recall that a spectrum is a sequence of spaces X_i for $i \in \mathbb{N}$ together with homotopy equivalences $X_i \simeq \Omega X_{i+1}$ for all $i \in \mathbb{N}$.

Definition 3.18.1. A map of spectra $X \to Y$ is a sequence of maps $f_i : X_i \to Y_i$ together with homotopies h_i filling the squares

$$\begin{tikzcd} X_i \arrow[r, "\simeq"] \arrow[d, "f_i"'] & \Omega X_{i+1} \arrow[d, "\Omega f_{i+1}"] \\ Y_i \arrow[r, "\simeq"] & \Omega Y_{i+1} \end{tikzcd}$$

for each i.

A homotopy between maps of spectra (f_i, h_i) and (f_i', h_i') consists of homotopies $f_i \Rightarrow f_i'$ such that the two homotopies between the two maps $X_i \to \Omega Y_{i+1}$ built using f_i and Ωf_{i+1} are homotopic.

The homotopies are part of the datum of a map, they are not just required to exist. Unfortunately this has the effect that maps of spectra can not be composed in a strictly associative way, only up to homotopy. We can thus form the *stable homotopy category* SHC also called the *homotopy category of spectra* whose objects are spectra and whose morphisms are homotopy classes of maps of spectra. Two objects are equivalent in SHC precisely if they are equivalent as in Definition 3.14.12. A map is an equivalence preicsely if it induces an isomorphism on homotopy groups.

Remark 3.18.2. It is clear that one can also define the notion of homotopy between homotopies of spectra etc. This way spectra very naturally form an ∞-category and this ∞-category Sp is really the 'correct' object to study. But we do not want to go into this here. We will thus not really speak of the category of spectra (which is an ∞-category) but only of the homotopy category.

Let us now examine some categorical properties of spectra. In many respects spectra behaves like abelian groups.

Proposition 3.18.3. *1. The category* SHC *is pointed, that is it has an object which is both initial and final given by the spectrum* $0 = (\mathrm{pt}, \mathrm{pt}, \mathrm{pt}, ...)$.

2. *The category* SHC *is semiadditive, that is it has biproducts $A_0 \oplus A_1$ for any pair of spectra.* [8]

3. *The category* SHC *is additive, that is the shear map is an equivalence. Equivalently every object is a commutative group, i.e. the forgetful functor from commutative groups is an equivalence.*

[8] A biproduct is an object together with maps $i_i : A_i \to A_0 \oplus A_1$ and $\pi_i : A_0 \oplus A_1 \to A_i$ such that the i_i exhibit it as the coproduct, the π_i exibit it as the product and such that the composition $\pi_j \circ i_j$ is δ_{ij}. Equivalently in a pointed category one always gets a map from the coproduct $A_0 \vee A_1 \to A_0 \times A_1$ and one requires this to be an isomorphism. An equivalent way of stating this is that the forgetful functor from commutative monoids in the category (wrt product) to the category is an equivalence of categories)

Proof. Done on the exercises. □

Now we want to say a few more words about the formal structure on spectra. To this end we have to roughly say what homotopy colimits are.

Definition 3.18.4. A square of spectra (not in SHC!)

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

filled by a homotopy is a homotopy pullback if for each n the induced square

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & Y_n \arrow[d] \\ Z_n \arrow[r] & W_n \end{tikzcd}$$

is a pullback of spaces.

Remark 3.18.5. One should of course rather define this using a universal property, but because of lacking technology we can't at the moment.

We also note that spectra have all pullbacks, given by the 'pointwise pullbacl',. as the formula clearly indicates.

In particular we get that $\Omega X = \mathrm{pt} \times_X \mathrm{pt}$ is given by the levelwise loopspace. Now note that we get an induced functor

$$\Omega : \mathrm{SHC} \to \mathrm{SHC}$$

induced from this operation. And it levelwise loops, i.e. shifts the spectrum. Thus the following result is more or less immediate:

Theorem 3.18.6. *The functor* $\Omega : \mathrm{SHC} \to \mathrm{SHC}$ *is an equivalence of categories.*

In fact, since we have pushouts we have also defined fibres of a map of spectra $\mathrm{fib}(f) \to X \to Y$. Clearly we get an induced long exact sequence

$$... \to \pi_{i+1}(Y) \to \pi_i(\mathrm{fib}(f)) \to \pi_i(X) \to \pi_i(Y) \to \pi_{i-1}(F) \to ...$$

In particular a morphism of spectra is an equivalence precisely if its fibre is equivalent to 0 (recall that all homtoopy groups are abelian and there are no π_0-issues).

Finally we want to compare the category of spectra to the category of cohomology theories. Therefore recall:

Definition 3.18.7. The category CohTh of cohomology theories is the category of pairs consisting of functors $E^* : \mathrm{Ho}(\mathrm{Kan}_*)^{op} \to \mathrm{grAb}$ and natural isomorphisms $E^*(-) \simeq E^{*+1}(\Sigma -)$ that form cohomology theories. A morphism is given by a natural transformation $E^* \to F^*$ that commutes with the suspension isos.

Clearly cohomology theories form a nice additive category. We have also seen that every spectrum yields a cohomology theory and it is also clear (?) that every morphism of spectra gives a map of cohomology theories. We want to compare SHC to the category of cohomology theories. That is, we get a a functor

$$\mathrm{SHC} \to \mathrm{CohTh}$$

sending E to the induced cohomology theory. We have already seen that this functor is essentially surjective. But what are maps between the associated cohomology theories with two spectra?

Let us expand a bit on that: a spectrum X gives a sequence of functors

$$E_n(-) = [-, X_n]_* :$$

and a map of spectra $X \to Y$ give natural transformations $E_n \to F_n$ induced by $X_n \to Y_n$. The compatibility with the suspension isomorphism translates into the diagram

$$\begin{tikzcd} E_{n+1}(\Sigma -) = [-, \Omega X_{n+1}]_* \arrow[r] \arrow[d] & E_n(-) = [-, X_n]_* \arrow[d] \\ F_{n+1}(\Sigma -) = [-, \Omega Y_{n+1}]_* \arrow[r] & F_n(-) = [-, Y_n]_* \end{tikzcd}$$

commuting, i.e. by Yoneda this means that the diagram

$$\begin{tikzcd} X_{n+1} \arrow[r] \arrow[d] & X_n \arrow[d] \\ Y_{n+1} \arrow[r] & Y_n \end{tikzcd}$$

commutes in $\mathrm{Ho}(\mathrm{Kan}_*)$. But the choice of filler is not part of the datum of a morphism. That is a morphism of cohomology theories represented by spectra X and Y translates into a sequence of maps $f_i : X_i \to Y_i$ such that these squares commute, but not prefered filler choosen. We see that whenever this is the case we can of course choose such a filler, that is the functor

$$\mathrm{SHC} \to \mathrm{CohTh}$$

is full, that is surjective on Homs. More precisely we have:

Theorem 3.18.8. *The functor* SHC $\to$ CohTh *is essentially surjective and full, but not faithful. More precisely for any pair of spectra X and Y we have a short exact sequence of abelian groups*

$$0 \to \varprojlim{}^1 [X_i, Y_{i-1}] \to \mathrm{Hom}_{\mathrm{SHC}}(X, Y) \to \mathrm{Hom}_{\mathrm{CohTh}}(X, Y) \to 0$$

Proof. We have already established that the functor is essentially surjective and full. The latter translates into the map

$$\mathrm{Hom}_{\mathrm{SHC}}(X, Y) \to \mathrm{Hom}_{\mathrm{CohTh}}(X, Y)$$

being surjective. It therefore remains to understand its kernel (note that we are dealing with a group homomorphism since the functor

$$\mathrm{SHC} \to \mathrm{CohTh}$$

is additive). An element in the kernel is by definition an class of maps of spectra $X \to Y$ such that all the underlying maps $X_i \to Y_i$ are nullhomotopic. So we can WLOG assume they are actually zero, otherwise we replace our map by an equivalent one. Then the datum of making it a map of spectra is actually precisely given by a sequence of maps

$$h_i : X_i \to \Omega^2 Y_{i+1} \simeq Y_{i-1}$$

Two such maps are homotopic, i.e. equal in SHC precisely if there are maps $\eta_i : X_i \to \Omega Y_i = Y_{i-1}$ such that

$$h_i + \eta_i \simeq \Omega \eta_{i+1}.$$

But this is exactly an element in $\varprojlim^1 [X_i, Y_{i-1}]$. □

Definition 3.18.9. Maps of spectra $E \to F$ that induced the zero map on associated cohomology theories are called *hyperphantoms*.

By definition, hyperphantoms are precisely those maps of spectra $X \to Y$ that are in the $\lim^1$-term in the short exact sequence

$$0 \to \varprojlim^1 [X_i, Y_{i-1}] \to \mathrm{Hom}_{\mathrm{SHC}}(X, Y) \to \mathrm{Hom}_{\mathrm{CohTh}}(X, Y) \to 0$$

It is a priori clear that hyperphantoms actually exist and it is non-trivial to construct one and a little bit behind the methods of the scope of this course. [9]

Let us just note that for many spectra X and Y the group $\varprojlim^1 [X_i, Y_{i-1}]$ is actually zero, so that maps of spectra and maps of the associated cohomology theories in fact agree.

We finally close this section by noting that the category SHC has much more structure. For example, there is a tensor product $\otimes = \otimes_{\mathbb{S}}$ which formally is a functor

$$\otimes : \mathrm{SHC} \times \mathrm{SHC} \to \mathrm{SHC}$$

that makes SHC a symmetric monoidal category. The unit is the sphere spectrum $\mathbb{S}$. This tensor product has the property that the suspension spectrum functor

$$\Sigma^\infty_+ : \mathrm{Ho(Kan)} \to \mathrm{SHC}$$

becomes symmetric monoidal, in particular

$$\Sigma^\infty_+ X \otimes \Sigma^\infty_+ Y \cong \Sigma^\infty_+ (X \times Y).$$

Moreover algebra objects in this structure, i.e. specrum E equipped with a maps

$$E \otimes E \to E, \mathbb{S} \to E$$

which are unital and associative in SHC induced multicative cohomology theories under the functor

$$\mathrm{SHC} \to \mathrm{CohTh} .$$

By multiplicatie cohomology theories, we mean cohomology theories (E^*, δ) equipped with the structure of a graded ring on $E^*(X)$ for each space X such that all maps $E^*(X) \to E^*(Y)$ become ring maps and such that the connecting homomorphism δ becomes a module map:

$$\delta(\alpha) \cup \beta = \delta(\alpha \cup i^*(\beta)) .$$

[9] For example for $X = \mathrm{KU}$ and $Y = H\mathbb{Z}[1]$ there are such hyperphantoms. In fact, in this case $\mathrm{Hom}_{\mathrm{CohTh}}(X, Y) = 0$ and $\varprojlim^1 [X_i, Y_{i-1}]$ is an uncountable $\mathbb{Q}$-vector space.

For example $H\mathbb{Z}$ is a ring spectrum (since $\mathbb{Z}$ is a ring) and this is the u homotopy theoretic formulation of the existence of the cup product on singular cohomology.

We finally note that we think of spectra as a generalization of abelian groups and therefore about the study of those as 'higher algebra'. Here is a dictionary that might be helpful for some of you:

Abelian groups	**Spectra**
The integers $\mathbb{Z}$	The sphere spectrum $\mathbb{S}$
Free abelian group $\mathrm{Fr}_{\mathbb{Z}}(S)$	Suspension spectra $\Sigma^\infty_+(X)$
Rings	Ring spectra
$\otimes_{\mathbb{Z}}$	$\otimes_{\mathbb{S}}$

Let us finally stress again, that one should really treat spectra as an ∞-category rather than taking homotopy groups. This will be core content of next semester.

3.19 Further results

We first note that one of the key topics of this lecture were higher homotopy groups. We have seen that even for spheres they are not as easy. A summary of what we have seen is the following:

- $\pi_1(S^1) = \mathbb{Z}$, and $\pi_k(S^1) = 0$ for $k > 1$. This follows from covering theory.
- $\pi_k(S^n) = 0$ for $k < n$. This follows from cellular approximation.
- $\pi_n(S^n) = \mathbb{Z}$. This follows from the Hurewicz theorem, which compares homotopy to homology, in the first degree where the homotopy is nontrivial.
- $\pi_3(S^2) \cong \mathbb{Z}$ by the long exact sequence of the Hopf-fibration.
- Under the suspension maps $\pi_k(S^n) \to \pi_{k+1}(S^{n+1})$, the sequence

  $$\ldots \to \pi_k(S^n) \to \pi_{k+1}(S^{n+1}) \to \ldots$$

  eventually stabilizes, i.e. the maps become isomorphisms for large enough indices.
- $\pi_k(S^n)$ is countable for all k and n. This follows since $\operatorname{Sing} S^n$ is equivalent to a Kan complex with countably many simplices, namely a Kan replacement of $\Delta^n/\partial\Delta^n$ and the process of Kan replacing stays countable (remember the small object argument). It does not stay finite!

The first major breakthrough in these computations is the following result:

Theorem 3.19.1 (Serre)**.** *The group $\pi_k(S^n)$ is finite, except if $k = n$ or n is even and $k = 2n-1$. In the latter case, $\pi_{2n-1}(S^n)$ is of the form $\mathbb{Z} \oplus$ finite.*

Corollary 3.19.2. *The stable homotopy groups $\pi_i(\mathbb{S})$ are finite for $i > 0$.*

The method employed and invented by Serre is the Serre spectral sequence, which also yields a methods of computing the homotopy groups (which is practically only implementable for low degrees). Roughly speaking, the Serre spectral sequence is a method for a fibre sequence

$$F \to E \to B$$

to compute the cohomology $H^*(E)$ from the cohomology $H^*(F)$ and $H^*(B)$ similar, but much more complicated, to long exact sequences on homotopy groups. It is a generalization of the Künneth theorem in the sense that for the fibre sequence

$$F \to F \times B \to B$$

Recall that if B is of finite type:

$$0 \to (H^*(B) \otimes H^*(F))_n \to H^n(F \times B) \to \bigoplus_{p+q=n+1} \mathrm{Tor}(H^p(B), H^q(F)) \to 0$$

We also have that

$$0 \to H^i(B) \otimes H^j(F) \to H^i(B, H^j(F)) \to \mathrm{Tor}(H^{i+1}(B), H^j(F)) \to 0$$

Combining these two statements gives the following Corollary.

Corollary 3.19.3. *For a finite type space B we have a (non-natural) isomorphism*

$$H^n(B \times F) \cong \bigoplus_{i+j=n} H^i(B, H^j(F))$$

Proof. We have isomorphisms

$$\begin{aligned} H^n(F \times B) &= \bigoplus_{i+j=n} (H^i(B) \otimes H^j(F)) \oplus \bigoplus_{i+j=n+1} \mathrm{Tor}(H^i(B), H^j(F)) \\ &= \bigoplus_{i+j=n} \big(H^i(B) \otimes H^j(F) \oplus \mathrm{Tor}(H^{i+1}(B), H^j(F))\big) \\ &= \bigoplus_{i+j=n} H^i(B, H^j(F)) \ . \end{aligned}$$

□

Remark 3.19.4. It turns out, that one can also show that this is true without the finite type assumption on B.

The general slogan now is, that the Serre spectral sequence for a fibre sequence

$$F \to E \to B$$

is a tool (similar to a long exact sequence) that let us compute $H^*(E)$ from $H^*(B, H^*(F))$ for a fibre seuqence $F \to E \to B$. In the case of the product the two are isomorphic (non-canonically). But for general fibre sequences they are not and one has 'differentials' that tell you how to go from $H^*(B, H^*(F))$ to $H^*(E)$. One writes this as

$$H^*(B, H^*(F)) \Rightarrow H^*(E)$$

One can also use this tool to learn somethings (sometimes everything) about the cohomogy of F from knowing the cohomology of E and B.

To apply this to the computation of $\pi_i(S^k)$. Serre proceeded roughly by first computing the cohomology of Eilenberg Mac-Lane spaces $K(A,n)$ inductively This is actually hard in this generality, but one can learn quite a bit using the fibre sequence

$$K(A, n-1) \to \mathrm{pt} \to K(A, n)$$

and so-called Steenrod operations with the transgression theorem. After that, he would go on to consider the fibre sequence

$$K(\mathbb{Z}, k-1) \to \tau_{\geq k+1} S^k \to S^k \to \tau_{\leq k} S^k = K(\mathbb{Z}, k)$$

and compute the homology/cohomology of $\tau_{\geq k+1} S^k$ by the left hand fibre sequence. The latter can then by Hurewicz be used to compute the lowest homotopy group $\pi_{k+1}(\tau_{\geq k+1} S^k) = \pi_{k+1}(S^k)$. In the next step we would continue looking at the fibre sequence

$$K(\pi_{k+1}(S^k), k) \to \tau_{\geq k+2} S^k \to \tau_{\geq k+1} S^k \to K(\pi_{k+1}(S^k), k+1)$$

and try to compute the (co)homology of $\tau_{\geq k+2} S^k$ and so forth.