The generalized Segal conjecture and the coalgebraic Frobenius

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Abstract

We prove a generalized Segal conjecture, extending work of Bergsaker and Rognes and confirming a conjecture by the author: for every finite group G and every bounded-below spectrum X, the Hill–Hopkins–Ravenel norm $N_e^G X$ becomes Borel-complete after completion at the augmentation ideal of the Burnside ring. Equivalently, for every non-trivial finite p-group G, the proper Tate diagonal $X \to (X^{\otimes G})^{\varphi G}$ is a p-completion. The proof reduces to the case $X = \mathbb{F}_p$, established by Bergsaker and Rognes, and removes the finiteness hypotheses from their theorem by formal manipulations.

As an application of this result, we refine the coalgebraic Frobenius to an oriented circle action (i.e. an action of the monoidal category $B\mathbb{N}$) on the category of bounded-below, p-complete $\mathbb{E}_\infty$-coalgebras, confirming another conjecture by the author. The construction is based on Yuan's approach to Frobenius actions via partial algebraic K-theory.

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1 Introduction

The classical Segal conjecture, now a theorem of Carlsson [7], is one of the deepest and most mysterious statements in stable homotopy theory. In the last years there has been quite some work on conceptually understanding and (re)proving versions of it. Some of these results, which we will discuss below, have led the author to formulate the generalized Segal conjecture in the ICM proceedings [22, Conjecture 7.10]. In this paper we prove this conjecture using crucial work of Bergsaker–Rognes.

Theorem 1.1. *For every prime p, every non-trivial finite p-group G and every bounded-below spectrum X, the proper Tate diagonal*

$$X \xrightarrow{\Delta_G} (X^{\otimes G})^{\varphi G}$$

exhibits the target as the p-completion of X. Here G acts on $X^{\otimes G} = X \otimes \ldots \otimes X$ by permuting the factors via left multiplication on G and $(-)^{\varphi G}$ denotes the proper Tate construction.

We will prove this statement via a (seemingly) more general completion formulation for arbitrary finite groups, as suggested in [22, Remark 7.11]. We will formulate this using the language of equivariant stable homotopy theory, notably the language of genuine G-spectra and the Hill–Hopkins–Ravenel (HHR) norm [16, §2.2.3].[1] Let $A(G)$ be the Burnside ring and $I(G) = \ker(A(G) \to \mathbb{Z})$ be its augmentation ideal.

Theorem 1.2. *For every finite group G and every bounded-below spectrum X, the $I(G)$-completion of the HHR norm $N_e^G X$ is Borel-complete. In particular we have the equivalence of ordinary spectra*

$$(X^{\otimes G})^{hG} \simeq \left((N_e^G X)^G\right)^\wedge_{I(G)} \ .$$

Theorems 1.1 and 1.2 are equivalent. In Section 2, we explain this equivalence and the $I(G)$-completion appearing here in detail.

Previously known cases

For $X = \mathbb{S}$, the norm $N_e^G\mathbb{S}$ is the genuine G-sphere. Thus Theorem 1.2 specializes to the classical Segal conjecture for finite groups, proved by Carlsson [7], building on important earlier work [18, 14, 1, 21]. In degree zero, the classical conjecture identifies the zeroth stable cohomotopy of BG with the completion of the Burnside ring $A(G)$ at its augmentation ideal.

For $G = C_p$, the proper Tate construction agrees with the ordinary Tate construction. Lunøe–Nielsen and Rognes [20, Theorem 5.13] proved the norm formulation of the generalized Segal conjecture in this case under the additional assumption that $H_*(X;\mathbb{F}_p)$ is finite-dimensional in each degree. The author and Scholze [24, Theorem III.1.7] subsequently removed this finiteness assumption by reducing to the case of $X = \mathbb{F}_p$ which is covered by [20], establishing the theorem for every bounded-below spectrum. Further proofs of the case $X = \mathbb{F}_p$ were given in [15, 8] for $p = 2$ and in [2] for odd primes and recently by the author [23] for all primes. Thus the case $G = C_p$ of the two theorems is already known. The case $G = C_{p^n}$ follows formally from the case $G = C_p$ using the Tate orbit lemma, see [22, discussion following Conjecture 7.10].

Bergsaker–Rognes [6, Theorem 1.1] proved the norm formulation for all finite p-groups under the same finiteness assumption on mod p homology. They also established the $I(G)$-completion formulation for arbitrary finite groups when $\pi_*(X)$ is finitely generated in each degree [6, Theorem 1.2]. In the present paper we remove these finiteness assumptions. Our proof builds on their isotropy-separation reduction [6, Proposition 2.2] and uses their p-group theorem for $X = \mathbb{F}_p$ as its input. We extend this case to all bounded-below spectra using exactness, Postnikov towers, and preservation of products on $\mathbb{F}_p$-vector spaces, following the strategy for $G = C_p$ in [24, Theorem III.1.7]. The isotropy defect used below is precisely the obstruction studied by Bergsaker–Rognes. We use its functorial properties to remove the finiteness assumptions.

We hope to also find an independent proof of the case $X = \mathbb{F}_p$ along the lines of [23] in future work, but for the moment have not succeeded in doing so.

While preparing this paper, we learned from Vignesh Subramanian that he had independently obtained a proof of the generalized Segal conjecture. We also became aware of a note by Wei Yang [26], dated July 28, 2026, which contains another proof.

[1] For earlier multiplicative norm maps, see Greenlees–May [13, §§4–7].

Application: The coalgebraic Frobenius

As an application of these results, we also prove [22, Conjecture 4.5] from the author's ICM address, already envisioned as a potential application there (and also by Yuan in unpublished work, see [27, §1.6]). To state this result precisely recall that an $\mathbb{E}_\infty$-coalgebra in p-complete spectra consists of a p-complete spectrum C with comultiplication $C \to C\widehat{\otimes}_p C$ and counit $C \to \mathbb{S}^\wedge_p$, which are coherently coassociative, counital and cocommutative. We write coCAlg_p for their ∞-category and $\mathrm{coCAlg}_p^{\mathrm{bb}}$ for the full subcategory of coalgebras whose underlying spectra are bounded below. For $C \in \mathrm{coCAlg}_p^{\mathrm{bb}}$, the *coalgebraic Frobenius* is the composite

$$\varphi_p : C \xrightarrow{\Delta^p} (C^{\otimes p})^{hC_p} \xrightarrow{\mathrm{can}} (C^{\otimes p})^{tC_p} \xrightarrow{\Delta_{C_p}^{-1}} C. \tag{1}$$

Here tensor powers are p-completed, the cyclic group acts by permuting the factors, Δ^p is the iterated comultiplication, and can is the canonical map from homotopy fixed points to the Tate construction. The inverse of the Tate diagonal exists by the C_p-case of Theorem 1.1, already proved in [24, Theorem III.1.7]. This is the construction of [22, Definition 4.3].

Conjecture 4.5 of [22] says that this endomorphism refines to an 'oriented circle action' on the category of coalgebras. More precisely let $B\mathbb{N}$ denote the category with one object and endomorphism monoid $(\mathbb{N}, +)$, equipped with the symmetric monoidal structure given by addition. We call it the *oriented circle*. For coalgebras with p-adically finite underlying spectra, a Frobenius action of this category follows by duality from Yuan's theorem for algebras [27, Theorem 6.8]. Combining his strategy with Theorem 1.2 removes this finiteness assumption:

Theorem 1.3. *For every prime p, the coalgebraic Frobenius refines to an action of the monoidal category $B\mathbb{N}$ on $\mathrm{coCAlg}_p^{\mathrm{bb}}$. More precisely, there is a strong monoidal functor*

$$B\mathbb{N} \longrightarrow \big(\mathrm{Fun}(\mathrm{coCAlg}_p^{\mathrm{bb}}, \mathrm{coCAlg}_p^{\mathrm{bb}}), \circ\big)$$

sending the unique object to the identity functor and $1 \in \mathbb{N}$ to the natural transformation φ_p.

As explained in [22, Remark 4.6] and [27, Remark 1.14], the additional content of a $B\mathbb{N}$-action over a natural endomorphism of each object is the coherent commutativity of the Frobenius with itself: the map from $\mathbb{N}$ to endomorphisms of the identity functor refines to a map of $\mathbb{E}_2$-monoids. The proof of Theorem 1.3 occupies Part II, where we use the generalized Segal conjecture to adapt Yuan's construction to coalgebras. The $B\mathbb{N}$-action constructed here will be used in forthcoming joint work with Riedel generalizing Yuan's and Mandell's theorems to non-finite settings.

A $B\mathbb{N}$-action on all bounded-below p-complete $\mathbb{E}_\infty$-coalgebras was also announced by Yuan [27, Remark 1.5 and §1.6].

Notation

Throughout, fibers and pullbacks are understood as (homotopy) fibers and pullbacks. We use the same notation for an abelian group and its associated Eilenberg–MacLane spectrum. Tensor products without further qualifier are always tensor products of spectra (i.e. relative to the sphere $\mathbb{S}$).

We let G be a finite group and write Sp^G for the ∞-category of genuine G-spectra (see e.g. [27, §2] for a review). We refer to those as genuine G-spectra or simply G-spectra. By a spectrum with G-action we simply mean an object of Sp^{BG}. We use the usual notation Y^H and $\Phi^H Y$ for the fixed points and geometric fixed points of a genuine G-spectrum Y. Unless otherwise stated, geometric fixed points are viewed as ordinary spectra. We let $\mathcal{P}$ be the family of proper subgroups of G and $\mathcal{T} = \{e\}$ the trivial family. The classifying G-anima of these families are denoted $E\mathcal{P}$ and $E\mathcal{T} = EG$ as usual.

We let $F(-,-)$ denote the inner hom in G-spectra. For a pointed G-anima K, we write $F(K,Y)$ for $F(\Sigma^\infty K, Y)$ to shorten notation. We write $Y^\wedge_{\mathcal{P}} = F(E\mathcal{P}_+, Y)$ and $Y^\wedge_{\mathcal{T}} = F(EG_+, Y)$ for $\mathcal{P}$-completion and Borel completion. For $G \neq e$, the inclusion $\mathcal{T} \subseteq \mathcal{P}$ gives natural maps $Y \longrightarrow Y^\wedge_{\mathcal{P}} \longrightarrow Y^\wedge_{\mathcal{T}}$. In this case, the first induces equivalences on proper fixed points and the second on the underlying spectrum. The category of Borel-complete G-spectra is equivalent to Sp^{BG}.

For a prime p, we write Sp_p and Sp^G_p for the categories of p-complete spectra and p-complete genuine G-spectra, respectively, equipped with the p-completed tensor product. A genuine spectrum is p-complete precisely if all the fixed point spectra are p-complete. We denote inflation from the trivial group by $\inf^G_e : \mathrm{Sp} \to \mathrm{Sp}^G$.

For a symmetric monoidal ∞-category $\mathcal{V}$, we write

$$\mathrm{coCAlg}(\mathcal{V}) := \mathrm{CAlg}(\mathcal{V}^{\mathrm{op}})^{\mathrm{op}}$$

for its category of counital $\mathbb{E}_\infty$-coalgebras. Generally all coalgebras appearing in this paper are cocommutative and counital.

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AI Statement

I used AI to search and understand the literature, assess the existing results, and refine the proofs and exposition, in particular by generating referee-style reports. Specifically, it was an AI system (Astra) which pointed out that a key step in the argument of Bergsaker–Rognes is the reduction to the p-adic vanishing of $\Psi^G(N^G_e X)$ (using different notation) [6, Proposition 2.2], after which it was easy for me to adapt the strategy from [24, Section III.1] to general groups and prove Theorem 1.2. For the second part, I used AI to improve parts of the proof of Proposition 6.8, which I still find somewhat unsatisfactory and hope to improve in the future.

Part I
The generalized Segal conjecture

We now prove Theorems 1.1 and 1.2. In Section 2, we reduce both statements to the p-group case, where it suffices to show that $(N^G_e X)^\wedge_p$ is Borel-complete. We approach this through the isotropy defect, whose basic properties are established in Section 3. In Section 4, we reduce its p-adic vanishing to the case $X = \mathbb{F}_p$, established by Bergsaker–Rognes [6].

2 Equivalent formulations

Recall that the Burnside ring $A(G) = \pi^G_0 \mathbb{S}_G$ is the group completion of isomorphism classes of finite G-sets under disjoint union. It comes with the augmentation $A(G) \to \mathbb{Z}$ induced

by cardinality. The augmentation ideal is the kernel of this map and is finitely generated $I(G) = (a_1, \dots, a_r)$. For any genuine G-spectrum Y we have

$$Y^\wedge_{I(G)} \simeq \varprojlim_n Y/(a_1^n, \dots, a_r^n),$$

where the quotient denotes iterated cofibers of multiplication by the respective elements. This induces the $I(G)$-completion on all fixed point spectra, using the module structure over $\mathbb{S}^G_G$ by restriction. In particular $Y \to Y^\wedge_{I(G)}$ is an equivalence on underlying spectra, since each a_i restricts to zero at the trivial subgroup. Borel completion commutes with this completion, since it preserves limits and is exact. Thus every Borel-complete spectrum is $I(G)$-complete, and the Borel comparison factors canonically through $Y^\wedge_{I(G)}$:

$$Y \to Y^\wedge_{I(G)} \to Y^\wedge_{\mathcal{T}} \ .$$

For bounded-below Y, the effect of completion on geometric fixed points is given by [25, Theorem 4.1 and Lemma 5.4]:

$$\Phi^H(Y^\wedge_{I(G)}) \simeq \begin{cases} \Phi^e Y = Y^e & \text{if } H = e, \\ (\Phi^H Y)^\wedge_p & \text{if } H \neq e \text{ is a } p\text{-group}, \\ 0 & \text{if } H \text{ is not a } p\text{-group for any } p. \end{cases} \tag{2}$$

We get similar statements for completion at the ideal (p), e.g. that p-completion of G-spectra is simply given by completing all the fixed point spectra. On geometric fixed points we find $\Phi^H(Y^\wedge_p) \simeq (\Phi^H Y)^\wedge_p$ provided that Y is bounded below. In fact a genuine G-spectrum Y is connective if and only if all $\Phi^H Y$ are connective and in this case if all homotopy groups of $\Phi^H Y$ are finite p-groups resp. derived p-complete then the same holds for all homotopy groups of Y^H. Both of these facts are standard and easily follow from the usual isotropy separation sequence by induction on the group order (using that the classifying anima are of finite type). We also note that the notion of being bounded below for a G-spectrum is equivalent to all fixed point spectra (or all geometric fixed point spectra) being bounded below. Whether or not one asks this to be uniform through all subgroups doesn't matter since there are only finitely many subgroups.

Recall also that the proper Tate construction of a spectrum M with H-action is obtained by applying Φ^H to its Borel-complete genuine refinement. It vanishes when H is not a p-group for any prime p [22, Proposition 7.7]. For a non-trivial p-group H, it kills p-invertible spectra as well: the ambidextrous adjunction between induction and restriction gives equivariant maps $M \to \mathrm{Ind}_e^H \mathrm{Res}_e^H M \to M$ whose composite is multiplication by $|H|$ on underlying spectra. If p acts invertibly, this composite is an equivalence, so M is a retract of an induced spectrum, which is killed by proper Tate.

We use the following identities for the HHR norm [6, Proposition 2.1]

$$\mathrm{Res}^G_H N_e^G X \simeq N_e^H(X^{\otimes [G:H]}), \qquad \Phi^H N_e^G X \simeq X^{\otimes [G:H]}. \tag{3}$$

In particular, $N_e^G X$ is bounded below whenever X is bounded below [6, Lemma 6.3].

Proposition 2.1. *Theorem 1.1 holds for all finite p-groups precisely if Theorem 1.2 holds for all finite groups*[2]*. They are also equivalent to the following p-group statement: for every prime p, every finite p-group G and every bounded-below spectrum X, the canonical map*

$$(N_e^G X)^\wedge_p \longrightarrow \big((N_e^G X)^\wedge_p\big)^\wedge_{\mathcal{T}} = \big((N_e^G X)^\wedge_{\mathcal{T}}\big)^\wedge_p \tag{4}$$

is an equivalence.

[2] We do not claim that they are equivalent if we fix the group G. What we precisely show is that Theorem 1.2 implies Theorem 1.1 for a fixed non-trivial finite p-group G and is conversely implied by Theorem 1.1 for all p-subgroups $H \leq G$.

Proof. We consider the completion map $N_e^G X \to (N_e^G X)^\wedge_{\mathcal{T}}$. For $H \leq G$, restriction commutes with Borel completion, so (3) identifies the map on geometric H-fixed points with

$$X^{\otimes[G:H]} \xrightarrow{\Delta_H} ((X^{\otimes[G:H]})^{\otimes H})^{\varphi H},$$

For $H = e$ this is the identity. If Theorem 1.1 holds, then this is a p-completion for H a non-trivial p-group, and its target is zero if H is not a p-group for a prime p. Comparing with (2) shows that $(N_e^G X)^\wedge_{I(G)} \to (N_e^G X)^\wedge_{\mathcal{T}}$ is an equivalence on all geometric fixed points, hence an equivalence. This proves Theorem 1.2.

Conversely assume Theorem 1.2 holds. Then apply Φ^G to that theorem for a non-trivial p-group G. The computation (2) identifies the resulting map with the p-completion of X, giving Theorem 1.1. This shows the equivalence between these two theorems.

Now assume Theorem 1.1 and let G be a p-group. The same calculation as above shows that $N_e^G X \to (N_e^G X)^\wedge_{\mathcal{T}}$ is a mod p equivalence on all geometric fixed points, hence a genuine mod p equivalence. Completing and commuting p-completion with Borel completion gives (4).

Finally, assume (4) and let $G \neq e$ be a p-group. The fiber of $N_e^G X \to (N_e^G X)^\wedge_p$ is p-invertible on underlying naive G-spectra, so proper Tate sends this map to an equivalence. Therefore

$$(X^{\otimes G})^{\varphi G} \simeq \Phi^G\big((N_e^G X)^\wedge_{\mathcal{T}}\big) \simeq \Phi^G\Big(\big((N_e^G X)^\wedge_p\big)^\wedge_{\mathcal{T}}\Big) \simeq \Phi^G\big((N_e^G X)^\wedge_p\big) \simeq X^\wedge_p.$$

The identifications respect the map from $X = \Phi^G N_e^G X$, so the Tate diagonal is the completion map. This proves Theorem 1.1. □

3 The isotropy defect

In this section let G be a non-trivial finite group. The (stable) isotropy separation sequence is

$$\Sigma^\infty_+ E\mathcal{P} \longrightarrow \mathbb{S}_G \longrightarrow \Sigma^\infty \widetilde{E}\mathcal{P}$$

where $\widetilde{E}\mathcal{P}$ is defined as the cofiber of $E\mathcal{P} \to *$. It is an idempotent algebra and we have $\Phi^G Y \simeq (\Sigma^\infty \widetilde{E}\mathcal{P} \otimes Y)^G$ as functors $\mathrm{Sp}^G \to \mathrm{Sp}$. There is a fully faithful embedding

$$\Xi^G : \mathrm{Sp} \longrightarrow \mathrm{Sp}^G, \qquad \Xi^G(X) = \Sigma^\infty \widetilde{E}\mathcal{P} \otimes X,$$

where tensoring with a spectrum X uses the action of spectra on G-spectra (implemented through colimits and stability). Φ^G is left adjoint to Ξ^G. The essential image of Ξ^G consists precisely of those Y for which $Y^H = 0$ for every proper subgroup $H \leqslant G$. On this image, categorical and geometric G-fixed points agree.

The functor Ξ^G admits a further right adjoint, described in [17, §2.6]. We recall its construction and collect the properties needed for our proof.

Definition 3.1. We denote the right adjoint of Ξ^G by Ψ^G and call it the *isotropy defect*. Thus

$$\Psi^G(Y) = F(\widetilde{E}\mathcal{P}, Y)^G \ .$$

Remark 3.2. *Ψ^G is exactly the obstruction also used by Bergsaker–Rognes [6, Proposition 2.2 and Definition 2.3], see a more precise discussion below. They write $\widetilde{E}\mathcal{P} = S^{\infty\rho}$ where $\rho = \mathbb{C}[G] \ominus \mathbb{C}$ is the reduced regular complex representation.*

Since Ψ^G is a right adjoint, it is exact and preserves all limits. In particular, it commutes with p-completion which will be used later. Applying $F(-, Y)$ to isotropy separation gives a fiber sequence

$$\Xi^G \Psi^G(Y) \longrightarrow Y \longrightarrow Y^\wedge_{\mathcal{P}} \tag{5}$$

which we call the coisotropy sequence. Thus $\Psi^G(Y) = 0$ if and only if Y is $\mathcal{P}$-complete, hence the name 'isotropy defect'. In particular, $\Psi^G(Y^\wedge_{\mathcal{T}}) = 0$.

Proposition 3.3. *For a proper subgroup $H \lneqq G$, the functor Ψ^G vanishes on $\mathrm{Ind}_H^G Z$. It preserves direct sums of genuine spectra with a common upper bound on their homotopy groups.*

Proof. Induction agrees with coinduction, so $F(E\mathcal{P}_+, \mathrm{Ind}_H^G Z) \simeq \mathrm{Ind}_H^G Z$, since the restriction of $E\mathcal{P}$ to H is contractible. The vanishing follows from (5).

Choose a G-CW model of $\widetilde{E}\mathcal{P}$ with finitely many cells in each dimension. For $Y \in \mathrm{Sp}^G_{\leq n}$ and fixed k, the group $\pi_k F(\widetilde{E}\mathcal{P}, Y)^G$ is computed from a finite skeleton, whose size depends only on n and k. Thus Ψ^G preserves direct sums with a common upper bound. □

To relate the isotropy defect to the proof strategy in [24, Section III.1], consider the functor

$$T_G : \mathrm{Sp} \longrightarrow \mathrm{Sp}, \qquad T_G(X) = \Phi^G\big((N_e^G X)^\wedge_{\mathcal{P}}\big).$$

This is the 'correct' generalization of the functor T_p considered there. Applying Φ^G to (5) and using (3) we get a natural fiber sequence

$$\Psi^G(N_e^G X) \longrightarrow X \longrightarrow T_G(X).$$

The proper Tate diagonal is the composite of $X \to T_G(X)$ with the natural comparison $T_G(X) \to (X^{\otimes G})^{\varphi G}$.

The significance of these functors is the following isotropy-separation reduction argument of Bergsaker–Rognes [6, Proposition 2.2], following Carlsson [7, Theorem A(b)] and Caruso–May–Priddy [9, Lemma 1.9]. We recall the argument, which applies here without finite-type assumptions.

Proposition 3.4. *The p-group statement (4) and hence Theorems 1.1 and 1.2 are equivalent to the assertion that $\Psi^G(N_e^G X)$ is p-adically trivial for each bounded-below spectrum X and each non-trivial finite p-group G. Equivalently, the map $X \to T_G(X)$ is a p-adic equivalence for all such G and X.*

Proof. If the p-completion of the norm is Borel-complete, then it is also $\mathcal{P}$-complete so that by the coisotropy sequence (5) the isotropy defect vanishes p-adically (using that Ψ^G commutes with p-completion).

Conversely, if $\Psi^G(N_e^G X)^\wedge_p = \Psi^G((N_e^G X)^\wedge_p) = 0$, then the spectrum $(N_e^G X)^\wedge_p$ is $\mathcal{P}$-complete. To show that it is Borel-complete it therefore suffices to show that the map

$$N_e^G X \to (N_e^G X)^\wedge_{\mathcal{T}}$$

is a p-adic equivalence on all fixed point spectra for proper subgroups, i.e. after restriction to proper subgroups. This follows by induction on $|G|$, the case $G = e$ being automatic, since these restrictions are themselves norms by (3). □

4 The vanishing of the defect

We now prove the p-adic vanishing of $\Psi^G(N_e^G X)$ by successive reductions, following the strategy of the proof of [24, Theorem III.1.7]. Our presentation avoids the explicit use of pro-isomorphisms. Instead, we use preservation of infinite products to pass from $\mathbb{F}_p$ to arbitrary $\mathbb{F}_p$-vector spaces, each of which is a retract of a product of copies of $\mathbb{F}_p$. The idea of reformulating the C_p-proof in this way is due to Phil Pützstück. Here we adapt it to arbitrary finite p-groups. It is however not very complicated to rewrite it in a way closer to [24] using pro-isomorphisms. We leave the details to the reader.

All arguments in this section are formal and elementary. The (hard) result of Bergsaker–Rognes [6, Theorem 1.1] is invoked only at the very last step to establish the vanishing for $\mathbb{F}_p$.

Lemma 4.1. *The functor $X \mapsto \Psi^G(N_e^G X)$ is exact.*

Proof. For a cofiber sequence $A \to B \to C$, the tensor-product cube gives a finite filtration of $N_e^G B$, starting with $N_e^G A$ and with final layer $N_e^G C$. Every intermediate layer is induced from stabilizers of nonempty proper subsets of the transitive G-set G. These stabilizers are proper subgroups, so Ψ^G kills the intermediate layers. Thus the sequence

$$\Psi^G(N_e^G A) \longrightarrow \Psi^G(N_e^G B) \longrightarrow \Psi^G(N_e^G C)$$

is a cofiber sequence. □

Lemma 4.2. *For a bounded-below spectrum X, the natural maps*

$$N_e^G X \xrightarrow{\simeq} \varprojlim_n N_e^G(\tau_{\leq n} X) \qquad \text{and} \qquad \Psi^G(N_e^G X) \xrightarrow{\simeq} \varprojlim_n \Psi^G\big(N_e^G(\tau_{\leq n} X)\big)$$

are equivalences.

Proof. By (3) and the usual yoga for total fibers the connectivity of the fibers on geometric fixed points of $N_e^G X \to N_e^G(\tau_{\leq n} X)$ tends to ∞, hence also on fixed points as recalled in Section 2. This shows the first claim. The second follows by limit preservation of Ψ^G. □

To prove preservation of products, we truncate the norm in genuine G-spectra. Since Ψ^G preserves limits, Postnikov completeness gives

$$\Psi^G(N_e^G X) \simeq \varprojlim_n \Psi^G(\tau_{\leq n} N_e^G X). \tag{6}$$

Lemma 4.3. *Every group $\pi_k \Psi^G(\tau_{\leq n} N_e^G \mathbb{F}_p)$ is a finite p-group.*

Proof. Finite tensor powers of $\mathbb{F}_p$ are connective and of finite type. By (3) and the facts recalled in Section 2, all fixed points of $N_e^G \mathbb{F}_p$ have finite p-groups as homotopy groups. The same holds for $Y = \tau_{\leq n} N_e^G \mathbb{F}_p$. Since Y is bounded above, $\pi_k \Psi^G(Y)$ is computed from a finite skeleton of $\widetilde{E}\mathcal{P}$, as in the proof of Proposition 3.3. Hence it is a finite p-group. □

Lemma 4.4. *The functor $V \mapsto \Psi^G(N_e^G V)$ from $\mathbb{F}_p$-vector spaces to spectra preserves arbitrary products.*

Proof. For a family $(V_i)_{i \in I}$ of vector spaces, distribute the norm of $\bigoplus_i V_i$ out. The constant functions $G \to I$ give the individual norms. All other summands are induced from proper stabilizers. Genuine Postnikov truncation commutes with direct sums and induction, so the truncated mixed summands are still induced from proper subgroups. They now have a common upper bound. Proposition 3.3 therefore gives

$$\Psi^G\left(\tau_{\leq n} N_e^G\left(\bigoplus_i V_i\right)\right) \simeq \bigoplus_i \Psi^G(\tau_{\leq n} N_e^G V_i).$$

Thus $V \mapsto \pi_k \Psi^G(\tau_{\leq n} N_e^G V)$ is infinitely additive and hence of the form

$$\pi_k \Psi^G(\tau_{\leq n} N_e^G V) \cong \pi_k \Psi^G(\tau_{\leq n} N_e^G \mathbb{F}_p) \otimes_{\mathbb{F}_p} V.$$

Here additivity gives the $\mathbb{F}_p$-vector space structure on the coefficient, which is finite-dimensional by Lemma 4.3. Thus every functor $V \mapsto \Psi^G(\tau_{\leq n} N_e^G V)$ preserves arbitrary products. The claim then follows from (6), since limits commute with products. □

Proof of Theorem 1.2. By Proposition 3.4, we need to prove that $\Psi^G(N_e^G X)/p = 0$ for each bounded-below X and each non-trivial p-group G. Using the Postnikov tower and Lemmas 4.1 and 4.2 reduces this to X concentrated in a single degree represented by an abelian group A. Then we can commute the mod p reduction inside and get an extension of the p-torsion in A and the non-derived mod p-reduction of A, both of which are $\mathbb{F}_p$-vector spaces.

Thus it suffices to show that $\Psi^G(N_e^G V) = 0$ for an $\mathbb{F}_p$-vector space V. Bergsaker–Rognes' result [6, Theorem 1.1] shows using Proposition 3.4 that $\Psi^G(N_e^G \mathbb{F}_p) = 0$. We use Lemma 4.4 to deduce that it vanishes on arbitrary products of $\mathbb{F}_p$. But then every $\mathbb{F}_p$-vector space $V = \bigoplus_{i\in I} \mathbb{F}_p$ is a retract of $\prod_{i\in I} \mathbb{F}_p$. Hence $\Psi^G(N_e^G V)$ also vanishes. □

Part II
The coalgebraic Frobenius

We now turn to the proof of Theorem 1.3, i.e. the construction of the oriented circle action on the category of bounded-below, p-complete coalgebras. Fix a prime p. Our input from Part I is the following consequence of the generalized Segal conjecture (Theorem 1.2): the p-completed norm of every bounded-below spectrum is Borel-complete for every finite p-group, by Proposition 2.1 and (4). We first use this to construct an adjunction on coalgebras. In Section 6 and Section 7 we assemble these adjunctions into coherent norm systems, and in Section 8 we apply Yuan's construction to obtain the Frobenius action.

5 The Frobenius through genuine coalgebras

We consider cocommutative coalgebras in genuine G-spectra Sp_p^G and refer to them as genuine coalgebras. Such a coalgebra has genuine equivariant comultiplication and counit maps $A \to A\otimes A$ and $A \to \mathbb{1}$, with the usual $\mathbb{E}_\infty$-coherences. No 'normed' or 'ultracommutative' structure is included in this definition. In particular, maps into indexed norms are not part of the data.

In this and the following sections, tensor products, inflation $\inf_e^G$, norms N_H^G, and geometric fixed points are taken in the p-complete sense. Thus $N_H^G Y$ means $(N_H^G Y)_p^\wedge$ and $\Phi^H Y$ means $(\Phi^H Y)_p^\wedge$. We will also consider genuine residual actions on geometric fixed points. For example Φ^H refines to a functor from genuine $(G \times H)$-spectra to genuine G-spectra.

Norms, inflation, and geometric fixed points preserve mod p equivalences. Hence the usual norm and base-change identities [4, §9.2] descend to the p-complete setting. These functors are also strong symmetric monoidal and hence induce functors on coalgebras. The tensor product of coalgebras is their categorical product, and the tensor unit is terminal, see e.g. [3, §2]. In particular, the induced functors preserve finite products.

Let G be a finite p-group, and write U for the forgetful functor from genuine spectra to spectra with G-action, i.e. functors BG to Sp.

Lemma 5.1. *Let $B \in \mathrm{coCAlg}(\mathrm{Sp}_p^G)$ be such that the underlying spectrum of $B^{\otimes n}$ is Borel-complete for every $n \geq 0$[3]. Then, for every $A \in \mathrm{coCAlg}(\mathrm{Sp}_p^G)$, restriction induces a natural equivalence*

$$\mathrm{Map}_{\mathrm{coCAlg}(\mathrm{Sp}_p^G)}(A, B) \simeq \mathrm{Map}_{\mathrm{Fun}(BG,\mathrm{coCAlg}_p)}(UA, UB).$$

In particular, restriction is fully faithful on the full subcategory of coalgebras satisfying this condition.

[3]For $n = 0$ this is the p-complete genuine sphere which is Borel-complete under our assumptions and conventions.

Proof. Using the operadic description, we can describe the mapping anima between $\mathbb{E}_\infty$-coalgebras as limits of mapping anima between tensor powers. All target tensor powers are Borel-complete, so restriction identifies every term with its counterpart in $\mathrm{Fun}(BG, \mathrm{Sp}_p)$. Taking the operadic limits gives the asserted equivalence. The case $n = 0$ includes the compatibility with the counit. □

We continue to let G be a finite p-group.

Lemma 5.2. *For any $A \in \mathrm{coCAlg}(\mathrm{Sp}_p^G)$ and any bounded-below p-complete coalgebra B, there is a natural equivalence*

$$\mathrm{Map}_{\mathrm{coCAlg}(\mathrm{Sp}_p^G)}(A, N_e^G B) \simeq \mathrm{Map}_{\mathrm{coCAlg}_p}(\mathrm{Res}_e^G A, B). \tag{7}$$

In particular, restriction is left adjoint to the norm on bounded-below coalgebras.

Proof. Every tensor power $B^{\otimes n}$ is bounded below. Thus $(N_e^G B)^{\otimes n} = N_e^G(B^{\otimes n})$ is Borel-complete by (4). Lemma 5.1 therefore identifies mapping anima from any A into $N_e^G B$ with the counterpart after applying U.

Using that the tensor product in counital cocommutative coalgebras is the categorical product we see that the underlying naive coalgebra $B^{\otimes G}$, with its permutation action, is coinduced. This identifies the remaining mapping anima with $\mathrm{Map}_{\mathrm{coCAlg}_p}(\mathrm{Res}_e^G A, B)$. This proves (7), naturally in both variables. Since restriction, inflation and the norm preserve bounded-below objects, this also gives the adjunction. □

If η denotes the unit of the previous adjunction, we get for every bounded-below, p-complete coalgebra C a canonical map

$$\bar{\Delta}_G := \eta_{\inf_e^G C} : \quad \inf_e^G C \longrightarrow N_e^G \mathrm{Res}_e^G \inf_e^G C = N_e^G C \ .$$

Concretely under the adjunction between inflation and fixed points this corresponds to a map of spectra $C \to (N_e^G C)^G$, which by completeness of the target is simply the map induced by comultiplication $\Delta^G : C \to (C^{\otimes G})^{hG}$. For a finite-dimensional $\mathbb{F}_p$-vector space V, viewed as an elementary abelian p-group, define

$$\varphi_V := \Phi^V \bar{\Delta}_V : \quad \mathrm{id}_{\mathrm{coCAlg}_p^{\mathrm{bb}}} \to \mathrm{id}_{\mathrm{coCAlg}_p^{\mathrm{bb}}}$$

using $\Phi^V \inf_e^V = \mathrm{id}$ and $\Phi^V N_e^V = \mathrm{id}$. We also define $\varphi_0 := \mathrm{id}$. For $V \neq 0$, comparison with Borel completion identifies its underlying map as

$$C \xrightarrow{\Delta^V} (C^{\otimes V})^{hV} \xrightarrow{\mathrm{can}} (C^{\otimes V})^{\varphi V} \xrightarrow{\Delta_V^{-1}} C. \tag{8}$$

Here φV denotes proper Tate fixed points and Δ_V is the proper Tate diagonal, which is an equivalence here by Theorem 1.1. For $V = \mathbb{F}_p$, the proper Tate construction agrees with the ordinary Tate construction, so $\varphi_{\mathbb{F}_p} = \varphi_p$ is precisely the coalgebraic Frobenius of (1).

Example 5.3. *For $V = \mathbb{F}_p \oplus \mathbb{F}_p$, the endomorphism φ_V is the composite of the coalgebraic Frobenius with itself:*

$$\varphi_{\mathbb{F}_p \oplus \mathbb{F}_p} \simeq \varphi_p \circ \varphi_p.$$

This follows since the V-fold comultiplication is obtained by applying the p-fold comultiplication twice. Using Lemma 5.2 gives the corresponding factorization of its genuine lift. Taking geometric fixed points successively in the two $\mathbb{F}_p$-directions thus shows the assertion, see also [27, Examples 2.11 and 2.12].

The idea of our proof of Theorem 1.3 (inspired by Yuan's work) is to first make such identifications $\varphi_{\mathbb{F}_p \oplus \mathbb{F}_p} \simeq \varphi_p \circ \varphi_p$ coherently compatible with maps and direct sums of vector spaces, thus isolating the ways in which the coalgebraic Frobenius "commutes with itself". This yields a monoidal action of a category built from finite-dimensional $\mathbb{F}_p$-vector spaces on $\mathrm{coCAlg}_p^{\mathrm{bb}}$, precisely the monoidal category $Q\mathrm{Vect}^{\mathrm{fd}}_{\mathbb{F}_p}$ discussed at the beginning of Section 8. Then, following Yuan, we use his partial K-theory analogue of Quillen's computation for $\mathbb{F}_p$ [27, Theorem 5.2], together with p-completeness, to descend this action to the oriented circle, i.e. to obtain an action of $B\mathbb{N}$.

6 Coalgebraic norm systems

We now recall the correspondence categories from Yuan's paper [27], with some modifications that we need for the present paper. Generally we review what we need from the work of Yuan, but assume familiarity with the ideas and methods.

We work with finite groupoids with elementary abelian p-isotropy, i.e. finite disjoint unions (possibly empty) of BV for finite-dimensional $\mathbb{F}_p$-vector spaces V, including $V = 0$. Here BV denotes the classifying anima of the additive group of V. We consider those as a $(2,1)$-category, defined as a full subcategory of the category of anima. A *finite covering* is a map between such groupoids whose fibers are finite discrete anima. For example, $* \to BV$ is a finite covering, but $BV \to *$ is not (unless $V = 0$).

Definition 6.1. The category Corr_p has these groupoids as objects and spans

$$a = (X \xleftarrow{f} M \xrightarrow{g} Y)$$

with g a finite covering as morphisms. Composition is by pullback. Formally this is a subcategory of the span category of anima (see e.g. [11, §2]).

Definition 6.2. The category Corr_p^+ has the same objects, but its morphisms are spans for which the joint map $(f,g) : M \to X \times Y$ is a finite covering. To compose $X \leftarrow M \to Y$ and $Y \leftarrow N \to Z$, factor the natural map as

$$M \times_Y N \longrightarrow P \longrightarrow X \times Z,$$

where the first map has connected fibers and the second is a finite covering. The composite is $X \leftarrow P \to Z$. Thus P is the relative 0-truncation of $M \times_Y N$ over $X \times Z$. This composition restricts to the usual composition on Corr_p, giving a wide subcategory $\mathrm{Corr}_p \subseteq \mathrm{Corr}_p^+$. A precise construction of this $(2,1)$-category is given in [27, Definition 2.13 and Convention 2.17], from which we obtain Corr_p^+ by restricting to the groupoids considered here.

Definition 6.3. The category Corr_p^* has pointed groupoids (X, x) as objects and spans in Corr_p with a chosen point in M and point-preserving maps to X and Y as morphisms. Composition is by pullback, i.e. this is a subcategory of the span category of pointed anima.

These are $(2,1)$-categories but we regard them as ∞-categories here. We call a morphism in Corr_p or Corr_p^* a backward morphism if $g = \mathrm{id}$ and a forward morphism if $f = \mathrm{id}$. We have functors

$$\mathrm{Corr}_p^* \longrightarrow \mathrm{Corr}_p \subseteq \mathrm{Corr}_p^+ .$$

Cartesian products of groupoids (which are not categorical products in the span categories) give symmetric monoidal structures on these categories so that these functors are symmetric monoidal.

Remark 6.4. *Our* Corr_p *and* Corr_p^+ *are the full subcategories of Yuan's* Glo *and* Glo^+*, respectively, on groupoids with elementary abelian p-isotropy [27, Variant 2.5 and Definition 2.20].* Corr_p^* *has not explicitly appeared in Yuan's work. We warn the reader that* Corr_p^* *is not pointed objects in* Corr_p *as the latter is already pointed.*

Construction 6.5. *We now assemble genuine coalgebras into a category over* Corr_p^**. For* $X \simeq \coprod_i BG_i$*, write* $\mathrm{Sp}_p^X := \prod_i \mathrm{Sp}_p^{G_i}$ *for the category of p-complete genuine spectra over X. The proof of [27, Theorem 2.18] assembles genuine spectra into a coCartesian fibration over* Corr_p^+ *with symmetric monoidal fibers and strong symmetric monoidal transition functors. Localizing this system at the mod p-equivalences as in Section 5 and applying* coCAlg *fiberwise gives a coCartesian fibration*

$$q^+ : \mathcal{E}^+ \longrightarrow \mathrm{Corr}_p^+, \qquad \mathcal{E}_X^+ = \mathrm{coCAlg}(\mathrm{Sp}_p^X).$$

The fiber transport along a span $(X \xleftarrow{f} M \xrightarrow{g} Y)$ *is*

$$g_\otimes f^* : \mathrm{coCAlg}(\mathrm{Sp}_p^X) \longrightarrow \mathrm{coCAlg}(\mathrm{Sp}_p^Y),$$

where f^* *is restriction and* $g_\otimes$ *is the multiplicative pushforward along g (which specializes to norms for finite coverings, in particular for subgroup inclusions, and to geometric fixed points for quotient homomorphisms, see [27, §2.1]). Pullback along the functor* $\mathrm{Corr}_p^* \to \mathrm{Corr}_p^+$ *yields a coCartesian fibration*

$$q : \mathcal{E} \longrightarrow \mathrm{Corr}_p^*, \qquad \mathcal{E}_{(X,x)} = \mathrm{coCAlg}(\mathrm{Sp}_p^X). \tag{9}$$

Thus an object of $\mathcal{E}$ *is a pointed groupoid* (X, x) *together with a genuine X-coalgebra and a morphism is a span of pointed groupoids* (f, g) *together with a coalgebra map* $g_\otimes f^* A \to B$*. Here g is a finite covering, so that* $g_\otimes$ *is indeed the norm for a covering.*

Definition 6.6. A *coalgebraic norm system*, or simply a *norm system*, is a section of q (as in (9)) which is Cartesian over backward morphisms. We write $\mathrm{NormSys}_p$ for the full ∞-category of these sections.

Remark 6.7.

- *Informally, a norm system packages coalgebras for all elementary abelian p-groups into one object: it assigns a genuine V-coalgebra* $s(BV)$ *to each such group V, together with maps relating these coalgebras under restriction and norms. Proposition 6.8 below shows that, when* $C = s(*)$ *is bounded below, this entire coherent system is determined by C: it is a coherent system of norm powers, with* $s(BV) \simeq N_e^V C$*. Therefore we think of these systems as controlling the coherence and functoriality of norms of elementary abelian p-groups.*

- *Unlike Yuan's global algebras [27, Definition 3.19], which inspired our definition, our norm systems are Cartesian rather than coCartesian over backward morphisms. Pointings are essential here: over* Corr_p*, Cartesianity over the backward spans* $X \leftarrow \varnothing \to \varnothing$ *would force every* $s(X)$ *to be terminal, since the fiber over* $\varnothing$ *is terminal. The chosen points exclude these spans and provide the finite coverings* $x : * \to X$ *used to form the norm powers below.*

Explicitly, for a pointed map $f : X \to Y$, Cartesianity over the backward morphism means that the section map $f^* s(Y) \to s(X)$ induces an equivalence

$$\mathrm{Map}_{\mathrm{coCAlg}(\mathrm{Sp}_p^Y)}(A, s(Y)) \simeq \mathrm{Map}_{\mathrm{coCAlg}(\mathrm{Sp}_p^X)}(f^* A, s(X))$$

for every A. Note that q is not a Cartesian fibration. In the following we also use forward coCartesianity over forward morphisms associated with a finite pointed covering $g : X \to Y$.

This then concretely means that $g_{\otimes}s(X) \to s(Y)$ is an equivalence. We will see below that this is automatic for a norm system, provided that $s(*)$ is bounded below.

For a pointed groupoid (X, x), the point $x : * \to X$ is a finite covering. Write $N_x := x_{\otimes}$ for the norm along x. On the marked component BG, $N_x C$ is $N_e^G C$, and on every other component it is the terminal coalgebra $\mathbb{1}$. Lemma 5.2 gives the equivalence

$$\mathrm{Map}_{\mathrm{coCAlg}(\mathrm{Sp}_p^X)}(A, N_x C) \simeq \mathrm{Map}_{\mathrm{coCAlg}_p}(x^* A, C) \tag{10}$$

for every genuine X-coalgebra A and bounded-below C. Its counit is projection to the distinguished factor of $x^* N_x C \simeq C^{\otimes \Omega X}$.

The main result about norm systems is the following:

Proposition 6.8. *Let* $\mathrm{NormSys}_p^{\mathrm{bb}}$ *denote the full subcategory of norm systems s for which $s(*)$ is bounded below. Evaluation at the point induces an equivalence*

$$\mathrm{ev}_* : \mathrm{NormSys}_p^{\mathrm{bb}} \xrightarrow{\simeq} \mathrm{coCAlg}_p^{\mathrm{bb}}.$$

Its inverse sends C to a section with value $N_x C$ at (X, x). Every such section is also coCartesian over forward morphisms.

7 Proof of Proposition 6.8

The proof has three steps. First, we prove a general extension statement for sections over span categories (Lemmas 7.1 and 7.2), then apply it to coalgebras in spectra with group action, i.e. 'naive' coalgebras (Lemma 7.3). For bounded-below coalgebras, we then use the generalized Segal conjecture in the form (4) and Lemma 5.1 to lift this system uniquely to genuine coalgebras finishing the proof of Proposition 6.8.

We first need a general description of Cartesian sections in a coCartesian fibration.

Lemma 7.1. *Let $q : \mathcal{E} \to B$ be a coCartesian fibration classified by the functor $F : B \to \mathrm{Cat}_\infty$. The assignment*

$$s \longmapsto (F_s \to F), \qquad F_s(b) = F(b)_{/s(b)},$$

identifies the category of sections that are q-Cartesian over a chosen class of morphisms in B with the full subcategory of $\mathrm{Fun}(B, \mathrm{Cat}_\infty)_{/F}$ consisting of natural transformations whose components are representable right fibrations and whose naturality squares over those morphisms are pullbacks.

Proof. Let $\mathrm{RSl} \subseteq \mathrm{Fun}(\Delta^1, \mathrm{Cat}_\infty)$ be the full subcategory of representable right fibrations. By [10, Corollaries 6.4–6.5], evaluation at the target $\mathrm{RSl} \to \mathrm{Cat}_\infty$ is the universal coCartesian fibration, so there is an equivalence $\mathcal{E} \simeq B \times_{\mathrm{Cat}_\infty} \mathrm{RSl}$ over B. Applying $\mathrm{Fun}(B, -)$, which preserves pullbacks, and taking the fiber over id_B gives

$$\mathrm{Fun}_{/B}(B, \mathcal{E}) \simeq \mathrm{Fun}(B, \mathrm{RSl}) \times_{\mathrm{Fun}(B, \mathrm{Cat}_\infty)} \{F\}.$$

The left-hand side is the category of sections. Since a functor into an arrow category is a natural transformation, the right-hand side is the full subcategory of $\mathrm{Fun}(B, \mathrm{Cat}_\infty)_{/F}$ whose components are representable right fibrations. Under this equivalence, s corresponds to $F_s \to F$. This correspondence is compatible with morphisms of sections.

The naturality square of $F_s \to F$ at a is a pullback precisely when

$$F(b)_{/s(b)} \longrightarrow F(b) \times_{F(c)} F(c)_{/s(c)}$$

is an equivalence. These are right fibrations over $F(b)$, so this can be tested on the fibers, where the comparison is

$$\mathrm{Map}_{F(b)}(z, s(b)) \longrightarrow \mathrm{Map}_{F(c)}(F(a)z, s(c)).$$

Equivalence for every z is exactly the q-Cartesian condition on $s(a)$. □

We now use methods for extending functors to span categories developed in the theory of six-functor formalisms, specifically [12, Theorem 3.3]. To this end let D be a category with pullbacks and P a class of morphisms containing all equivalences and closed under pullbacks and composition. Suppose that $g, gf \in P$ implies $f \in P$, and that each morphism in P is n-truncated for some n that is allowed to depend on the morphism. Let $F : D^{\mathrm{op}} \to \mathrm{Cat}_\infty$ be a functor. We write $f^* = F(f)$ and assume that f^* has a right adjoint f_* for $f \in P$ and that these adjoints satisfy Beck–Chevalley for pullback squares. Let

$$q : \mathcal{E} \longrightarrow \mathrm{Span}_{\mathrm{all},P}(D)$$

be the coCartesian fibration associated to the span extension $\overline{F}$ of F, with transport $g_* f^*$ along the span $X \xleftarrow{f} M \xrightarrow{g} Y$ (using [5, Section 11] or [11, Corollary 2.6(3)]).

Lemma 7.2 (Extension to spans)**.** *Restriction to D^{op} induces an equivalence between the full subcategories of sections of q and $q|_{D^{\mathrm{op}}}$ that are, in both cases, Cartesian over backward morphisms associated to maps in P. Every such section of q is also coCartesian over all forward morphisms.*

This equivalence restricts to an equivalence between the full subcategories of sections Cartesian over all backward morphisms.

Proof. We want to apply the first equivalence of [12, Theorem 3.3] with $E = P$ and I the equivalences in D. The assumptions on P make this a Nagata setup in the sense of [12, Definition 2.1]. Thus the cited theorem gives an equivalence between certain subcategories of $\mathrm{Fun}(\mathrm{Span}_{\mathrm{all},P}(D), \mathrm{Cat}_\infty)$ and $\mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Cat}_\infty)$.

By Lemma 7.1, the categories of sections of q and $q|_{D^{\mathrm{op}}}$ are equivalent to full subcategories of $\mathrm{Fun}(\mathrm{Span}_{\mathrm{all},P}(D), \mathrm{Cat}_\infty)_{/\overline{F}}$ and $\mathrm{Fun}(D^{\mathrm{op}}, \mathrm{Cat}_\infty)_{/F}$, respectively. We claim that the subcategories encoding sections Cartesian over backward morphisms associated to maps in P correspond under the equivalence induced by [12, Theorem 3.3] on the slice categories over $\overline{F}$ and F.

For $f : X \to Y$ in P, backward Cartesianity says that the adjoint structure map $s(Y) \to f_* s(X)$ is an equivalence. Hence f_* induces a right adjoint on slices, compatibly with the forgetful functors. Using their conservativity, one checks that the required mate conditions reduce to Beck–Chevalley for F, including stable (Pr-II) of [12, Definition 2.6]. The same argument applies to maps between sections. Conversely, right adjointability of the forgetful squares implies backward Cartesianity by evaluation at terminal objects.

The equivalence therefore restricts as claimed, since restriction changes neither objectwise representability nor the conditions on backward morphisms. Forward transport on slices is right adjoint to backward transport and thus preserves terminal objects, proving forward coCartesianity. □

We now specialize to naive coalgebras. Let D be the category of pointed finite groupoids with elementary abelian p-isotropy and P the finite coverings, so that $\mathrm{Corr}^*_p = \mathrm{Span}_{\mathrm{all},P}(D)$. Consider

$$F(X) = \mathrm{coCAlg}(\mathrm{Fun}(X, \mathrm{Sp}_p)) \simeq \mathrm{Fun}(X, \mathrm{coCAlg}_p),$$

where $\mathrm{Fun}(X, \mathrm{Sp}_p)$ carries the pointwise tensor product. Restriction along $f : X \to Y$ gives $f^* : F(Y) \to F(X)$. For $f \in P$, its right adjoint f_* on coalgebras is computed by products

in coCAlg_p over the finite fibers, hence by tensor products of coalgebras, and satisfies Beck–Chevalley. The resulting coCartesian fibration $q_{\mathrm{naive}} : \mathcal{E}_{\mathrm{naive}} \to \mathrm{Corr}_p^*$ has transport $g_* f^*$ along a span $X \xleftarrow{f} M \xrightarrow{g} Y$.

Lemma 7.3. *Evaluation at $*$ induces an equivalence from the full category of backward Cartesian sections of q_{naive} to coCAlg_p. Its inverse sends C to a section σ_C with $\sigma_C(X, x) = x_* C$. These sections are automatically coCartesian over forward morphisms.*

Proof. Finite coverings satisfy the hypotheses of Lemma 7.2, which reduces the claim to Cartesian sections of $q_{\mathrm{naive}}|_{D^{\mathrm{op}}}$ and gives forward coCartesianity. Since $*$ is terminal in D^{op} and every pointing $x : * \to X$ is a finite covering, relative right Kan extension from $*$ exists and has values $x_* C$. By cancellation of Cartesian morphisms, these extensions are precisely the Cartesian sections. Hence evaluation at $*$ is an equivalence [19, Proposition 4.3.2.15 and Remark 4.3.2.18]. □

We note that under the previous equivalence for a pointed map $f : (X, x) \to (Y, y)$, the backward structure map $f^* y_* C \to x_* C$ is adjoint to the counit

$$x^* f^* y_* C \simeq y^* y_* C \longrightarrow C.$$

Proof of Proposition 6.8. Let $\mathcal{E}_{\mathrm{B}} \subseteq \mathcal{E}$ (here the latter as in (9)) be the full subcategory on objects whose underlying genuine spectra and all their finite tensor powers, including the zeroth power, are Borel-complete. Restriction to spectra with action and applying coCAlg gives a functor over Corr_p^*

$$r : \mathcal{E}_{\mathrm{B}} \longrightarrow \mathcal{E}_{\mathrm{naive}},$$

see also [27, Lemma 3.29] for the analogous relative Borel-completion functor on spectra. This functor is fully faithful on the total categories since over a span $a = (X \xleftarrow{f} M \xrightarrow{g} Y)$, Lemma 5.1, applied componentwise, gives

$$\mathrm{Map}_{\mathrm{coCAlg}(\mathrm{Sp}_p^Y)}(g_\otimes f^* A, B) \simeq \mathrm{Map}_{\mathrm{coCAlg}(\mathrm{Fun}(Y, \mathrm{Sp}_p))}(g_* f^* r A, r B).$$

Here restriction identifies the genuine norm with the indexed tensor product, which is the indexed product in coCAlg_p. The comparison only requires completeness of target tensor powers. It does not require $g_\otimes f^* A$ to belong to $\mathcal{E}_{\mathrm{B}}$.

For bounded-below C, the generalized Segal conjecture (4) applied on each component of X implies that $(N_x C)^{\otimes n} \simeq N_x(C^{\otimes n})$ is Borel-complete for each $n \geq 0$. Moreover, $r(N_x C) \simeq x_* C$. Thus every value of the naive section σ_C belongs to the essential image of r. Full faithfulness therefore lifts the entire section.

The lifted section is backward Cartesian since its backward maps lift the backward structure maps described after Lemma 7.3. The adjunction (10) identifies their universal property, for every genuine A, as

$$\mathrm{Map}_{\mathrm{coCAlg}(\mathrm{Sp}_p^Y)}(A, N_y C) \simeq \mathrm{Map}_{\mathrm{coCAlg}_p}(y^* A, C) \simeq \mathrm{Map}_{\mathrm{coCAlg}(\mathrm{Sp}_p^X)}(f^* A, N_x C).$$

Conversely, if s is a backward Cartesian genuine section with $C = s(*)$ bounded below, its backward point maps identify $s(X) \simeq N_x C$. These identifications carry the backward point maps to the point-adjunction counits, whose restrictions are the usual coinduction counits. Hence s takes values in $\mathcal{E}_{\mathrm{B}}$, and $r(s)$ is Cartesian over backward point maps. Every backward map followed by the backward point map is again a backward point map, so cancellation shows that $r(s)$ is backward Cartesian everywhere.

Since a fully faithful functor induces fully faithful functors on diagram categories, this identifies the genuine sections in question with the naive backward Cartesian sections whose

value at $*$ is bounded below. Lemma 7.3 therefore proves the claimed evaluation equivalence. Finally, for a pointed finite covering $g : (X, x) \to (Y, y)$, transitivity of norms gives $g_{\otimes} N_x C \simeq N_y C$. Thus the source and target of the section's structure map belong to $\mathcal{E}_{\mathrm{B}}$. This map is an equivalence naively by forward coCartesianity, hence genuinely by full faithfulness. This proves automatic forward coCartesianity as well. □

8 The oriented circle action

Write $Q\mathrm{Vect}^{\mathrm{fd}}_{\mathbb{F}_p}$ for Quillen's Q-construction applied to the exact category $\mathrm{Vect}^{\mathrm{fd}}_{\mathbb{F}_p}$ of finite-dimensional $\mathbb{F}_p$-vector spaces with injections and surjections. Concretely its morphisms are spans $U \twoheadleftarrow W \hookrightarrow V$ and the 2-morphisms are given by equivalences of spans. This is a priori a $(2,1)$-category, but it is equivalent to a 1-category. It is (symmetric) monoidal with respect to direct sum.

We apply Yuan's general construction of actions on categories of sections [27, Proposition A.10] to the coalgebra fibration q^+ and its pullback q, see Construction 6.5. His construction works for any coCartesian fibration. Yuan applies it to the fibration of genuine spectra in the proof of [27, Theorem 3.23]. Here is the required action on the base. A vector space V acts by $(X, x) \mapsto (X \times BV, (x, *))$, and a morphism in the Q-construction acts by the pointed product span

$$X \times BU \longleftarrow X \times BW \longrightarrow X \times BV.$$

After applying the functor $i : \mathrm{Corr}^*_p \to \mathrm{Corr}^+_p$, these endofunctors have augmentations to i given by the spans

$$X \times BV \xleftarrow{\mathrm{id}} X \times BV \longrightarrow X \qquad \text{in } \mathrm{Corr}^+_p .$$

The augmentations are compatible with morphisms in $Q\mathrm{Vect}^{\mathrm{fd}}_{\mathbb{F}_p}$: since $W \to U$ is surjective, the fibers of $BW \to BU$ are the connected anima $B\ker(W \to U)$, which disappear under relative 0-truncation in the composite span. As in the proof of [27, Proposition 3.24], functoriality of this factorization and products gives a monoidal family of augmented endofunctors.

Proposition A.10 is stated for a subcategory inclusion, but its proof applies to our functor i as well: it uses only precomposition and pointwise coCartesian transport along the augmentations. Applying it to q^+ gives an action on all sections of q. Transport along the displayed augmentation is Φ^V on coalgebras, so the action is

$$(A_V s)(X) = \Phi^V s(X \times BV).$$

Since $Q\mathrm{Vect}^{\mathrm{fd}}_{\mathbb{F}_p}$ is symmetric monoidal, we regard the resulting right action as a left action. This action is defined on all sections, before any Cartesianity or boundedness condition is imposed.

Remark 8.1. *This construction is where we use the genuine equivariant nature of the norm systems and why we cannot work entirely in the Borel world. For $V \neq 0$, the augmentation $X \times BV \to X$ is not a finite covering, so the construction requires the extension q^+ over Corr^+_p. This extension gives transports by geometric fixed points, which are strong symmetric monoidal. The corresponding proper Tate construction on Borel spectra is only lax symmetric monoidal and therefore does not by itself induce the required functors on coalgebras. Thus the naive fibration used in Lemma 7.3 does not directly give this action. Compare [27, Remark 3.36] for a similar phenomenon in the world of algebras used in Yuan's work.*

Proposition 8.2. *This action preserves the full subcategory $\mathrm{NormSys}^{\mathrm{bb}}_p$ of norm systems whose value at $*$ is bounded below. Via Proposition 6.8, it induces a strong monoidal functor*

$$\Theta : Q\mathrm{Vect}^{\mathrm{fd}}_{\mathbb{F}_p} \longrightarrow (\mathrm{Fun}(\mathrm{coCAlg}^{\mathrm{bb}}_p, \mathrm{coCAlg}^{\mathrm{bb}}_p), \circ).$$

It has $\Theta(V) = \Phi^V N_e^V \simeq \mathrm{id}$ *and* $\Theta(U \twoheadleftarrow W \hookrightarrow V) = \varphi_{\ker(W \to U)}$. *In particular, it inverts the forward morphisms.*

Proof. Let $s \in \mathrm{NormSys}_p^{\mathrm{bb}}$ and put $C = s(*)$. We have

$$(A_V s)(X) = N_x\big(\Phi^V N_e^V C\big)$$

where the components away from the marked component are terminal. In particular, its value at $*$ is $\Phi^V N_e^V C \simeq C$, so it is again bounded below.

For backward Cartesianity it suffices to check the point maps $x : * \to X$ since a backward map followed by the backward pointing map is the backward pointing map for its source, and Cartesian edges have the relevant cancellation property. On the marked component BG the relevant map before applying Φ^V is

$$\mathrm{Res}_V^{G\times V}\, N_e^{G\times V} C = (N_e^V C)^{\otimes G} \longrightarrow N_e^V C.$$

It is projection to the identity G-coordinate: this is the Cartesian counit for the pointed inclusion $BV \to BG \times BV$. Geometric fixed points preserve finite products of coalgebras and therefore take it to the same projection for $\Phi^V N_e^V C$. Using $x^*\Phi^V \simeq \Phi^V(x \times \mathrm{id}_{BV})^*$ the adjunction (10) identifies the resulting edge as Cartesian. Thus $A_V s \in \mathrm{NormSys}_p^{\mathrm{bb}}$.

For the formula on arrows, put $K = \ker(W \to U)$. Norm–inflation base change identifies the source of the section map as

$$N_W^V \inf_U^W N_e^U C \simeq N_K^V \inf_e^K C.$$

Under the adjunction of Lemma 5.2, both the section map and $N_K^V \bar{\Delta}_K$ have adjunct $C^{\otimes(V/K)} \to C$, given by projection to the factor indexed by the zero coset K. Hence the two maps agree. Applying Φ^V gives $\Phi^K \bar{\Delta}_K = \varphi_K$. □

Let $\mathcal{W}'$ be the backward morphisms in $Q\mathrm{Vect}_{\mathbb{F}_p}^{\mathrm{fd}}$. We write $K^{\mathrm{part}}(\mathbb{F}_p)$ for Yuan's partial algebraic K-theory, which is an $\mathbb{E}_\infty$-monoid characterized by the symmetric monoidal equivalence

$$Q\mathrm{Vect}_{\mathbb{F}_p}^{\mathrm{fd}}[(\mathcal{W}')^{-1}] \simeq BK^{\mathrm{part}}(\mathbb{F}_p)$$

[27, Theorem 4.14 and Corollary 4.23.1]. Here BM denotes the one-object category with endomorphism anima M. The group completion of $K^{\mathrm{part}}(\mathbb{F}_p)$ is ordinary algebraic K-theory $K(\mathbb{F}_p)$ [27, Corollary 4.9.2]. Dimension identifies $\pi_0 K^{\mathrm{part}}(\mathbb{F}_p)$ with $\mathbb{N}$ and hence gives a canonical map $K^{\mathrm{part}}(\mathbb{F}_p) \to \mathbb{N}$ [27, Proposition 4.10]. We will use the following forward-localization variant.

Lemma 8.3. *Let* $\mathcal{W}$ *be the class of forward morphisms in* $Q\mathrm{Vect}_{\mathbb{F}_p}^{\mathrm{fd}}$, *induced by inclusions. There is a monoidal equivalence*

$$Q\mathrm{Vect}_{\mathbb{F}_p}^{\mathrm{fd}}[\mathcal{W}^{-1}] \simeq BK^{\mathrm{part}}(\mathbb{F}_p).$$

The dimension map $K^{\mathrm{part}}(\mathbb{F}_p) \to \mathbb{N}$ *is an* $\mathbb{F}_p$*-homology equivalence.*

Proof. Duality of finite-dimensional vector spaces induces a monoidal self-equivalence of $Q\mathrm{Vect}_{\mathbb{F}_p}^{\mathrm{fd}}$ interchanging $\mathcal{W}'$ and $\mathcal{W}$. This construction is covariant on the Q-category. Explicitly it takes $U \twoheadleftarrow W \hookrightarrow V$ to $U^\vee \twoheadleftarrow U^\vee \times_{W^\vee} V^\vee \hookrightarrow V^\vee$. It therefore identifies the forward localization with $Q\mathrm{Vect}_{\mathbb{F}_p}^{\mathrm{fd}}[(\mathcal{W}')^{-1}] \simeq BK^{\mathrm{part}}(\mathbb{F}_p)$ as described above. The assertion about the dimension map is [27, Theorem 5.2]. □

Proof of Theorem 1.3. Proposition 8.2 and Lemma 8.3 give a monoidal action of $BK^{\mathrm{part}}(\mathbb{F}_p)$ on $\mathrm{coCAlg}_p^{\mathrm{bb}}$. Equivalently, this is a map of $\mathbb{E}_2$-monoids from $K^{\mathrm{part}}(\mathbb{F}_p)$ to $\mathrm{End}(\mathrm{id}_{\mathrm{coCAlg}_p^{\mathrm{bb}}})$. The underlying anima of the latter is $\mathbb{F}_p$-homology-local because it is a limit of coalgebra mapping anima, which are limits of mapping anima between bounded-below p-complete spectra. The map therefore factors uniquely through the $\mathbb{F}_p$-homology equivalence $K^{\mathrm{part}}(\mathbb{F}_p) \to \mathbb{N}$. Since mod-$p$ homology equivalences are preserved by finite products, this factorization takes place in $\mathbb{E}_2$-monoids. Thus the action descends uniquely to a $B\mathbb{N}$-action (see also [27, §6.2]). To identify the generator, consider the spans

$$a_V = (0 \leftarrow 0 \hookrightarrow V), \qquad b_V = (0 \leftarrow V \xrightarrow{=} V).$$

In the forward localization, a_V is inverted and the loop $a_V^{-1} b_V$ acts by φ_V. Under the duality used in the proof of Lemma 8.3, this is the usual partial K-theory class of $V^\vee$. Taking $V = \mathbb{F}_p$ identifies the generator $1 \in \mathbb{N}$ with $\varphi_{\mathbb{F}_p} = \varphi_p$ by (8) and (1). This proves the theorem. □

References

[1] J. F. Adams, J. H. Gunawardena and H. R. Miller, *The Segal conjecture for elementary abelian p-groups*, Topology **24** (1985), no. 4, 435–460. Author's online version.

[2] G. Angelini-Knoll and F. Zou, *Spoke topological Hochschild homology*, preprint (2025). arXiv:2512.11338.

[3] T. Bachmann and R. Burklund, *$\mathbb{E}_\infty$-coalgebras and p-adic homotopy theory*, Selecta Math. 32 (2026), 32.

[4] T. Bachmann and M. Hoyois, *Norms in motivic homotopy theory*, arXiv:1711.03061.

[5] C. Barwick, *Spectral Mackey functors and equivariant algebraic K-theory (I)*, Advances in Mathematics **304** (2017), 646–727. arXiv:1404.0108.

[6] H. S. Bergsaker and J. Rognes, *The Segal conjecture for smash powers*, Journal of Topology **16** (2023), 567–587. arXiv:2207.13408.

[7] G. Carlsson, *Equivariant stable homotopy and Segal's Burnside ring conjecture*, Annals of Mathematics (2) **120** (1984), no. 2, 189–224. doi:10.2307/2006940.

[8] C. Carrick, *Cofreeness in Real bordism theory and the Segal conjecture*, Proceedings of the American Mathematical Society **150** (2022), no. 7, 3161–3175. arXiv:2101.04891.

[9] J. Caruso, J. P. May and S. B. Priddy, *The Segal conjecture for elementary abelian p-groups. II. p-adic completion in equivariant cohomology*, Topology **26** (1987), no. 4, 413–433. doi:10.1016/0040-9383(87)90040-1.

[10] H. Chu and R. Haugseng, *Free algebras through Day convolution*, Algebraic & Geometric Topology **22** (2022), 3401–3458. doi:10.2140/agt.2022.22.3401.

[11] B. Cnossen, T. Lenz and M. Ramzi, *Universality of Barwick's unfurling construction*, International Mathematics Research Notices **2025** (2025), no. 18, rnaf280. doi:10.1093/imrn/rnaf280.

[12] A. Dauser and J. Kuijper, *Uniqueness of six-functor formalisms*, arXiv:2412.15780v2.

[13] J. P. C. Greenlees and J. P. May, *Localization and completion theorems for MU-module spectra*, Annals of Mathematics (2) **146** (1997), no. 3, 509–544. doi:10.2307/2952455.

[14] J. H. Gunawardena, *Cohomotopy of some classifying spaces*, Ph.D. thesis, University of Cambridge, 1981.

[15] J. Hahn and D. Wilson, *Real topological Hochschild homology and the Segal conjecture*, Advances in Mathematics **387** (2021), Paper No. 107839. arXiv:1911.05687.

[16] M. A. Hill, M. J. Hopkins and D. C. Ravenel, *On the nonexistence of elements of Kervaire invariant one*, Annals of Mathematics (2) **184** (2016), no. 1, 1–262. doi:10.4007/annals.2016.184.1.1.

[17] P. Hu, I. Kriz and P. Somberg, *On some adjunctions in equivariant stable homotopy theory*, Algebraic & Geometric Topology **18** (2018), 2419–2442. arXiv:1802.08998.

[18] W. H. Lin, *On conjectures of Mahowald, Segal and Sullivan*, Mathematical Proceedings of the Cambridge Philosophical Society **87** (1980), no. 3, 449–458. doi:10.1017/S0305004100056887.

[19] J. Lurie, *Higher topos theory*, Annals of Mathematics Studies 170, Princeton University Press, 2009. Author's online version.

[20] S. Lunøe–Nielsen and J. Rognes, *The topological Singer construction*, Documenta Mathematica **17** (2012), 861–909. arXiv:1010.5633.

[21] J. P. May and J. E. McClure, *A reduction of the Segal conjecture*, Canadian Mathematical Society Conference Proceedings **2**, Part 2 (1982), 209–222. Online version.

[22] T. Nikolaus, *Frobenius homomorphisms in higher algebra*, Proceedings of the ICM 2022. doi:10.4171/ICM2022/159.

[23] T. Nikolaus, *A short proof of the Segal conjecture for C_p*, preprint (2026). arXiv:2610.39440.

[24] T. Nikolaus and P. Scholze, *On topological cyclic homology*, Acta Mathematica **221** (2018), no. 2, 203–409. arXiv:1707.01799.

[25] K. Ragnarsson, *Completion of G-spectra and stable maps between classifying spaces*, arXiv:1001.0771.

[26] W. Yang, *The smash-power Segal conjecture without finite-type hypotheses*, unpublished note, July 28, 2026. Author's online version.

[27] A. Yuan, *Integral models for spaces via the higher Frobenius*. References to numbered statements use the November 2021 version. arXiv:1910.00999v2.