Group completion commutes with products

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1 Introduction

The question of whether the group completion functor

$$(-)^{\mathrm{gp}} : \mathbb{E}_\infty\text{-Mon} \to \mathbb{E}_\infty\text{-Grp} \simeq \mathrm{Sp}_{\geq 0}$$

preserves arbitrary products has interested the K-theory community for some time. Known product-preservation results for the K-theory of stable and additive categories suggest a positive answer. In fact, these results would follow formally from product preservation for group completion. Such a theorem would also imply analogous results in related theories, notably Grothendieck–Witt and L-theory.

Several people have worked on this problem, but so far no one has been able to produce a correct proof. Our goal was to settle the question. Our initial strategy was to reduce it either to an infinite version of the Künneth theorem or to the product-preservation theorem for the K-theory of additive ∞-categories, which follows from work of Kasprowski–Winges [KW19] and Córdova Fedeli [CF23]. To our surprise, the first approach could not work, since the infinite Künneth statement we had conjectured turned out to be incorrect (the counterexample will be presented in a different chart), but the second approach succeeded: building on this K-theoretic input, we used an AI system to prove the following:

Theorem A. *The group completion functor*

$$(-)^{\mathrm{gp}} : \mathbb{E}_\infty\text{-Mon} \to \mathbb{E}_\infty\text{-Grp}$$

preserves infinite products.

Although the proof was unexpectedly elegant, we were not satisfied with it conceptually. We wanted product preservation by group completion to serve as an *algebraic starting point* explaining the corresponding behaviour of K-theory and the related functors GW and L.

We therefore tried to generalize. It was already expected that the full strength of $\mathbb{E}_\infty$-commutativity would not be necessary and that $\mathbb{E}_2$-commutativity would suffice (the $\mathbb{E}_1$-case being false even in the classical 1-categorical setting[1]). After some back and forth, we obtained a technical reduction of the $\mathbb{E}_2$-case to the $\mathbb{E}_\infty$-case. Conceptually, this was still unsatisfying, since the argument continued to depend on the K-theoretic statement. We then found a homological argument that settles the $\mathbb{E}_2$-case directly. With a bit of extra work, this generalized to the following statement about rigid monoidal ∞-categories (where every object admits a left and a right dual; in fact, one-sided duals suffice, see Corollary 1.1).

Theorem B. *The groupoid completion functor*

$$|-| : \mathbb{E}_1\text{-Mon}(\mathrm{Cat}_\infty)_{\mathrm{rig}} \longrightarrow \mathbb{E}_1\text{-Mon}$$

from rigid $\mathbb{E}_1$*-monoidal* ∞*-categories to* $\mathbb{E}_1$*-monoids in spaces preserves infinite products.*

Applied to the classifying ∞-category $\mathbf{B}M$ of an $\mathbb{E}_2$-monoid M, this recovers the $\mathbb{E}_2$-case (Corollary 1.2).

The proof of this result is still based on a homological calculation. We believed there should be a purely combinatorial/homotopy-theoretic proof instead, and in discussion with the AI, we came up with the following statement and a new proof, further generalizing the previous two results:

Theorem C. *Let* $(M_{i,\bullet})_{i\in I}$ *be a small family of simplicial* $\mathbb{E}_1$*-monoids in spaces. If* $\left|\prod_{i\in I} M_{i,\bullet}\right|$ *is grouplike, i.e. its monoid of components is a group, the canonical map*

$$\left|\prod_{i\in I} M_{i,\bullet}\right| \longrightarrow \prod_{i\in I} |M_{i,\bullet}| \tag{$*$}$$

is an equivalence.

This theorem also has an unexpected connection with an approach already present in the literature. Carlsson's original strategy for proving that K-theory preserves infinite products [Car95] involved a class of simplicial monoids equipped with pre-inverses and a controlled homotopy. Our result can be viewed as a more conceptual, ∞-categorical version of this approach, although the proof is strikingly different.

The implications Theorem C $\Longrightarrow$ Theorem B $\Longrightarrow$ Theorem A are simple and short enough to include here:

Corollary 1.1 (C $\Rightarrow$ B)**.** *Assuming Theorem C, groupoid completion preserves arbitrary small products of small monoidal* ∞*-categories with left duals. The same holds with right duals. In particular, Theorem B holds.*

Proof. A monoidal ∞-category $\mathcal{C}$ is a monoid object in Cat_∞. The Rezk nerve preserves products and realizes to groupoid completion [MG19, Definition 2.1 and Proposition 2.4]. Thus, for a family $(\mathcal{C}_i)_{i\in I}$ of monoidal ∞-categories, $M_{i,n} = \mathrm{Fun}([n], \mathcal{C}_i)^{\simeq}$ are simplicial monoids with $\left|\prod_i M_{i,\bullet}\right| \simeq \left|\prod_i \mathcal{C}_i\right|$. If every $\mathcal{C}_i$ has left duals, so does $\prod_i \mathcal{C}_i$, and evaluation and coevaluation make $\left|\prod_i \mathcal{C}_i\right|$ grouplike. Theorem C therefore proves the assertion. The right-dual case is identical. □

Corollary 1.2 (B $\Rightarrow$ A)**.** *Assuming Theorem B, group completion of* $\mathbb{E}_2$*-monoids in spaces preserves arbitrary small products. In particular, Theorem A holds.*

[1] For the free monoid F on two letters a, b, the family $\left((ab^{-1})^n\right)_{n\in\mathbb{N}}$ in $\prod_n F^{\mathrm{gp}}$ is not in the image of $\left(\prod_n F\right)^{\mathrm{gp}}$: in each coordinate, an element of the latter is a product of a fixed number k of positive and inverted positive words, so its reduced form has at most $k-1$ sign changes, whereas $(ab^{-1})^n$ has $2n-1$.

Proof. For an $\mathbb{E}_2$-monoid A, the one-object ∞-category $\mathbf{B}A$ is monoidal, its only object is the tensor unit, and $\mathbf{B}$ preserves products [Leh26, Proposition 2.6]. Since the unit is dualizable, $\mathbf{B}A$ is rigid, so Theorem B and the bar–loop formula $A^{\mathrm{gp}} \simeq \Omega|BA|$ [Leh26, Proposition 2.9] give

$$\left(\prod_i A_i\right)^{\mathrm{gp}} \simeq \Omega\left|\prod_i \mathbf{B}A_i\right| \simeq \prod_i \Omega|\mathbf{B}A_i| \simeq \prod_i A_i^{\mathrm{gp}}.$$

The composite is the canonical comparison. For $\mathbb{E}_\infty$-monoids, note that the forgetful functor to $\mathbb{E}_2$-monoids preserves products and commutes with group completion. □

In particular, Theorem C gives a proof of Theorem A that is independent of the K-theoretic input. In Figure 1 below we summarize the various statements and implications we have shown.

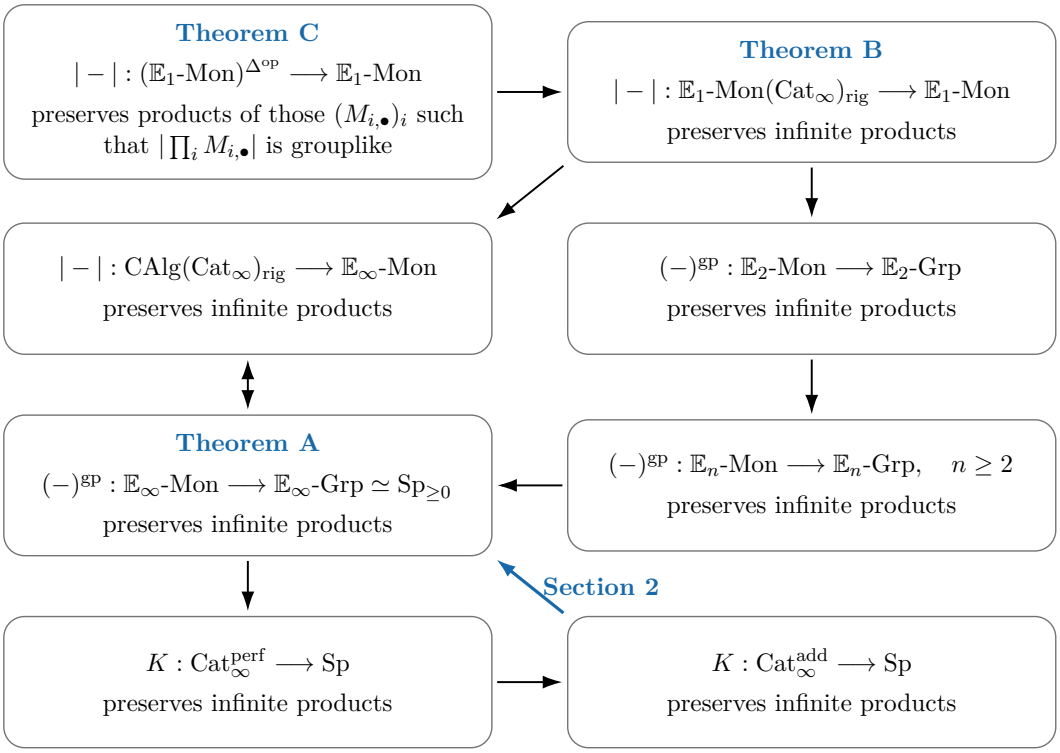


Figure 1: Various implications of certain functors preserving infinite products.

In the following sections, we present the three proofs of Theorems A, B and C, partly based on initial AI-generated drafts:

- Section 2 proves Theorem A (Theorem 2.1) directly by a reduction to the K-theory of additive categories;
- Section 3 proves Theorem B (Theorem 3.2), as well as the $\mathbb{E}_2$-case (Theorem 3.1), by a direct homological argument;
- Section 4 proves Theorem C (Corollary 4.3) by reducing it to the case of powers (Theorem 4.1); the deductions of Theorems B and A from it are given above (Corollaries 1.1 and 1.2).

After this, there is an appendix (Appendix A) answering four questions about our methodology and the process of arriving at these results.

Furthermore, building on this chart, we plan to write an article that streamlines these arguments, makes them more conceptual, and presents them in a more reader-friendly form.

2 Group completion of $\mathbb{E}_\infty$-monoids commutes with infinite products: A reduction to K-theory

Group completion of $\mathbb{E}_\infty$-monoids in spaces preserves all small products. We first prove this for products of the free $\mathbb{E}_\infty$-monoid on one point, using a trace retraction and the product theorem for additive K-theory. A simultaneous induction on Postnikov degree gives the result in this case. Free resolutions then give the general case; the required exchanges of products with colimits and realizations are justified explicitly.

2.1 Statement and inputs

Write

$$(-)^{\mathrm{gp}} \colon \mathbb{E}_\infty\text{-Mon} \longrightarrow \mathrm{Sp}_{\geq 0}$$

for group completion, identifying grouplike $\mathbb{E}_\infty$-monoids with connective spectra. All limits and colimits are taken in the relevant ∞-categories. We work in a fixed universe: every index set, space, and auxiliary category below is small in that universe.

Theorem 2.1 (Theorem A)**.** *For every small set I and every family $(M_i)_{i\in I}$ of $\mathbb{E}_\infty$-monoids in spaces, the canonical map*

$$\left(\prod_{i\in I} M_i\right)^{\mathrm{gp}} \longrightarrow \prod_{i\in I} M_i^{\mathrm{gp}} \tag{1}$$

is an equivalence of connective spectra.

For an additive ∞-category $\mathcal{B}$, its core $\mathcal{B}^{\simeq}$ is an $\mathbb{E}_\infty$-monoid under direct sum. Write $K^{\oplus}(\mathcal{B}) = (\mathcal{B}^{\simeq})^{\mathrm{gp}}$ for its additive K-theory. Unless otherwise specified, additive categories carry the split exact structure. For every small additive ∞-category, $K^{\oplus}(\mathcal{B})$ agrees naturally with its connective split-exact K-theory by [Lur14, Theorem 10]; neither idempotent completeness nor discreteness of the mapping spaces is required. We use two external inputs.

1. **Products in additive K-theory.** For small additive ∞-categories $\mathcal{B}_i$, the canonical map

$$\left(\prod_i \mathcal{B}_i^{\simeq}\right)^{\mathrm{gp}} \xrightarrow{\simeq} \prod_i (\mathcal{B}_i^{\simeq})^{\mathrm{gp}} \tag{2}$$

   is an equivalence. This follows from Córdova Fedeli's Corollary 2.11.1 [CF23], with the clarification concerning π_0 in the remark below. The same formulation is used in [KM24, §2 and the proof of Proposition 3.2].

2. **Coherent Euler characteristic.** If $\mathcal{D}$ is a small, idempotent-complete, rigid symmetric monoidal stable ∞-category, with tensor product exact in each variable, then

$$\mathcal{D}^{\simeq} \longrightarrow \Omega^\infty \mathrm{End}_{\mathcal{D}}(\mathbf{1}), \qquad P \longmapsto \mathrm{tr}(\mathrm{id}_P), \tag{3}$$

   is a natural map of $\mathbb{E}_\infty$-monoids for direct sum and addition. This is the additive part of the Chern pre-character of Hoyois–Scherotzke–Sibilla [HSS17, (6.4)], restricted to constant loops; see also their Theorem 6.5 and Remark 6.6. We need this map on the entire core, with its additive coherences.

Remark. Strictly speaking, the argument for [CF23, Corollary 2.11.1] establishes (2) directly in positive degrees; there is a subtlety at π_0. If K_{cn} denotes connective K-theory without idempotent completion and $\mathbb{K}$ denotes Morita-invariant nonconnective K-theory, then for a stable ∞-category $\mathcal{C}$ one has

$$\tau_{\geq 0}\mathbb{K}(\mathcal{C}) \simeq K_{\mathrm{cn}}(\mathcal{C}^{\mathrm{idem}}),$$

and passage to $\mathcal{C}^{\mathrm{idem}}$ can change K_0; see [KW19, §2.1]. In positive degrees this causes no problem: combine [CF23, Proposition 2.11] with the split-exact comparison above and the Gillet–Waldhausen equivalence for stable envelopes [SW26, Theorem 1.8].

The remaining degree can be checked directly. Set $M_i = \pi_0(\mathcal{B}_i^{\simeq})$. On π_0, the map in (2) is the canonical homomorphism

$$\left(\prod_i M_i\right)^{\mathrm{gp}} \longrightarrow \prod_i M_i^{\mathrm{gp}}$$

for ordinary commutative monoids. It is surjective because a family of differences $a_i - b_i$ is the difference of the families (a_i) and (b_i). It is injective because, if $a_i - b_i = 0$ in every M_i^{gp}, one can choose $c_i \in M_i$ with $a_i + c_i = b_i + c_i$; the family (c_i) witnesses equality in the product monoid. Thus (2) holds in degree zero as well, without any idempotent-completeness assumption. This elementary calculation for monoids in sets is independent of the theorem being proved.

The free $\mathbb{E}_\infty$-monoid on one point is

$$\mathrm{Fin}^{\simeq} \simeq \coprod_{r\geq 0} B\Sigma_r.$$

Its universal property gives $(\mathrm{Fin}^{\simeq})^{\mathrm{gp}} \simeq \mathbb{S}$. We first prove the theorem for arbitrary products of $\mathrm{Fin}^{\simeq}$.

2.2 Two truncation lemmas

A map of spaces or connective spectra is an *n-equivalence* if its nth Postnikov truncation is an equivalence. For spaces this includes every component and every basepoint.

Lemma 2.2. *For $n \geq 0$, group completion preserves n-equivalences.*

Proof. Truncation of spaces preserves finite products, so it induces a reflector $\tau_{\leq n}\colon \mathbb{E}_\infty\text{-Mon} \to \mathbb{E}_\infty\text{-Mon}(\mathrm{An}_{\leq n})$, computed on underlying spaces. For $T \in \mathrm{Sp}_{[0,n]}$, adjunction gives

$$\begin{aligned}\mathrm{Map}_{\mathrm{Sp}_{[0,n]}}(\tau_{\leq n}M^{\mathrm{gp}}, T) &\simeq \mathrm{Map}_{\mathbb{E}_\infty\text{-Mon}}(M, \Omega^\infty T)\\ &\simeq \mathrm{Map}_{\mathbb{E}_\infty\text{-Mon}}(\tau_{\leq n}M, \Omega^\infty T).\end{aligned}$$

Thus $\tau_{\leq n}M^{\mathrm{gp}}$ depends only on $\tau_{\leq n}M$. □

This does not say that group completion preserves truncated objects: $\mathrm{Fin}^{\simeq}$ is 1-truncated, whereas $(\mathrm{Fin}^{\simeq})^{\mathrm{gp}} \simeq \mathbb{S}$ has higher homotopy groups.

Lemma 2.3. *Let $f\colon \mathcal{B} \to \mathcal{E}$ be essentially surjective. Suppose that for all $a, a' \in \mathcal{B}$, the map*

$$\mathrm{Map}_{\mathcal{B}}(a, a') \longrightarrow \mathrm{Map}_{\mathcal{E}}(fa, fa')$$

is an n-equivalence, where $n \geq 0$. Then $\mathcal{B}^{\simeq} \to \mathcal{E}^{\simeq}$ is an $(n+1)$-equivalence. If f is additive between additive ∞-categories, the induced map $(\mathcal{B}^{\simeq})^{\mathrm{gp}} \to (\mathcal{E}^{\simeq})^{\mathrm{gp}}$ is also an $(n+1)$-equivalence.

Proof. The induced functor on homotopy categories is an equivalence. Consequently, the cores have the same components, and f preserves and reflects the invertible components of each endomorphism space. At $a \in \mathcal{B}$,

$$\Omega_a\mathcal{B}^{\simeq} \simeq \mathrm{Aut}_{\mathcal{B}}(a),$$

the union of those components of $\mathrm{Map}_{\mathcal{B}}(a, a)$. The map of these loop spaces is an n-equivalence, proving the first assertion. Apply Lemma 2.2 for the second. □

2.3 The I-indexed Burnside ∞-category

Consider $\mathrm{Span}(\mathrm{Fin}^I)$. Its objects are families $P = (P_i)_{i\in I}$ of finite sets; their cardinalities need not have a common bound. A morphism is a span $P \leftarrow T \rightarrow Q$, with isomorphisms of spans as 2-morphisms. Composition is by pullback.

Coordinatewise disjoint union is a finite biproduct. Taking the fibers of $T_i \to P_i \times Q_i$ identifies the mapping $\mathbb{E}_\infty$-monoid as

$$\mathrm{Map}_{\mathrm{Span}(\mathrm{Fin}^I)}(P, Q) \simeq \prod_{i\in I} (\mathrm{Fin}^{\simeq})^{P_i \times Q_i}. \tag{4}$$

Addition here is disjoint union of the middle objects of spans.

The *I-indexed Burnside ∞-category* $\mathcal{A}_I$ is obtained from $\mathrm{Span}(\mathrm{Fin}^I)$ by keeping its objects and group-completing its mapping $\mathbb{E}_\infty$-monoids. This is a change of enrichment: composition is additive in each variable, so it extends to the group completions, together with its unit and associativity coherences. Equivalently, use the symmetric monoidal group-completion functor from $\mathbb{E}_\infty$-monoids with their tensor product for biadditive maps to connective spectra. Thus

$$\mathrm{map}_{\mathcal{A}_I}(P, Q) \simeq \left(\prod_{i\in I} (\mathrm{Fin}^{\simeq})^{P_i \times Q_i} \right)^{\mathrm{gp}}. \tag{5}$$

Finite biproducts survive, since group completion preserves finite products. Indeed, finite products are also finite coproducts in both $\mathbb{E}_\infty$-Mon and $\mathrm{Sp}_{\geq 0}$, and $(-)^{\mathrm{gp}}$ is a left adjoint. The mapping monoids of $\mathcal{A}_I$ are grouplike, so $\mathcal{A}_I$ is additive.

Coordinatewise Cartesian product makes $\mathrm{Span}(\mathrm{Fin}^I)$ and $\mathcal{A}_I$ symmetric monoidal, with tensor product additive in each variable and unit $\mathbf{1} = (*)_{i\in I}$. Every object P is self-dual: coevaluation and evaluation are the spans

$$\mathbf{1} \longleftarrow P \xrightarrow{\Delta} P \times P, \qquad P \times P \xleftarrow{\Delta} P \longrightarrow \mathbf{1}. \tag{6}$$

The triangle identities follow from the corresponding pullback identifications. In particular,

$$\mathrm{End}_{\mathcal{A}_I}(\mathbf{1}) \simeq ((\mathrm{Fin}^{\simeq})^I)^{\mathrm{gp}}. \tag{7}$$

The coordinate projections give an additive symmetric monoidal functor

$$\mathcal{A}_I \longrightarrow \prod_{i\in I} \mathcal{A}_{\{*\}}. \tag{8}$$

It is essentially surjective: both sides have the same object families. On mapping spectra it induces

$$\left(\prod_{i\in I} (\mathrm{Fin}^{\simeq})^{P_i \times Q_i} \right)^{\mathrm{gp}} \longrightarrow \prod_{i\in I} ((\mathrm{Fin}^{\simeq})^{P_i \times Q_i})^{\mathrm{gp}} \simeq \prod_{i\in I} \prod_{P_i \times Q_i} \mathbb{S}. \tag{9}$$

This is the canonical product comparison indexed by $\coprod_{i\in I}(P_i \times Q_i)$, using finite-product preservation within each coordinate and $(\mathrm{Fin}^{\simeq})^{\mathrm{gp}} \simeq \mathbb{S}$.

2.4 The trace retraction

The canonical functor from finite-set families and bijections to $\mathcal{A}_I^{\simeq}$ preserves disjoint union and direct sum. It induces

$$((\mathrm{Fin}^{\simeq})^I)^{\mathrm{gp}} \longrightarrow (\mathcal{A}_I^{\simeq})^{\mathrm{gp}}.$$

Proposition 2.4. *The canonical map $((\mathrm{Fin}^{\simeq})^I)^{\mathrm{gp}} \to (\mathcal{A}_I^{\simeq})^{\mathrm{gp}}$ admits a retraction, natural under coordinate projections.*

Proof. First place $\mathcal{A}_I$ in its idempotent-complete stable envelope $\mathcal{D}_I$. One model is the thick subcategory generated by the representable spectral presheaves on $\mathcal{A}_I$. Spectral Yoneda is fully faithful, and Day convolution extends the tensor product exactly in each variable. The representables are dualizable by (6); their finite cofibers, shifts, and retracts are also dualizable. Hence $\mathcal{D}_I$ is rigid, and

$$\mathrm{End}_{\mathcal{D}_I}(\mathbf{1}) \simeq \mathrm{End}_{\mathcal{A}_I}(\mathbf{1}) \simeq \big((\mathrm{Fin}^{\simeq})^I\big)^{\mathrm{gp}}.$$

The coherent Euler characteristic (3) restricts to a map of $\mathbb{E}_\infty$-monoids

$$\mathcal{A}_I^{\simeq} \longrightarrow \mathcal{D}_I^{\simeq} \longrightarrow \Omega^\infty\big((\mathrm{Fin}^{\simeq})^I\big)^{\mathrm{gp}}.$$

Since the target is grouplike, group completion gives a map

$$(\mathcal{A}_I^{\simeq})^{\mathrm{gp}} \longrightarrow \big((\mathrm{Fin}^{\simeq})^I\big)^{\mathrm{gp}}.$$

To show that this is a retraction, compute the trace in $\mathrm{Span}(\mathrm{Fin}^I)$:

$$\mathrm{tr}(\mathrm{id}_P) = (\mathbf{1} \leftarrow P \rightarrow \mathbf{1}). \tag{10}$$

The pullback in the trace formula is $P \times_{P\times P} P \cong P$. A bijection $P \to Q$ induces that same bijection on the middle objects, naturally in composites of bijections.

This identification also respects the full additive structure. For a finite family $(P_s)_{s\in S}$, the diagonal of $\coprod_s P_s$ has only diagonal blocks; the mixed blocks are empty. Consequently its trace is canonically the disjoint union of the traces of the P_s. More explicitly, for a map of finite pointed sets $\alpha\colon S_+ \to T_+$, the middle object in component $t \in T$ is

$$\coprod_{s\in\alpha^{-1}(t)} P_s.$$

These descriptions agree with composition by the canonical identifications of iterated disjoint unions, including empty fibers, permutations, and summands sent to the basepoint. They therefore identify the associated Fin_*-diagrams, not just their components.

By naturality of trace under $\mathrm{Span}(\mathrm{Fin}^I) \to \mathcal{A}_I \to \mathcal{D}_I$, the composite

$$(\mathrm{Fin}^{\simeq})^I \longrightarrow \mathcal{A}_I^{\simeq} \longrightarrow \Omega^\infty\big((\mathrm{Fin}^{\simeq})^I\big)^{\mathrm{gp}}$$

is the group-completion unit as a map of $\mathbb{E}_\infty$-monoids. Its extension to group completions is the identity, proving the retraction. All constructions commute with coordinate projections, giving the claimed naturality, including that of the retraction homotopy. □

Remark. Only the map $\mathcal{A}_I^{\simeq} \to \mathcal{D}_I^{\simeq}$ and the coherent trace on $\mathcal{D}_I$ enter this construction. No identification of $K^{\oplus}(\mathcal{A}_I)$ or $(\mathcal{D}_I^{\simeq})^{\mathrm{gp}}$ with the ordinary stable K-theory of $\mathcal{D}_I$ is needed.

2.5 Products of $\mathrm{Fin}^{\simeq}$

Taking group completions of cores in (8) and applying the additive product theorem (2) gives

$$(\mathcal{A}_I^{\simeq})^{\mathrm{gp}} \longrightarrow \left(\prod_i \mathcal{A}_{\{*\}}^{\simeq}\right)^{\mathrm{gp}} \xrightarrow{\simeq} \prod_i (\mathcal{A}_{\{*\}}^{\simeq})^{\mathrm{gp}}.$$

The natural retractions give a commutative diagram

$$\begin{array}{ccccc} \big((\mathrm{Fin}^{\simeq})^I\big)^{\mathrm{gp}} & \longrightarrow & (\mathcal{A}_I^{\simeq})^{\mathrm{gp}} & \longrightarrow & \big((\mathrm{Fin}^{\simeq})^I\big)^{\mathrm{gp}} \\ \downarrow & & \downarrow & & \downarrow \\ \prod_i \mathbb{S} & \longrightarrow & \prod_i (\mathcal{A}_{\{*\}}^{\simeq})^{\mathrm{gp}} & \longrightarrow & \prod_i \mathbb{S}. \end{array} \tag{11}$$

Both horizontal composites are identities. Hence the outer vertical comparison is a retract of the middle vertical map in the arrow ∞-category of spectra.

Proposition 2.5. *For every small set I, the canonical map*

$$\big((\mathrm{Fin}^{\simeq})^I\big)^{\mathrm{gp}} \longrightarrow \prod_{i\in I} \mathbb{S}$$

is an equivalence.

Proof. For each $n \geq 0$, we prove simultaneously for all small sets I that this comparison is an n-equivalence.

For $n = 0$, the source has the Grothendieck group of $\mathbb{N}^I$ as its group of components, and the comparison induces an isomorphism

$$\pi_0\big((\mathrm{Fin}^{\simeq})^I\big)^{\mathrm{gp}} \xrightarrow{\cong} \mathbb{Z}^I.$$

Every integer family is the difference of its nonnegative positive and negative parts; injectivity follows from cancellation in $\mathbb{N}^I$.

Suppose the assertion holds through degree n for every I. By (9), the functor (8) is an n-equivalence on all mapping spaces. It is essentially surjective, so Lemma 2.3 makes its map on group-completed cores an $(n+1)$-equivalence. By the additive product theorem, the middle vertical map in (11) is therefore an $(n+1)$-equivalence. Its retract, the outer comparison, has the same property.

Thus the comparison induces isomorphisms on all homotopy groups of connective spectra, and is an equivalence. □

Simultaneous induction is essential: its hypothesis applies to every set $\coprod_{i\in I}(P_i \times Q_i)$, even when the sizes of the finite sets P_i and Q_i are unbounded across I.

2.6 Passage to arbitrary $\mathbb{E}_\infty$-monoids

Let $L \dashv U$ be the free–forgetful adjunction between spaces and $\mathbb{E}_\infty$-monoids. On underlying spaces,

$$L(X) \simeq \coprod_{r\geq 0} (X^r)_{h\Sigma_r}, \qquad L(*) = \mathrm{Fin}^{\simeq}.$$

The functor U creates products and sifted colimits, including filtered colimits and geometric realizations. For the latter, use the Segal description of commutative monoids and the fact that sifted colimits of spaces commute with finite products.

2.6.1 Two exchanges with products

Lemma 2.6. *For each $i \in I$, let Λ_i be a small nonempty directed poset and let $X_{i,-}$ be a diagram of spaces or of spectra. Then*

$$\operatorname*{colim}_{(\lambda_i)\in\prod_i \Lambda_i} \prod_i X_{i,\lambda_i} \xrightarrow{\simeq} \prod_i \operatorname*{colim}_{\lambda_i\in\Lambda_i} X_{i,\lambda_i}. \tag{12}$$

Proof. The product poset is directed, by choosing upper bounds coordinatewise. For sets, a family of colimit elements has representatives at a family of stages; equality can likewise be witnessed at a family of later stages. This proves bijectivity. For spaces, take strict diagrams of Kan complexes: filtered colimits are computed degreewise, remain Kan, and compute the colimits of spaces. The degreewise set calculation applies. For spectra, apply π_d, which commutes with both products and filtered colimits, and use the same calculation for groups. □

Lemma 2.7. *For any family of simplicial connective spectra $X_{i,\bullet}$,*

$$\left|\prod_i X_{i,\bullet}\right| \xrightarrow{\simeq} \prod_i |X_{i,\bullet}|\,. \tag{13}$$

Proof. The mth stage $|\mathrm{sk}_m X|$ of the skeletal filtration of realization is built from finitely many latching attachments, using finite direct diagrams and the finite simplicial pairs $(\Delta^r, \partial\Delta^r)$. Products of spectra commute with finite colimits, so

$$\left|\mathrm{sk}_m\left(\prod_i X_{i,\bullet}\right)\right| \simeq \prod_i |\mathrm{sk}_m X_{i,\bullet}|\,.$$

The cofiber of the attachment in degree r is $\Sigma^r \operatorname{cofib}(L_r X \to X_r)$, where $L_r X$ is the latching object. It is r-connective. Hence $\operatorname{cofib}(|\mathrm{sk}_m X| \to |X|)$ is $(m+1)$-connective, uniformly in the simplicial connective spectrum X. Products preserve this connectivity. For any fixed degree $d \geq 0$, choose $m > d$: both sides of (13) have the same π_d as their common mth skeletal stage. □

2.6.2 Free monoids on sets

For a finite set A, $L(A) \simeq (\mathrm{Fin}^{\simeq})^A$. For an arbitrary set A,

$$L(A) \simeq \operatorname*{colim}_{\substack{B \subseteq A \\ B \text{ finite}}} (\mathrm{Fin}^{\simeq})^B.$$

For a family (A_i), all colimits below run over families of finite subsets $B_i \subseteq A_i$, chosen independently in each coordinate. This is the full product of the directed posets of finite subsets. Lemma 2.6, preservation of colimits by group completion, and Proposition 2.5 give

$$\begin{aligned}\left(\prod_i L(A_i)\right)^{\mathrm{gp}} &\simeq \operatorname*{colim}_{(B_i)} \left(\prod_i (\mathrm{Fin}^{\simeq})^{B_i}\right)^{\mathrm{gp}} \\ &\simeq \operatorname*{colim}_{(B_i)} \prod_i \left((\mathrm{Fin}^{\simeq})^{B_i}\right)^{\mathrm{gp}} \\ &\simeq \prod_i L(A_i)^{\mathrm{gp}}.\end{aligned} \tag{14}$$

The middle comparison is Proposition 2.5 for the set $\coprod_i B_i$, using finite-product preservation in each coordinate.

2.6.3 Free monoids on spaces

Represent each space X_i by a Kan complex. Applying L to its sets of simplices gives a simplicial $\mathbb{E}_\infty$-monoid $L(X_{i,\bullet})$ whose realization is $L(X_i)$. For these resolutions we also have

$$\left|\prod_i L(X_{i,\bullet})\right| \simeq \prod_i L(X_i). \tag{15}$$

Here is a model that justifies this exchange with an arbitrary product. Use the bisimplicial sets

$$B^i_{p,q} = \coprod_{r \geq 0} (E\Sigma_r)_q \times_{\Sigma_r} X^r_{i,p}.$$

For fixed p, the q-direction is the nerve of the groupoid of finite $X_{i,p}$-labelled sets. It is Kan and represents $UL(X_{i,p})$. The diagonal is

$$\operatorname{diag} B^i = \coprod_{r \geq 0} E\Sigma_r \times_{\Sigma_r} X^r_i.$$

Each summand is a Kan fibration over $B\Sigma_r$ with fiber X_i^r, so $\operatorname{diag} B^i$ is Kan and represents $UL(X_i)$.

The diagonal of a bisimplicial set represents the realization of its rows. Products of the Kan rows represent products of spaces, and

$$\operatorname{diag}\left(\prod_i B^i\right) = \prod_i \operatorname{diag}(B^i).$$

The last product is again a product of Kan complexes. This proves (15) on underlying spaces, and hence in $\mathbb{E}_\infty$-Mon, since U creates realizations and detects equivalences.

Now use (14) degreewise, together with Lemma 2.7:

$$\begin{aligned}\left(\prod_i L(X_i)\right)^{\mathrm{gp}} &\simeq \left|\left(\prod_i L(X_{i,\bullet})\right)^{\mathrm{gp}}\right| \\ &\simeq \left|\prod_i L(X_{i,\bullet})^{\mathrm{gp}}\right| \\ &\simeq \prod_i L(X_i)^{\mathrm{gp}}.\end{aligned} \tag{16}$$

2.6.4 The free bar resolution

Completion of the proof of Theorem 2.1. For $M \in \mathbb{E}_\infty$-Mon, use the augmented free bar resolution

$$B_n(M) = (LU)^{n+1}M.$$

Every term is free on a space. After applying U, the augmentation has an extra degeneracy, so it is a split augmented simplicial object. Taking the coordinatewise extra degeneracies gives a splitting of

$$\prod_i UB_\bullet(M_i) \longrightarrow \prod_i UM_i.$$

A split augmented simplicial object realizes to its augmentation object. Because U creates products and realizations, it follows that

$$\left|\prod_i B_\bullet(M_i)\right| \simeq \prod_i M_i.$$

Apply (16) in each simplicial degree and then Lemma 2.7:

$$\begin{aligned}\left(\prod_i M_i\right)^{\mathrm{gp}} &\simeq \left|\left(\prod_i B_\bullet(M_i)\right)^{\mathrm{gp}}\right| \\ &\simeq \left|\prod_i B_\bullet(M_i)^{\mathrm{gp}}\right| \\ &\simeq \prod_i |B_\bullet(M_i)^{\mathrm{gp}}| \simeq \prod_i M_i^{\mathrm{gp}}.\end{aligned}$$

These equivalences commute with the coordinate projections, so the resulting equivalence is the canonical map (1). □

3 Group completion of $\mathbb{E}_2$-monoids commutes with infinite products: A homological approach

In this section we prove the following two results by a direct homological argument:

Theorem 3.1 ($\mathbb{E}_2$-version of Theorem A)**.** *For every family of $\mathbb{E}_2$-monoids in spaces, the canonical map*

$$\left(\prod_{i\in I} M_i\right)^{\mathrm{gp}} \longrightarrow \prod_{i\in I} M_i^{\mathrm{gp}}$$

is an equivalence, where $(-)^{\mathrm{gp}}$ denotes the group completion to an $\mathbb{E}_2$-group.

Theorem 3.2 (Theorem B)**.** *For every family of small rigid $\mathbb{E}_1$-monoidal ∞-categories, the canonical map*

$$\left|\prod_i \mathcal{C}_i\right| \longrightarrow \prod_i |\mathcal{C}_i|$$

is an equivalence. Here $|\mathcal{C}|$ denotes groupoid completion. Rigidity means that every object has a left and a right dual.

As explained in the introduction (see Corollary 1.2), Theorem 3.2 actually formally implies Theorem 3.1. Theorem 3.1 also implies a categorical version, namely that for any family of $\mathbb{E}_2$-monoidal ∞-categories $\mathcal{C}_i$ we have that the canonical map

$$\left|\prod_i \mathcal{C}_i\right|^{\mathrm{gp}} \longrightarrow \prod_i (|\mathcal{C}_i|^{\mathrm{gp}})$$

is an equivalence. If the $\mathcal{C}_i$'s are moreover rigid, then their groupoid completions are already grouplike, so that we get the conclusion of Theorem 3.2 under the more restrictive assumption that the categories are $\mathbb{E}_2$ rather than just $\mathbb{E}_1$. We will see that the two theorems have very similar proofs and even though Theorem 3.1 is implied by Theorem 3.2, since the proof is quicker and Theorem 3.1 is the one we ultimately care about we include proofs of both for completeness.

3.1 Auxiliary results

Fix a set I and write $J = (J_i)_{i\in I}$. Chains are integral, graded homologically, and tensor products are derived. Write $\mathrm{Mon}_{\mathrm{free}}$ and $\mathrm{Ab}_{\mathrm{free}}$ for the ordinary categories of free monoids and free abelian groups on small sets. A map $\mathrm{Free}^{\mathrm{mon}}(S) \to \mathrm{Free}^{\mathrm{mon}}(J)$ assigns a finite word to each generator of S; composition is substitution.

The following lemma is standard.

Lemma 3.3 (Universal property)**.** *For every ordinary additive category $\mathcal{A}$, abelianized group completion induces an equivalence*

$$\mathrm{Fun}^{\times}\big((\mathrm{Ab}_{\mathrm{free}}^{I})^{\mathrm{op}}, \mathcal{A}\big) \xrightarrow{\simeq} \mathrm{Fun}^{\times}\big((\mathrm{Mon}_{\mathrm{free}}^{I})^{\mathrm{op}}, \mathcal{A}\big).$$

Here $\mathrm{Fun}^{\times}$ denotes finite-product-preserving functors.

Let Lin be the left adjoint to restriction

$$\mathrm{Fun}^{\times}\big((\mathrm{Ab}_{\mathrm{free}}^{I})^{\mathrm{op}}, \mathcal{D}\mathbb{Z}\big) \longrightarrow \mathrm{Fun}\big((\mathrm{Set}^{I})^{\mathrm{op}}, \mathcal{D}\mathbb{Z}\big)$$

along $\mathbb{Z}[-]$. It is not quite left Kan extension since the latter does not land in additive functors.

Lemma 3.4 (Linearization)**.** *For spaces X_i and free abelian groups P_i,*

$$\mathrm{Lin}\left[J \longmapsto C_*\left(\prod_i X_i^{J_i}\right)\right](P) \simeq \prod_i \mathrm{Hom}_{\mathbb{Z}}(P_i, C_*(X_i)). \tag{17}$$

At $P_i = \mathbb{Z}[J_i]$, the comparison is induced by coordinate projections.

Proof. For sets S_i, additive extension sends the free representable to the additive representable:

$$\mathbb{Z}[\mathrm{Hom}_{\mathrm{Set}^I}(-, S)] \longmapsto \mathrm{Hom}_{\mathrm{Ab}^I_{\mathrm{free}}}(-, \mathbb{Z}[S]).$$

Both are projective, so this also computes the derived extension and gives the stated comparison. For spaces, use Kan simplicial-set models and realize degreewise. Products of connective chain complexes commute with simplicial realization: each total degree uses only finitely many simplicial degrees, and products in abelian groups are exact. The same holds for $\mathrm{Hom}_{\mathbb{Z}}(P_i, -)$, since P_i is free. □

Lemma 3.5 (Comparison criterion)**.** *A natural transformation between finite-product-preserving functors $(\mathrm{Mon}^I_{\mathrm{free}})^{\mathrm{op}} \to \mathrm{An}_*$ with connected values is an equivalence if the induced transformation of functors $(\mathrm{Mon}^I_{\mathrm{free}})^{\mathrm{op}} \to \mathcal{D}\mathbb{Z}$ after composition with singular chains becomes an equivalence after restriction to $(\mathrm{Set}^I)^{\mathrm{op}}$ and application of* Lin*.*

Proof. Let D be the relative chain diagram. Then $H_k D = 0$ for $k < 1$ and $\mathrm{Lin}(D) = 0$. If $D \neq 0$, choose the least n with $A = H_n D \neq 0$. Splitting pointed chains into their unit and reduced parts, Künneth gives, canonically via the projections,

$$A(J \sqcup K) \cong A(J) \oplus A(K).$$

Indeed, besides the two individual relative complexes, the product comparison has a mixed term filtered by tensor products of a relative complex and reduced chains of a connected space. These start in degrees n and 1, so the mixed term contributes nothing in degree n. Also $A(\varnothing) = 0$.

Lemma 3.3 extends A additively. Adjunction then makes the canonical truncation $D \to A[n]$ null, since $\mathrm{Lin}(D) = 0$, contradicting its identity action on $H_n D$. Thus the comparison is a homology equivalence. Word concatenation makes its values connected H-spaces, so homological Whitehead [May83, Theorem B] makes it an equivalence. □

Only the rigid case needs the following lemma. The reader interested solely in Theorem 3.1 can safely skip this result.

Lemma 3.6 (Uniform zigzag)**.** *Objects in the same component of the groupoid completion of a rigid monoidal ∞-category admit a zigzag*

$$x \longleftarrow \bullet \longrightarrow \bullet \longleftarrow \bullet \longrightarrow y.$$

Consequently, groupoid completion of products of such categories preserves π_0.

Proof. Write tensor products by juxtaposition. For a zigzag $1 = x_0 \leftrightarrow \cdots \leftrightarrow x_n = y$, choose a suitable dual d_j of x_{j-1}, with $d_1 = 1$. Evaluation and coevaluation give

given arrow	induced arrows	
$x_{j-1} \to x_j$	$1 \to d_j x_j$	$x_{j-1} d_j \to 1$
$x_j \to x_{j-1}$	$d_j x_j \to 1$	$1 \to x_{j-1} d_j$.

Turn these arrows into spans by inserting identities. Tensoring the first spans connects 1 to

$$(d_1x_1)\cdots(d_nx_n) \simeq (x_1d_2)\cdots(x_{n-1}d_n)y;$$

tensoring the second spans connects this object to y. Only parentheses change. For general x, y, choose a dual d with $1 \to dx$ and $xd \to 1$. Then dy lies in the unit component. Apply the unit case to dy, tensor on the left by x, and compose the last arrow with $xdy \to y$. Empty zigzags use identities.

Every component contains an object, giving surjectivity on π_0. The fixed-shape zigzags assemble coordinatewise, giving injectivity. □

3.2 Proofs of the theorems

Proof of Theorem 3.1. Write $\mathrm{Bar}(M) = |[q] \mapsto M^q|$. By Dunn additivity [Lur17, Theorem 5.1.2.2], each M_i is an $\mathbb{E}_1$-monoid in $\mathbb{E}_1$-monoids. Consider

$$\mathrm{Bar}\left(\prod_i M_i^{J_i}\right) \longrightarrow \prod_i \mathrm{Bar}(M_i)^{J_i}.$$

The second multiplication evaluates word substitutions as maps preserving the first $\mathbb{E}_1$-structure. Taking bar in the first direction therefore makes this a natural transformation on $(\mathrm{Mon}^I_{\mathrm{free}})^{\mathrm{op}}$. Words are evaluated before bar, so their lengths may be unbounded. Both diagrams are connected and preserve finite products.

Lemma 3.4, applied degreewise to the bar and directly to the target, identifies both additive extensions compatibly with the comparison. Lemma 3.5 makes the bar comparison an equivalence. Set every $J_i = \{*\}$ and loop, using $\Omega\,\mathrm{Bar}(M) \simeq M^{\mathrm{gp}}$. □

For $\mathbb{E}_2$-monoidal ∞-categories this also gives $|\prod_i \mathcal{C}_i|^{\mathrm{gp}} \simeq \prod_i |\mathcal{C}_i|^{\mathrm{gp}}$: apply Theorem 3.1 degreewise to their nerves and realize. Group completion preserves realization, and realization of simplicial grouplike $\mathbb{E}_1$-spaces preserves products, as it is modeled by diagonals of bisimplicial groups.

Proof of Theorem 3.2. Duality makes the groupoid completions grouplike, and Lemma 3.6 identifies their components. Translation therefore reduces the proof to unit components. Replace each category by its full unit-component subcategory; it remains rigid monoidal, and the same lemma shows that all repeated products have connected groupoid completion.

Consider

$$\left|\prod_i \mathcal{C}_i^{J_i}\right| \longrightarrow \prod_i |\mathcal{C}_i|^{J_i}.$$

Tensor-word functors, formed before realization, give the free-monoid operations without any bound on word lengths. Both diagrams preserve finite products. The nerve $[q] \mapsto \mathrm{Fun}([q], \mathcal{C})^{\simeq}$ preserves products and realizes to $|\mathcal{C}|$, so Lemma 3.4 applied degreewise identifies the additive extensions of both chain diagrams, just as above. Lemma 3.5 proves the result on unit components; translation finishes the proof. □

4 Realization of simplicial monoids commutes with products

In this section, we deduce Theorem C from the following special case of it:

Theorem 4.1. *Let $M_\bullet$ be a simplicial monoid in spaces and J a small set. If $|M_\bullet^J|$ is grouplike, the canonical map*

$$|M_\bullet^J| \longrightarrow |M_\bullet|^J \tag{$*$}$$

is an equivalence.

The proof of this special case essentially uses the combinatorics of free monoids and groups to reduce to the well-known statements that simplicial group objects are Kan, and that realization of Kan simplicial objects commutes with products [Lur11, Corollary 4.2.8]. As shown in the introduction, Theorem C (Corollary 4.3) formally implies Theorems B and A (Corollaries 1.1 and 1.2). We begin by showing how to deduce Theorem C from Theorem 4.1.

4.1 Deduction of Theorem C from Theorem 4.1

To prove Theorem C using Theorem 4.1, we will require the following lemma clarifying the hypotheses of both theorems:

Lemma 4.2. *Let $(M_{i,\bullet})_{i\in I}$ be a small family of simplicial monoids in spaces, and let*

$$R_i = \mathrm{Img}((d_1, d_0)\colon \pi_0 M_{i,1} \longrightarrow \pi_0 M_{i,0} \times \pi_0 M_{i,0}).$$

The following are equivalent:

(1) $|\prod_{i\in I} M_{i,\bullet}|$ is grouplike.

(2) There exists some $r \geq 0$ such that for all but finitely many $i \in I$ and every $a \in \pi_0 M_{i,0}$ there exists some $b \in \pi_0 M_{i,0}$ so that ab is connected to 1 by a zigzag of at most r edges in R_i, traversed in either direction. Moreover, each $|M_{i,\bullet}|$ is grouplike.

Proof. Each R_i is reflexive. Since π_0 preserves colimits and products,

$$\pi_0 \left|\prod_{i\in I} M_{i,\bullet}\right| = \left(\prod_{i\in I} \pi_0 M_{i,0}\right) / \left\langle \prod_{i\in I} R_i \right\rangle,$$

where brackets denote the generated equivalence relation.

Assume (1). Each $|M_{i,\bullet}|$ is a monoid retract of the product realization, hence grouplike. If the uniform bound in (2) fails, choose distinct indices i_n and elements $a_{i_n} \in \pi_0 M_{i_n,0}$ for which no b admits such a witness of length at most n. Complete these to a tuple by inserting units elsewhere. A right inverse of its class would supply a finite zigzag of tuples, contradicting these choices.

Conversely, assume (2), let $F \subseteq I$ be the finite set of indices excluded there, and fix a tuple $(a_i)_{i\in I}$. Choose right-inverse witnesses of length at most r outside F. Grouplikeness supplies witnesses at the finitely many remaining indices, so all chosen lengths have a finite bound s. Reflexive steps pad these zigzags into a common alternating pattern of length $2s$, giving a zigzag of tuples. Thus every element of the monoid of components has a right inverse, so this monoid is a group. □

Corollary 4.3 (Theorem C)**.** *Let $(M_{i,\bullet})_{i\in I}$ be a small family of simplicial monoids in spaces. If $|\prod_{i\in I} M_{i,\bullet}|$ is grouplike, the canonical map*

$$\left|\prod_{i\in I} M_{i,\bullet}\right| \longrightarrow \prod_{i\in I} |M_{i,\bullet}| \tag{$*$}$$

is an equivalence.

Proof of Corollary 4.3. Since realization commutes with finite products, Lemma 4.2 lets us reduce to the case that there exists some $r \geq 0$ so that for all $i \in I$ and $a \in \pi_0 M_{i,0}$ there is some $b \in \pi_0 M_{i,0}$ so that ab is connected to 1 by a zigzag of at most r edges in R_i (in the notation of said lemma). Another application of Lemma 4.2 shows that the assumptions of Theorem 4.1 are satisfied for $J = I$ and $M_\bullet = \prod_{i\in I} M_{i,\bullet}$, and hence that $|M_\bullet^I| \xrightarrow{\simeq} |M_\bullet|^I$. Including each factor by inserting units elsewhere, and projecting, exhibits the comparison $(*)$ as a retract of the I-th power comparison for $M_\bullet$:

$$\begin{array}{ccccc} |\prod_{i\in I} M_{i,\bullet}| & \longrightarrow & |\prod_{i\in I} M_\bullet| & \longrightarrow & |\prod_{i\in I} M_{i,\bullet}| \\ \downarrow & & \downarrow\simeq & & \downarrow \\ \prod_{i\in I} |M_{i,\bullet}| & \longrightarrow & \prod_{i\in I} |M_\bullet| & \longrightarrow & \prod_{i\in I} |M_{i,\bullet}| \end{array}$$

and thus we conclude that also the outer vertical map is an equivalence, as desired. □

4.2 Setup and proof outline of Theorem 4.1

Write An for the ∞-category of spaces and y for the Yoneda embedding. Let $\mathrm{Mon}_{\mathrm{free}}$ and $\mathrm{Grp}_{\mathrm{free}}$ be the categories of free ordinary monoids and groups on small sets, with objects $\mathrm{Free}^{\mathrm{mon}}(J)$ and $\mathrm{Free}^{\mathrm{grp}}(J)$. We let $\mathcal{P}_\Sigma(-) \subseteq \mathrm{Fun}^\times((-)^{\mathrm{op}}, \mathrm{An})$ denote the free small sifted cocompletion: the full subcategory of small sifted colimits of representables [Lur09, Section 5.5.8].

Let $u : \mathrm{Mon}_{\mathrm{free}} \to \mathrm{Grp}_{\mathrm{free}}$ be group completion, and write $u_!$ for left Kan extension along u^{op} of space-valued presheaves. On small sifted cocompletions it is determined by $u_!(yF) = y(uF)$ [Lur09, Proposition 5.5.8.15]. We will need to consider presheaves on $\mathrm{Mon}_{\mathrm{free}}$ and $\mathrm{Grp}_{\mathrm{free}}$ and will restrict to ranks $< \kappa$ for uncountable regular cardinals κ to avoid size issues. Word sets and free resolutions stay within this bound, and letting κ vary gives the unrestricted statements.

Definition 4.4 (Product-preserving and grouplike presheaves)**.** For a small set J, write j_1 and j_2 ($j \in J$) for the generators of $\mathrm{Free}^{\mathrm{mon}}(J \amalg J)$ coming from the two copies of J. The *shears* are the endomorphisms

$$\sigma_J : \mathrm{Free}^{\mathrm{mon}}(J \amalg J) \longrightarrow \mathrm{Free}^{\mathrm{mon}}(J \amalg J), \qquad j_1 \longmapsto j_1, \quad j_2 \longmapsto j_1 j_2.$$

Let X be a presheaf on $\mathrm{Mon}_{\mathrm{free}}$.

1. X is *product-preserving* if it preserves all small products as a functor $\mathrm{Mon}_{\mathrm{free}}^{\mathrm{op}} \to \mathrm{An}$, i.e. sends coproducts in $\mathrm{Mon}_{\mathrm{free}}$ to products in An.

2. If X preserves finite products, the maps $\mathrm{Free}^{\mathrm{mon}}(J) \to \mathrm{Free}^{\mathrm{mon}}(J \amalg J)$, $j \mapsto j_1 j_2$, and $\mathrm{Free}^{\mathrm{mon}}(J) \to \mathrm{Free}^{\mathrm{mon}}(\varnothing)$ give each $X(\mathrm{Free}^{\mathrm{mon}}(J))$ a monoid structure. We call X *grouplike* if all these monoids are grouplike. Equivalently, X is grouplike if and only if it inverts the shears σ_J for all J, as σ_J acts on $X(\mathrm{Free}^{\mathrm{mon}}(J))^2$ by $(a, b) \mapsto (a, ab)$.

On $\mathrm{Grp}_{\mathrm{free}}$, $j \mapsto j^{-1}$ also supplies inverses.

4.2.1 Proof outline

The finite case is automatic, so assume J is infinite. We treat the main question uniformly in J by considering the presheaves $J \mapsto M_n^J$ and $J \mapsto |M_\bullet^J|$, which automatically preserve finite products. The monoid structure allows one to formally upgrade these to presheaves defined on $\mathrm{Mon}_{\mathrm{free}}$, e.g. we get $Y_n = Y_{M_n} \in \mathcal{P}_\Sigma(\mathrm{Mon}_{\mathrm{free}})$ (Definition 4.5) with $Y_n(\mathrm{Free}^{\mathrm{mon}}(J)) \simeq M_n^J$. Now we would like to show that $X = |Y_\bullet| \in \mathcal{P}_\Sigma(\mathrm{Mon}_{\mathrm{free}})$ is product-preserving (Definition 4.4),

giving Theorem 4.1 for all J at once. The hypothesis and Lemma 4.2 make X grouplike in the sense of Definition 4.4.

The technical heart of the proof shows that group completion $u\colon \mathrm{Mon}_{\mathrm{free}} \to \mathrm{Grp}_{\mathrm{free}}$ is actually a Dwyer–Kan localization in the ∞-categorical sense, so X being grouplike means that the unit $X \to u^* u_! X$ is an equivalence. Since u^* clearly preserves product-preserving presheaves, it suffices to show that $u_! X = |u_! Y_\bullet| \in \mathcal{P}_\Sigma(\mathrm{Grp}_{\mathrm{free}})$ preserves products.

We compute that $u_! Y_n$ is the presheaf $\mathrm{Free}^{\mathrm{grp}}(J) \mapsto (M_n^{\mathrm{gp}})^J$ (Lemma 4.7), which gives equivalences natural in J:

$$(u_! X)(\mathrm{Free}^{\mathrm{grp}}(J)) \simeq |(M_\bullet^{\mathrm{gp}})^J| \simeq |M_\bullet^{\mathrm{gp}}|^J \simeq |M_\bullet|^J,$$

where the second equivalence uses the classical fact that realization of simplicial group objects commutes with products, and the third uses that group completion preserves colimits and that $|M_\bullet|$ is already grouplike. Thus $u_! X$ is product-preserving, as required.

4.3 Group Completion as a Localization

Definition 4.5. For a monoid M in spaces, let Y_M be the presheaf on $\mathrm{Mon}_{\mathrm{free}}$ given by

$$Y_M(H) = \mathrm{Map}_{\mathbb{E}_1\text{-}\mathrm{Mon}}(H, M).$$

Note that we have equivalences $Y_M(\mathrm{Free}^{\mathrm{mon}}(J)) \simeq M^J$ natural in J.

Lemma 4.6. *For every monoid M in spaces, there exists a simplicial resolution by free monoids $F_\bullet \to M$ so that $|yF_\bullet| \simeq Y_M$. In particular, we have equivalences $|F_\bullet^J| \simeq M^J$ natural in sets J.*

Proof. This can be deduced from the classical free simplicial-monoid resolutions discussed in [Qui94, Section Q.5, pp. 95–96]: applied to a Kan simplicial-monoid model of M, these give a degreewise free resolution by a trivial fibration, whose arbitrary powers are again trivial fibrations between Kan complexes.

Alternatively, this can be done using the latching–matching induction of [Lur09, proof of Lemma 5.5.8.13], carried out in monoids in spaces over M. Suppose the construction has been done below degree n, and let $L_n F$ and N_n be the latching and relative matching objects, respectively, with $N_0 = M$. The matching objects are formed using homotopy limits; for example, $N_1 = F_0 \times_M F_0$. Choose points representing every component of N_n, giving a π_0-surjection $\mathrm{Free}^{\mathrm{mon}}(S_n) \twoheadrightarrow N_n$, and set

$$F_n = L_n F \amalg \mathrm{Free}^{\mathrm{mon}}(S_n),$$

where the coproduct is taken in monoids in spaces. The factorization $L_n F \to F_n \to N_n$ supplies the degeneracies and compatible faces needed to extend the construction. The split induction expresses $L_n F$ as a coproduct of the previously chosen $\mathrm{Free}^{\mathrm{mon}}(S_k)$, indexed by proper order-preserving surjections $[n] \twoheadrightarrow [k]$. Thus each F_n is an ordinary free monoid, and $F_n \to N_n$ is surjective on π_0.

Mapping out of $\mathrm{Free}^{\mathrm{mon}}(J)$ preserves limits, so the relative matching maps of $yF_\bullet \to Y_M$, evaluated at $\mathrm{Free}^{\mathrm{mon}}(J)$, are

$$F_n^J \longrightarrow N_n^J.$$

These are surjective on π_0, hence effective epimorphisms. Consequently, $yF_\bullet \to Y_M$ is a pointwise hypercover and $|yF_\bullet| \simeq Y_M$, as desired. □

Recall that $u\colon \mathrm{Mon}_{\mathrm{free}} \to \mathrm{Grp}_{\mathrm{free}}$ denotes group completion.

Lemma 4.7 (Kan extension along group completion)**.** *For a monoid M in spaces, the presheaf Y_M belongs to $\mathcal{P}_\Sigma(\mathrm{Mon}_{\mathrm{free}})$, and*

$$u_! Y_M \simeq \mathrm{Map}_{\mathbb{E}_1\text{-}\mathrm{Grp}}(-, M^{\mathrm{gp}})|_{\mathrm{Grp}_{\mathrm{free}}}.$$

The unit at $\mathrm{Free}^{\mathrm{mon}}(J)$ is $M^J \to (M^{\mathrm{gp}})^J$. Moreover, u^* is fully faithful.

Proof. Choose a resolution $yF_\bullet$ of Y_M by Lemma 4.6. We know that $u_! yF_i = yF_i^{\mathrm{gp}}$, hence

$$(u_! Y_M)(\mathrm{Free}^{\mathrm{grp}}(J)) \simeq |u_! yF_\bullet|(\mathrm{Free}^{\mathrm{grp}}(J)) \simeq |(F_\bullet^{\mathrm{gp}})^J| \simeq |F_\bullet^{\mathrm{gp}}|^J \simeq (M^{\mathrm{gp}})^J,$$

using that realization of simplicial groups preserves products ([Lur11, Corollary 4.2.8, p. 112]), and group completion preserves colimits. The maps are induced by group completion, identifying the unit. For a free group G, $u^* yG = Y_G$. Applying the preceding calculation to $M = G$ identifies the counit $u_! u^* yG \to yG$ as an equivalence, since $G^{\mathrm{gp}} \simeq G$. As $u_!$ and u^* preserve colimits, the counit is an equivalence on all presheaves, so u^* is fully faithful. □

4.3.1 Grouplike presheaves

This subsection is the technical heart needed to prove that $u\colon \mathrm{Mon}_{\mathrm{free}} \to \mathrm{Grp}_{\mathrm{free}}$ is a Dwyer–Kan localization at the shear maps in Theorem 4.11 below. There we first reduce to showing that it is a Dwyer–Kan localization in the 1-categorical sense. For this, we will show here that a presheaf $X\colon \mathrm{Mon}_{\mathrm{free}}^{\mathrm{op}} \to \mathrm{Set}$ preserving finite products is restricted from $\mathrm{Grp}_{\mathrm{free}}^{\mathrm{op}}$ if and only if it is grouplike, i.e. inverts the shear maps. We first need the following auxiliary lemma.

Lemma 4.8 (Fixed points)**.** *Let $X : \mathrm{Mon}_{\mathrm{free}}^{\mathrm{op}} \to \mathrm{Set}$ preserve finite products and be grouplike. Suppose $p : P \to P \amalg \{*\}$, extended by $p(*) = *$, satisfies $p^{r(x)}(x) = *$ for some finite $r(x)$ at each $x \in P$. Write $[x]$ for the corresponding generator and $[*] = 1$. If $S : \mathrm{Free}^{\mathrm{mon}}(P) \to \mathrm{Free}^{\mathrm{mon}}(P)$ sends $[x]$ to $[p(x)]$, then the only fixed point of $X(S)$ is the identity.*

Proof. Choose $r(x)$ least and define an endomorphism T of $\mathrm{Free}^{\mathrm{mon}}(P)$ by

$$T([x]) = [p^{r(x)-1}(x)] \cdots [p(x)][x].$$

The identity $S(T([x]))[x] = T([x])$ gives $X(T)(X(S)(a))\, a = X(T)(a)$. If $X(S)(a) = a$, cancellation in the group $X(\mathrm{Free}^{\mathrm{mon}}(P))$ gives $a = 1$. □

Lemma 4.9 (Discrete extension)**.** *Let $X : \mathrm{Mon}_{\mathrm{free}}^{\mathrm{op}} \to \mathrm{Set}$ preserve finite products and be grouplike. Then $X \to u^* u_! X$ is an isomorphism, where $u_!$ here denotes Set-valued left Kan extension.*

Proof. By Lemma 4.7, u^* is fully faithful also on Set-valued presheaves. Its essential image is closed under retracts, so a natural retraction of the unit suffices. Put $X_I = X(\mathrm{Free}^{\mathrm{mon}}(I))$. The colimit formula for Kan extension [Rie16, Theorem 6.2.1] describes $(u_! X)(\mathrm{Free}^{\mathrm{grp}}(J))$ by classes $[f, a]$, where $f : \mathrm{Free}^{\mathrm{grp}}(J) \to \mathrm{Free}^{\mathrm{grp}}(I)$ and $a \in X_I$, subject to

$$[u\alpha \circ f, b] = [f, X(\alpha)(b)] \qquad (\alpha : \mathrm{Free}^{\mathrm{mon}}(I) \to \mathrm{Free}^{\mathrm{mon}}(K),\ b \in X_K).$$

The unit sends $a \in X_J$ to $[\mathrm{id}, a]$. For a J-indexed family w of signed words in I, set $\mathrm{ev}_w(a) = X(\bar{w})(a, a^{-1})$, where $\bar{w} : \mathrm{Free}^{\mathrm{mon}}(J) \to \mathrm{Free}^{\mathrm{mon}}(I \amalg I)$ replaces negative letters by the second copy. We show that evaluation is independent of the chosen words and respects the relations.

Concatenation satisfies $\mathrm{ev}_{(v_j)_j}(a)\mathrm{ev}_{(w_j)_j}(a) = \mathrm{ev}_{(v_j w_j)_j}(a)$. Deleting at most one adjacent inverse pair in each word preserves evaluation: partition J according to the signs ii^{-1}, $i^{-1}i$, or the absence of a deletion, and use the group identities.

Let P be all signed words in I, with reduction $\mathrm{red} : P \to P$. Define $p : P \to P \amalg \{*\}$ by deleting the leftmost adjacent inverse pair in a nonreduced word and sending reduced words to $*$. Let $s = X(S)$ as in Lemma 4.8. Put

$$b = \mathrm{ev}_{\mathrm{id}_P}(a)\mathrm{ev}_{\mathrm{red}}(a)^{-1} \in X_P.$$

Split P into reduced and nonreduced words. On the first factor, $b = s(b) = 1$; on the second, one-step invariance and $\mathrm{red}(p(w)) = \mathrm{red}(w)$ give $s(b) = b$. Lemma 4.8 gives $b = 1$. Reindexing proves $\mathrm{ev}_w = \mathrm{ev}_{\mathrm{red}\circ w}$ for every family. Thus ev_f is well defined. Reducing ww^{-1} also gives $\mathrm{ev}_{w^{-1}}(a) = \mathrm{ev}_w(a)^{-1}$, so substitution of positive words yields

$$\mathrm{ev}_{u\alpha\circ f}(b) = \mathrm{ev}_f\big(X(\alpha)(b)\big).$$

For $\gamma : \mathrm{Free}^{\mathrm{mon}}(K) \to \mathrm{Free}^{\mathrm{mon}}(J)$, the same rule gives

$$\mathrm{ev}_{f\circ u\gamma}(a) = X(\gamma)(\mathrm{ev}_f(a)).$$

Thus $[f, a] \mapsto \mathrm{ev}_f(a)$ defines a natural retraction of the unit, since $\mathrm{ev}_{\mathrm{id}} = \mathrm{id}$. □

4.3.2 The Dwyer–Kan localization

Lemma 4.10 (A spectral criterion)**.** *Let $v : \mathcal{C} \to \mathcal{D}$ be a functor from a small ∞-category to a small ordinary category. Suppose that $hv : h\mathcal{C} \to \mathcal{D}$ is an equivalence and that v^* is fully faithful on space-valued presheaves. If every mapping space of $\mathcal{C}$ admits a grouplike $\mathbb{E}_1$-structure, then v is an equivalence.*

Proof. On spectrum-valued presheaves, restriction v^* remains fully faithful [Lur17, Remark 1.4.2.25] and is t-exact for the pointwise t-structures. Its left adjoint $v_!$ preserves connectivity. The induced adjunction on hearts is an equivalence, since the hearts are presheaves of abelian groups on the homotopy categories. Thus $v_!$ is conservative on bounded-below presheaves [KS17, Lemmas A.3.1 and A.3.2(ii)]. For connective E, the cofiber of $E \to v^*v_!E$ is connective and killed by $v_!$, so vanishes.

Apply this to $E = \Sigma^\infty_+ y_b$. Left Kan extension commutes with Σ^∞_+ and sends representables to representables, giving

$$\Sigma^\infty_+ \mathrm{Map}_{\mathcal{C}}(a, b) \xrightarrow{\sim} \Sigma^\infty_+ \mathrm{Hom}_{\mathcal{D}}(va, vb).$$

The bijection on components shows that each source component has trivial reduced suspension spectrum. Let G be the identity component of a mapping space. Its suspension is simply connected, so Freudenthal [Hat02, Corollary 4.24] gives $\Sigma G \simeq *$. Moreover, G is a loop space [Lur17, Theorem 5.2.6.10], hence a retract of $\Omega\Sigma G$ by the suspension–loop adjunction. Thus G is contractible, and translations identify all components with G. This proves full faithfulness; essential surjectivity follows from hv. □

Theorem 4.11. *Group completion $u : \mathrm{Mon}_{\mathrm{free}} \to \mathrm{Grp}_{\mathrm{free}}$ is the ∞-categorical localization at the shears σ_J.*

Proof. Let $q : \mathrm{Mon}_{\mathrm{free}} \to \mathcal{C}$ be the localization at the shears and $v : \mathcal{C} \to \mathrm{Grp}_{\mathrm{free}}$ the induced functor. The identity of $\mathrm{Free}^{\mathrm{mon}}(K)$ is a retract of σ_K in the arrow category: the sections send k to k_2 and k_1k_2, respectively, and both retractions send k_1 to 1 and k_2 to k. Thus $\sigma_J \amalg \mathrm{id}$ is a retract of $\sigma_J \amalg \sigma_K \cong \sigma_{J\amalg K}$, so q preserves finite coproducts [GLP26, Lemma 4.12, dualized].

Each restricted representable $F \mapsto \mathrm{Map}_{\mathcal{C}}(qF, A)$ therefore preserves finite products and is grouplike, since its shears are invertible. Put $X_A(-) = \mathrm{Hom}_{h\mathcal{C}}(q-, A) \colon \mathrm{Mon}^{\mathrm{op}}_{\mathrm{free}} \to \mathrm{Set}$, which

is finite-product preserving and grouplike. By Lemma 4.9 the unit map $X_A \xrightarrow{\cong} u^* u_! X_A$ is a natural isomorphism of Set-valued presheaves. Since hq is an ordinary localization, we compute

$$u_! X_A \cong (hv)_!(hq)_! X_A \cong (hv)_! y_A \cong y_{vA},$$

and under this identification the unit map becomes the effect of hv on homs:

$$\mathrm{Hom}_{h\mathcal{C}}(q-, A) = X_A \xrightarrow{\cong} u^* y_{vA} = \mathrm{Hom}_{\mathrm{Grp}_{\mathrm{free}}}(vq-, vA).$$

Thus hv is fully faithful, and clearly also essentially surjective.

Lemma 4.7 and $u^* = q^* v^*$ make v^* fully faithful. The mapping spaces of $\mathcal{C}$ are grouplike by the preceding restricted-representable argument, so Lemma 4.10 applies. □

Proposition 4.12 (Grouplike extension)**.** *The unit $X \to u^* u_! X$ is an equivalence for every finite-product-preserving grouplike presheaf X on ranks $< \kappa$, and for every grouplike $X \in \mathcal{P}_\Sigma(\mathrm{Mon}_{\mathrm{free}})$ without a rank bound.*

Proof. Being grouplike, X inverts the shears (see Definition 4.4). Theorem 4.11 puts X in the image of the fully faithful u^*, proving the unit equivalence. For $X \in \mathcal{P}_\Sigma(\mathrm{Mon}_{\mathrm{free}})$, choose κ to contain a small sifted presentation of X and the rank at which the unit is tested. □

4.4 Proof of Theorem 4.1

Proof of Theorem 4.1. The finite case is automatic, so suppose J is infinite. By Lemma 4.2 we deduce that $|M_\bullet^I|$ is grouplike for all sets I. By Lemma 4.7, $X = |Y_{M_\bullet}| \in \mathcal{P}_\Sigma(\mathrm{Mon}_{\mathrm{free}})$ satisfies $X(\mathrm{Free}^{\mathrm{mon}}(I)) \simeq |M_\bullet^I|$, so X is grouplike. Proposition 4.12 and Lemma 4.7 give

$$\begin{aligned} |M_\bullet^J| &\simeq (u_! X)(\mathrm{Free}^{\mathrm{grp}}(J)) \simeq |(M_\bullet^{\mathrm{gp}})^J| \\ &\simeq |M_\bullet^{\mathrm{gp}}|^J \simeq |M_\bullet|^J. \end{aligned}$$

Here $u_!$ and group completion preserve realizations; the third equivalence is [Lur11, Corollary 4.2.8], and the last uses grouplikeness of $|M_\bullet|$. The composite is the canonical comparison. □

A Four questions on the methodology for this chart

1. **What worked** (The productive ideas, methods, and uses of AI tools.)

   We first explored the question of whether $(-)^{\mathrm{gp}} : \mathbb{E}_\infty\text{-Mon} \to \mathbb{E}_\infty\text{-Grp}$ commutes with infinite products with the help of AI. The decisive step from our side was to direct the search toward a reduction to the analogous statement about the K-theory of additive categories, which was known. The two central ideas of the resulting proof—the inductive argument and the trace trick—were supplied by the AI; neither had been suggested in our prompts. Both were unexpected to us.

   Reaching this argument took approximately 1.5 hours of AI computation using Astra 6.0 with ultra reasoning, spread across several chats and guided by repeated human interventions. Requests to finish open steps, reconsider an approach, and simplify the argument helped move the exploration forward. Once the proof was available, we used independent versions of Astra to verify the solution, and then reorganized it further into a more conceptual, human-understandable form.

   The first $\mathbb{E}_2$-exploration took approximately three hours of computation and required several changes of strategy. The prompt that seemed decisive was to combine the $\mathbb{E}_\infty$-result with a homological analysis using the plus construction.

   Once both results had been obtained, we considered the case of rigid symmetric monoidal ∞-categories, which took roughly forty-five minutes of computation. The subsequent case of rigid $\mathbb{E}_1$-monoidal ∞-categories was even quicker: an initial result took around ten minutes of computation. We can only speculate that these gains in efficiency arose because the AI could build on an established proof strategy for a less general case. We have subsequently refined this result to its current state.

   The simplicial monoid case grew out of our wish for a proof that avoids homological algebra altogether and is purely categorical and combinatorial in nature. Finding the right statement, Theorem C, took a back-and-forth chat of about thirty minutes; a proof was then found by Astra 6.0 with ultra reasoning within approximately sixty minutes of computation. We subsequently simplified and canonicalized this proof considerably. In particular, many ad hoc constructions were replaced by canonical ones, turning statements about *structure* into statements about *properties*. For example, instead of manually constructing a lift of a grouplike presheaf X on $\mathrm{Mon}_{\mathrm{free}}$ along the restriction u^*, where $u\colon \mathrm{Mon}_{\mathrm{free}} \to \mathrm{Grp}_{\mathrm{free}}$ is group completion, we prove that the canonical unit map $X \to u^* u_! X$ is an equivalence (Proposition 4.12).

   Specific questions were more effective than general requests for further explanation. Combining human and AI understanding also proved very useful for simplifying and rewriting the arguments for intermediate steps. Of course, AI was also very helpful in the writing this document, for example with the figure in the introduction.

2. **What did not work** (Informative failures, blocked approaches, and output that did not yield understanding.)

   We explored a possible Künneth-type description of integral chains on infinite products of spaces through products of $\mathbb{E}_\infty$-coalgebras. This route did not yield the desired result; the proposed use of integral homology localization also ran into counterexamples.

   The early $\mathbb{E}_2$-output was difficult to assess. A substantial, typeset proof appeared before its central lemma and dependencies had become clear to us. Only after several generalizations did we obtain a short, more digestible proof. Similar problems emerged during the work on the generalization to simplicial monoids; however, the more general context allowed us to distil the core arguments more easily.

We also encountered a difficulty that was new to us: the AI had generated so many proofs and variants that it became hard to decide which to examine first. Understanding and dissecting any given proof required substantial effort, so we had to choose where to focus our attention. Here communication and coordination within the team turned out to be quite important.

3. **Difficulty landscape** (What was hard to discover, produce, verify, or understand?)

   The problem had already prompted plausible but incorrect arguments. The speed of this exploration should therefore not be taken as evidence that the problem was intrinsically easy. In the initial approach, a substantial part of the difficulty was carried by an existing K-theoretic product theorem: the result for additive ∞-categories used here is due to Córdova Fedeli [CF23, Corollary 2.11.1], while Kasprowski–Winges prove the related result for stable ∞-categories [KW19].

   The productive step was finding a way to bring that theorem to bear on the original question. Once this initial result had been established, it seemed that the AI, with further guidance from us, could build on its earlier successes to find generalizations, including ones that no longer relied on the difficult K-theoretic input.

4. **Where this points next**

   The immediate task is to record the product-preservation statement in the case of simplicial $\mathbb{E}_1$-monoids, Theorem C, together with all of its deductions, as sketched in Figure 1, in a more conceptual and human-written way. The reduction of the question about $\mathbb{E}_\infty$-monoids to the K-theoretic statement is also worth recording: its proof is aesthetically pleasing and much easier to understand than the more general argument. Other points raised during the investigation include the following.

   - The self-similarity of $\mathrm{Span}(\mathrm{Fin}^I)$ and its mapping monoids that appeared in the $\mathbb{E}_\infty$-argument suggests a broader question: can the argument be understood systematically by moving up categorical levels? One possibility is to investigate a suitable notion of a categorical spectrum with $\mathrm{Span}(\mathrm{Fin}^I)$ at its lowest level. This might lead to a more conceptual explanation of the degree-raising argument.
   - The subsequent generalizations to $\mathbb{E}_2$-monoids and then simplicial $\mathbb{E}_1$-monoids suggest that an algebraic functor, such as group completion, can be studied by analysing multi-coloured Lawvere theories through categories of free models of arbitrary rank, such as $\mathrm{Mon}_{\mathrm{free}}$ and $\mathrm{Grp}_{\mathrm{free}}$. This has parallels, for example, in condensed mathematics. The most surprising observation for us was that the group completion functor between the ordinary categories

     $$(-)^{\mathrm{gp}} : \mathrm{Mon}_{\mathrm{free}} \to \mathrm{Grp}_{\mathrm{free}}$$

     is in fact a Dwyer–Kan localization when these are viewed as ∞-categories.
   - Product preservation by group completion has immediate implications for homological stability, for example for automorphism groups arising from $\mathbb{E}_2$-monoidal structures, including mapping class groups and related families of groups. These implications need to be explored further.

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